Nonlinear Dynamics and Systems Theory, **22** (1) (2022) 46–57

A Geometric Study of Relative Operator Entropies

A. El Hilali [1], B. El Wahbi [1] and M. Chergui [2*]

[1] *Department of Mathematics, Faculty of Sciences, Ibn Tofail University, LAGA-Lab Kenitra, Morocco.*

[2] *Department of Mathematics, CRMEF-Kenitra, EREAM Team, LaREAMI-Lab, Kenitra, Morocco.*

Received: August 6, 2021; Revised: January 22, 2022

Abstract: This paper investigates geometrical properties for relative operator entropies acting on positive definite matrices by the use of the log-determinant metric. Particularly, we prove that both entropies $S_p(A|B)$ and $T_p(A|B)$ lie inside the sphere centered at the geometric mean of A and B with the radius equal to half the log-determinant distance between A and B.

Keywords: *parametric relative operator entropy; Tsallis relative operator entropy; general perturbation schemes; general systems.*

Mathematics Subject Classification (2010): 54C70, 47A63, 70K60, 93A10.

1 Introduction

Let $\mathcal{M}_n$ be the algebra of $n \times n$ matrices over $\mathbb{R}$, and $\mathbb{P}_n$ denote the cone of symmetric positive definite elements of $\mathcal{M}_n$. The identity matrix will be denoted by I. We recall that for any two matrices A and B from $\mathbb{P}_n$, we set $A \leq B$ to mean that $B - A \geq 0$, i.e., $B - A$ is a positive semi-definite matrix. This order, known in the literature by the Löwner order, is partial.

Kamei and Fujii introduced in [7, 8] the relative operator entropy $S(A|B)$ for two positive definite matrices A and B, by the following formula:

$$S(A|B) = A^{\frac{1}{2}} \log\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}, \tag{1}$$

which represents an extension of the operator entropy defined by Nakamura and Umegaki [18] and of the relative operator entropy introduced by Umegaki [21]. Later, a generalized parametric extension of the relative operator entropy was stated by Furuta in [10] as

$$S_p(A|B) = A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^p \log(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}, \qquad p \in \mathbb{R}. \tag{2}$$

* Corresponding author: mailto:chergui_m@yahoo.fr

