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A Geometric Study of Relative Operator Entropies

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Abstract: This paper investigates geometrical properties for relative operator entropies acting on positive definite matrices by the use of the log-determinant metric. Particularly, we prove that both entropies $S_p(A|B)$ and $T_p(A|B)$ lie inside the sphere centered at the geometric mean of A and B with the radius equal to half the log-determinant distance between A and B.

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1 Introduction

Let $\mathcal{M}_n$ be the algebra of $n \times n$ matrices over $\mathbb{R}$, and $\mathbb{P}_n$ denote the cone of symmetric positive definite elements of $\mathcal{M}_n$. The identity matrix will be denoted by I. We recall that for any two matrices A and B from $\mathbb{P}_n$, we set $A \leq B$ to mean that $B - A \geq 0$, i.e., $B - A$ is a positive semi-definite matrix. This order, known in the literature by the Löwner order, is partial.

Kamei and Fujii introduced in [7, 8] the relative operator entropy $S(A|B)$ for two positive definite matrices A and B, by the following formula:

$$S(A|B) = A^{\frac{1}{2}} \log\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}, \tag{1}$$

which represents an extension of the operator entropy defined by Nakamura and Umegaki [18] and of the relative operator entropy introduced by Umegaki [21]. Later, a generalized parametric extension of the relative operator entropy was stated by Furuta in [10] as

$$S_p(A|B) = A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^p \log(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}, \qquad p \in \mathbb{R}. \tag{2}$$

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The last generalization is to be understood as follows:

$$\lim_{p\to 0} S_p(A|B) = S_0(A|B) = S(A|B).$$

Applying the concept of the Tsallis relative entropy for matrices, Yanagi, Kuriyama and Furuichi presented in [20] another parametric extension of relative operator entropy as follows:

$$T_p(A|B) = \frac{A\sharp_p B - A}{p}, \qquad p \in [-1,1] \setminus \{0\}, \tag{3}$$

where $A\sharp_p B := A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^p A^{\frac{1}{2}}$ for all $p \in \mathbb{R}$ is the p-weighted geometric mean of A and B, which coincides, when taking $p = 1/2$, with the well known geometric mean that will be simply denoted in the sequel by $A\sharp B$. The last extension is justified by the following result proved in [8]:

$$\lim_{p\to 0} T_p(A|B) = S(A|B).$$

Some algebraic properties and inequalities involving the two parametric extensions of the relative operator entropy can be found, for instance, in [6, 9, 15, 16]. The representation of the Tsallis relative operator by means has allowed to derive some inequalities related to this operator.

Following the Kubo-Ando theory [12], it is known that for the representing function $f_\sigma(x) = 1\sigma x$ for an operator mean σ acting on positive matrices, the scalar inequality $f_{\sigma_1}(x) \le f_{\sigma_2}(x)$, $(x > 0)$ is equivalent to the operator one $A\sigma_1 B \le A\sigma_2 B$, for all positive definite matrices A and B. It is also worth recalling that for any non-negative monotone function f on $(0, +\infty)$, the binary map defined for two positive matrices A and B by $A\sigma B = A^{\frac{1}{2}} f\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}$ is a Kubo-Ando mean in the sense stated in [12].

The concept of entropy is widely used in estimating uncertainty existing in the state of a dynamic system and in measuring the degree of chaos in a deterministic system. Further details and approaches related to these notions can be found in [3, 11, 19], for example. Another very interesting topic in this field is the measure of the distance between the states of a dynamic system. It represents an essential tool to describe the evolution of quantum systems. Intensive studies have been carried out in the last few years concerning this point, see [1, 13, 14] for instance.

There are many definitions of the distance between states. Among those, we recall the log-determinant metric d_l, widely used in machine learning and quantum information, and defined for any matrices A and B from $\mathbb{P}_n$ as follows [17]:

$$d_l(A,B) = \log\det\left(A\nabla B\right) - \frac{1}{2}\log\det(AB), \tag{4}$$

where $A\nabla B := \dfrac{A+B}{2}$ denotes the arithmetic mean of A and B. We recall the following property [17] that holds true for every three matrices A, B and C from $\mathbb{P}_n$:

$$d_l(C\,A\,C, C\,B\,C) = d_l(A,B). \tag{5}$$

Recently, many authors employ this metric in developing various concepts and establishing interesting properties concerning some parametric means. For details, we refer the reader to [2, 4, 5] and the references therein.

In this paper, by the use of the log-determinant metric, we estimate some distances involving the relative operator entropies recalled in (1), (2) and (3). The present paper is organized in the following manner. In the second section, we present some preliminary tools and in the third one, we focus on stating our main results with respect to the log-determinant metric. The findings presented in this paper have geometric interpretations.

2 Preliminaries

In this section, we set out some preliminary results that will be needed in the sequel. We begin by presenting some results about some real functions that will be used as ingredients for our main results.

Lemma 2.1 *We have the following results:*

i) $\forall t > 0\,,\quad 1 - \dfrac{1}{t} \leq \log t \leq t - 1.$

ii) *The function* $p \longmapsto \dfrac{x^p - 1}{p}$ *is increasing on* $(0,1]$ *for each* $x > 0$.

iii) *The function* $p \longmapsto (1+p)^{\frac{1}{p}}$ *is decreasing on* $(0,1]$ *and* $\sup\limits_{p\in(0,1]} (1+p)^{\frac{1}{p}} = e$.

Proof. These statements are routine exercises in mathematical real analysis.

The following proposition will give an efficient tool in determining some results for the Tsallis relative operator entropy with respect to the log-determinant metric.

Proposition 2.1 *Let* C *be a positive definite matrix. If* $e\,I \leq C$*, then the map defined by*

$$p \longmapsto \left(\frac{C^p - I}{p}\right)^{1/2} \nabla \left(\frac{C^p - I}{p}\right)^{-1/2}$$

is increasing on $(0,1]$.

Proof. Let $x \geq e$. The functions $t \overset{h}{\longmapsto} t^{1/2} \nabla t^{-1/2}$ and $p \overset{k}{\longmapsto} \dfrac{x^p - 1}{p}$ are both increasing on $[1,\infty)$ and $(0,1]$, respectively. Since $x \geq e$ and using the third assertion in Lemma 2.1, we can set $x \geq (p+1)^{1/p}$ for every p in $(0,1]$.

So, for all $p \in (0,1]$, we get $\dfrac{x^p - 1}{p} \geq 1$. By taking the appropriate composition of the functions h and k, we deduce that the function

$$p \longmapsto \left(\frac{x^p - 1}{p}\right)^{1/2} \nabla \left(\frac{x^p - 1}{p}\right)^{-1/2}$$

is increasing on $(0,1]$.

Finally, if the condition $C \geq e\,I$ is satisfied, then the map $p \longmapsto \left(\dfrac{C^p - I}{p}\right)^{1/2} \nabla \left(\dfrac{C^p - I}{p}\right)^{-1/2}$ is increasing on $(0,1]$, thanks to the connection well known in the theory of Kubo-Ando between the inequalities satisfied by the representing functions and associate means operators. (For simplicity, this detail will be omitted in the following proofs.)

The main goal of the next three lemmas is to characterize some constants which will be needed in establishing appropriate conditions for our main results.

Lemma 2.2 *The function f defined on $(1,\infty)$ by $f(x) = x^{\frac{1}{2}} + x^{\frac{-1}{2}} - (\log x)^{\frac{1}{2}} - (\log x)^{\frac{-1}{2}}$ is strictly increasing. Moreover, there exists a unique α satisfying $f(\alpha) = 0$ and $1,76 < \alpha < 1,77$.*

Proof. For all $x > 1$, we have

$$f'(x) = \frac{(x-1)\sqrt{\log x}\log x + \sqrt{x}(1-\log x)}{2x\sqrt{x}\sqrt{\log x}\log x}.$$

For $1 < x \le e$, the inequality $(x-1)\sqrt{\log x}\log x + \sqrt{x}(1-\log x) \ge 0$ is simple to deduce.

If $x > e$, then we have

$$\begin{aligned}(x-1)\sqrt{\log x}\log x + \sqrt{x}(1-\log x) &> (x-1)\log x + \sqrt{x}(1-\log x)\\ &= (x-\sqrt{x})\log x + \sqrt{x} - \log x.\end{aligned}$$

One can easily check that $(x-\sqrt{x})\log x > 0$ and $\sqrt{x} - \log x > 0$. This implies that f increases strictly on $(1,\infty)$. In addition, since f is continuous on $(1,\infty)$, there is a bijection from $(1,\infty)$ onto $(\lim_{x\downarrow 1} f(x), \lim_{x\to+\infty} f(x)) = (-\infty,+\infty)$. This confirms the existence and uniqueness of α. Finally, by checking that $f(1,76) < 0 < f(1,77)$, the proof is ended.

Lemma 2.3 *The function g defined on $[1,\infty)$ by $g(x) = \log x - 1 + \dfrac{1}{x+1}$ is strictly increasing and there exists a unique $\beta > 1$ such that $g(\beta) = 0$. Moreover, $1,93 < \beta < 1,94$.*

Proof. It suffices to study the variations of the function g on $[1,\infty)$ and to deduce the results in a similar way as in the proof of Lemma 2.2.

Lemma 2.4 *The function h defined on $(1,\infty)$ by $h(x) = x + \dfrac{1}{x} - x^{\frac{1}{2}}\log x - (\log x)^{-1}$ is strictly increasing and there exists a unique $\sigma > 1$ satisfying $h(\sigma) = 0$. Moreover, $1,91 < \sigma < 1,92$.*

Proof. For all $x > 1$, we have

$$h'(x) = \frac{2\sqrt{x} - \log x - 2}{2\sqrt{x}} + \frac{1}{x(\log x)^2} - \frac{1}{x^2}.$$

By simple computations, one can check that

$$\frac{2\sqrt{x} - \log x - 2}{2\sqrt{x}} \ge 0 \quad \text{and} \quad \frac{1}{x(\log x)^2} - \frac{1}{x^2} \ge 0.$$

So h increases strictly on $(1,\infty)$. On the other hand, h is continuous on $(1,\infty)$, so it establishes a bijection from $(1,\infty)$ onto $\big(\lim_{x\downarrow 1} h(x), \lim_{x\to\infty} h(x)\big) = (-\infty,\infty)$. This proves the existence and the uniqueness of σ. To end the proof, it suffices to note that $h(1,91) < 0 < h(1,92)$.

Now we are in a position to state our findings, and hereafter, for any given two positive definite matrices A and B, we will constantly set $C = A^{-1/2}\,B\,A^{-1/2}$.

3 Statement of Findings

In this section, we aim to establish inequalities for the relative operator entropy and its generalizations with respect to the log-determinant metric.

Theorem 3.1 *Let $A, B \in \mathbb{P}_n$ be two positive definite matrices such that $\alpha A \leq B$. We have the following inequality:*

$$d_l\big(A, S(A|B)\big) \leq d_l(A, B), \tag{6}$$

where α is the fixed real number defined in Lemma 2.2.

Proof. Using Lemma 2.2, for every $x \geq \alpha$, we have

$$(\log x)^{\frac{1}{2}} + (\log x)^{\frac{-1}{2}} \leq x^{\frac{1}{2}} + x^{\frac{-1}{2}}.$$

So, if $\alpha A \leq B$, then $\alpha\, I \leq C$ and we get

$$(\log C)^{\frac{1}{2}} + (\log C)^{\frac{-1}{2}} \leq C^{\frac{1}{2}} + C^{\frac{-1}{2}}.$$

The last inequality combined with the monotonicity of the logarithm and the determinant gives

$$\log\det\left(\frac{1}{2}\big((\log C)^{\frac{1}{2}} + (\log C)^{\frac{-1}{2}}\big)\right) \leq \log\det\left(\frac{1}{2}\big(C^{\frac{1}{2}} + C^{\frac{-1}{2}}\big)\right),$$

that is,

$$d_l(I, \log C) \leq d_l(I, C),$$

or

$$d_l\big(A, S(A|B)\big) \leq d_l(A, B).$$

By this, the proof is concluded.

Now we will deal with the generalization of the last result for any operator $S_p(A|B)$ with $p \in (0, \frac{1}{2}]$.

Theorem 3.2 *Let A and B be two positive definite matrices such that $\sigma A \leq B$. The following inequality holds true:*

$$d_l(A, S_p(A|B)) \leq d_l(A, B) \tag{7}$$

for all $p \in (0, \frac{1}{2}]$. σ denotes the constant number in Lemma 2.4.

Proof. By the condition $\sigma A \leq B$ and for all $0 < p \leq \dfrac{1}{2}$, we have

$$\log C \leq C^p \log C \leq C^{\frac{1}{2}} \log C,$$

so

$$(C^p \log C)^{-1} \leq (\log C)^{-1}.$$

Hence

$$C^p \log C + (C^p \log C)^{-1} \leq C^{\frac{1}{2}} \log C + (\log C)^{-1}.$$

Since $C \geq \sigma\, I$, thanks to Lemma 2.4 we obtain the following inequalities:

$$C^p \log C + (C^p \log C)^{-1} \leq C^{\frac{1}{2}} \log C + (\log C)^{-1} \leq C + C^{-1},$$

and we can deduce that

$$(C^p \log C) + (C^p \log C)^{-1} + 2I \leq C + C^{-1} + 2I,$$

or equivalently,

$$\frac{(C^p \log C)^{\frac{1}{2}} + (C^p \log C)^{\frac{-1}{2}}}{2} \leq \frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}.$$

Thus,

$$\log\det\left(\frac{\left(C^p \log C\right)^{\frac{1}{2}} + \left(C^p \log C\right)^{\frac{-1}{2}}}{2}\right) \leq \log\det\left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right),$$

that is,

$$d_l(I, C^p \log C) \leq d_l(I, C).$$

The last inequality is equivalent to (2.4).

Remark 3.1 The proof of Theorem 3.2 is also valid for $p = 0$. So, it gives $d_l\big(A, S(A|B)\big) \leq d_l(A, B)$ for two definite positive matrices A and B when $\sigma A \leq B$. But since $\sigma > \alpha$, the result given by Theorem 3.1 is better than the one given in Theorem 3.2 for this case.

Theorem 3.3 *Let A and B be two positive definite matrices such that $eA \leq B$. For all two positive numbers $p, q \in [0, 1]$ such that $p \leq q$, we have*

$$d_l\big(A, S_p(A|B)\big) \leq d_l\big(A, S_q(A|B)\big). \tag{8}$$

Proof. One can easily check that for a given $x \geq e$, the function $p \longmapsto \big(x^p(\log x)\big)^{\frac{1}{2}} + \big(x^p(\log x)\big)^{\frac{-1}{2}}$ is increasing on $[0, 1]$. So, if $eA \leq B$, then for all $p, q \in [0, 1]$ such that $p \leq q$, we have

$$\frac{\big(C^p(\log C)\big)^{\frac{1}{2}} + \big(C^p(\log C)\big)^{\frac{-1}{2}}}{2} \leq \frac{\big(C^q(\log C)\big)^{\frac{1}{2}} + \big(C^q(\log C)\big)^{\frac{-1}{2}}}{2}.$$

Thus,

$$\log\det\left(\frac{\big(C^p(\log C)\big)^{\frac{1}{2}} + \big(C^p(\log C)\big)^{\frac{-1}{2}}}{2}\right) \leq \log\det\left(\frac{\big(C^q(\log C)\big)^{\frac{1}{2}} + \big(C^q(\log C)\big)^{\frac{-1}{2}}}{2}\right),$$

which means that the following inequalities hold:

$$d_l\big(I, C^p(\log C)\big) \leq d_l\big(I, C^q(\log C)\big),$$

or equivalently,

$$d_l\big(A, S_p(A|B)\big) \leq d_l\big(A, S_q(A|B)\big).$$

Remark 3.2 If the conditions stated in Theorems 3.1, 3.2 and 3.3 are not fulfilled by the matrices A and B, then the inequalities (6), (7) and (8) are no longer valid. This fact can be highlighted by the following counter-example.

Let us consider the following two positive definite matrices:

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}.$$

Computing with Matlab software, we find the following values:

- $d_l(A,B) = 0,0145 < d_l(A, S(A|B)) = 0,5080.$

- $d_l(A,B) < d_l\big(A, S_{1/2}(A|B)\big) = 0.4391 < d_l\big(A, S_{1/4}(A|B)\big) = 0,4729.$

Theorem 3.4 *Let A and B be two positive definite matrices such that $\beta\, A \leq B$. The following inequality holds for any $0 < p \leq 1$:*

$$d_l\big(A, T_p(A|B)\big) \leq d_l(A,B) \tag{9}$$

with β being the constant defined in Lemma 2.3.

Proof. Inequality (9) is equivalent to the following:

$$\log\det\left(\frac{(\frac{C^p-I}{p})^{\frac{1}{2}} + (\frac{C^p-I}{p})^{\frac{-1}{2}}}{2}\right) \leq \log\det\left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right). \tag{10}$$

So, let us prove that we have, for any $x \geq \beta$,

$$\left(\frac{x^p-1}{p}\right)^{\frac{1}{2}} + \left(\frac{x^p-1}{p}\right)^{\frac{-1}{2}} \leq x^{\frac{1}{2}} + x^{\frac{-1}{2}}, \tag{11}$$

or equivalently,

$$\frac{x^p-1}{p} + (\frac{x^p-1}{p})^{-1} \leq x + \frac{1}{x}.$$

For any $x \geq \beta$, we have $\log x \leq \dfrac{x^p-1}{p} \leq x-1$ and, consequently, we get

$$\frac{x^p-1}{p} + \left(\frac{x^p-1}{p}\right)^{-1} \leq x - 1 + (\log x)^{-1}.$$

Furthermore, according to Lemma 2.3, we have

$$x - 1 + (\log x)^{-1} \leq x + \frac{1}{x}.$$

So, if $\beta\, A \leq B$, then $\beta\, I \leq C$ and, consequently, the inequality (11) is established. This ends the proof.

Theorem 3.5 *Let A and B be two matrices from $\mathbb{P}_n$. If $e\, A \leq B$, then the following inequality*

$$d_l(A, T_p(A|B)) \leq d_l(A, T_q(A|B)) \tag{12}$$

holds true if $0 < p \leq q \leq 1$.

Proof. If $e\, A \leq B$, then $e\, I \leq C$. Employing Proposition 2.1, we get

$$\left(\frac{C^p-I}{p}\right)^{1/2} \nabla \left(\frac{C^p-I}{p}\right)^{-1/2} \leq \left(\frac{C^q-I}{p}\right)^{1/2} \nabla \left(\frac{C^q-I}{p}\right)^{-1/2}$$

for every $p, q \in (0,1]$ and $p \le q$. So,

$$\log\det\left[\left(\frac{C^p - I}{p}\right)^{1/2} \nabla \left(\frac{C^p - I}{p}\right)^{-1/2}\right] \le \log\det\left[\left(\frac{C^q - I}{q}\right)^{1/2} \nabla \left(\frac{C^q - I}{q}\right)^{-1/2}\right].$$

After some minor computations the last inequality implies

$$d_l\left(I, \frac{C^p - I}{p}\right) \le d_l\left(I, \frac{C^q - I}{q}\right)$$

which is equivalent to the desired one (12).

In what follows we focus on estimating the log-determinant distance between the geometric mean and different relative entropy operators.

Theorem 3.6 *Let A and B be two positive definite matrices such that $e\,A \le B$. We have the following inequality:*

$$d_l(A\sharp B, S(A|B)) \le \frac{1}{2} d_l(A, B). \tag{13}$$

Proof. The inequality

$$d_l(A\sharp B, S(A|B)) \le \frac{1}{2} d_l(A, B)$$

is equivalent to

$$\log\det\left(\frac{\left(C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}}\right)^{\frac{1}{2}} + \left(C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}}\right)^{\frac{-1}{2}}}{2}\right) \le \log\det\left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right)^{\frac{1}{2}}.$$

So, let us prove that if $C \ge e\,I$, then we have

$$\frac{\left(C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}}\right)^{\frac{1}{2}} + \left(C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}}\right)^{\frac{-1}{2}}}{2} \le \left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right)^{\frac{1}{2}},$$

or equivalently,

$$C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}} + \left(C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}}\right)^{-1} + 2I \le 2(C^{\frac{1}{2}} + C^{\frac{-1}{2}}). \tag{14}$$

Let us set for $x \ge e$, $\varphi(x) = 2x^{\frac{1}{2}} + 2x^{\frac{-1}{2}} - x^{\frac{-1}{2}}\log x - \left(x^{\frac{-1}{2}}\log x\right)^{-1} - 2$.

One can simply check that

$$\varphi'(x) = \frac{2(x - \log^2 x) + (\log x - 1)\log^2 x + (x-1)\log^2 x + x\,\log x\,(\log x - 1)}{2x\sqrt{x}(\log x)^2}.$$

It is clear that for $x \ge e$ one has $\varphi'(x) \ge 0$. So φ is increasing on $[e, \infty)$ and this implies that

$$\varphi(x) \ge \varphi(e) \ge 0.$$

Thus we get the inequality

$$\left(x^{\frac{-1}{2}}\log x\right) + \left(x^{\frac{-1}{2}}(\log x)\right)^{-1} + 2 \le 2(x^{\frac{1}{2}} + x^{\frac{-1}{2}}),$$

which can be rephrased as follows:

$$x^{\frac{-1}{4}}(\log x)x^{\frac{-1}{4}} + \left(x^{\frac{-1}{4}}(\log x)x^{\frac{-1}{4}}\right)^{-1} + 2 \le 2(x^{\frac{1}{2}} + x^{\frac{-1}{2}}).$$

By this, the inequality (14) is established and the proof of the desired result is ended.

A generalization of this result will be recited in the following theorem.

Theorem 3.7 *Let A and B be two positive definite matrices such that $\mu A \leq B$. If $\mu \geq 3,3$, then we have for all $p \in (0, \frac{1}{2}]$ the following inequality:*

$$d_l(A\sharp B, S_p(A|B)) \leq \frac{1}{2} d_l(A, B). \tag{15}$$

Proof. We consider the function l defined on $[3, \infty[$ by

$$l(x) = 2x^{\frac{1}{2}} + 2x^{\frac{-1}{2}} - \log x - \left(x^{\frac{-1}{2}} \log x\right)^{-1} - 2.$$

Let $x \geq 3,3$. We have

$$l'(x) = \frac{2(x - \log^2 x) + \sqrt{x} \log x [\, 2(\sqrt{x} - 1) \log x - \sqrt{x}\,]}{2x\sqrt{x}(\log x)^2}.$$

We can by routine computations show that $x - \log^2 x \geq 0$ and $2(\sqrt{x} - 1) \log x - \sqrt{x} \geq 0$. So, the function l is increasing on $[\mu, \infty)$. Consequently, for all $x \geq \mu$, we get

$$2x^{\frac{1}{2}} + 2x^{\frac{-1}{2}} \geq \log x + \left(x^{\frac{-1}{2}} \log x\right)^{-1} + 2. \tag{16}$$

On the other hand, since the map $p \mapsto x^p$ is increasing on $(0, \frac{1}{2}]$, we have for any $x \geq \mu$ that

$$\left(x^{p-\frac{1}{4}} (\log x) x^{\frac{-1}{4}}\right) \leq \left(x^{\frac{1}{4}} (\log x) x^{\frac{-1}{4}}\right) \leq \log x.$$

These inequalities added to the following ones:

$$\left(x^{p-\frac{1}{4}} (\log x) x^{\frac{-1}{4}}\right)^{-1} \leq \left(x^{\frac{-1}{4}} (\log x) x^{\frac{-1}{4}}\right)^{-1} \leq \left(x^{\frac{-1}{2}} \log x\right)^{-1},$$

enable us via (16) to deduce that

$$\left(x^{p-\frac{1}{4}} (\log x) x^{\frac{-1}{4}}\right) + \left(x^{p-\frac{1}{4}} (\log x) x^{\frac{-1}{4}}\right)^{-1} + 2 \leq 2(x^{\frac{1}{2}} + x^{\frac{-1}{2}}).$$

So, if we suppose that $\mu A \leq B$, then we can deduce from the last inequality that

$$\left(C^{p-\frac{1}{4}} (\log C) C^{\frac{-1}{4}}\right) + \left(C^{p-\frac{1}{4}} (\log C) C^{\frac{-1}{4}}\right)^{-1} + 2I \leq 2\left(C^{\frac{1}{2}} + C^{\frac{-1}{2}}\right),$$

which is equivalent to

$$\frac{\left(C^{p-\frac{1}{4}} (\log C) C^{\frac{-1}{4}}\right)^{\frac{1}{2}} + \left(C^{p-\frac{1}{4}} (\log C) C^{\frac{-1}{4}}\right)^{\frac{-1}{2}}}{2} \leq \left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right)^{\frac{1}{2}}.$$

So, this yields

$$\log \det \left(\frac{\left(C^{p-\frac{1}{4}} (\log C) C^{\frac{-1}{4}}\right)^{\frac{1}{2}} + \left(C^{p-\frac{1}{4}} (\log C) C^{\frac{-1}{4}}\right)^{\frac{-1}{2}}}{2}\right) \leq \log \det \left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right)^{\frac{1}{2}},$$

that is,

$$d_l(A\sharp B, S_p(A|B)) \leq \frac{1}{2} d_l(A, B).$$

Now, to estimate the distance $d_l(A\sharp B, T_p(A|B))$, we need the result quoted in the following lemma.

Lemma 3.1 *The function v defined by*

$$v(x) = x^{\frac{1}{2}} + 3x^{\frac{-1}{2}} - \left(x^{\frac{-1}{2}} \log x\right)^{-1} - 2$$

is strictly increasing on $[3, \infty)$ and there exists a unique $\lambda \geq 3$ satisfying $v(\lambda) = 0$. Moreover, $3,61 < \lambda < 3,62$.

Proof. For every $x \in (3, \infty)$, we have $v'(x) = \dfrac{w(x)}{2x\sqrt{x}(\log x)^2}$, where

$$t \in [3, \infty) \longmapsto w(t) := 2t - t\log t + (t-3)(\log t)^2.$$

Computing the first derivative of w on $(3, \infty)$, we find

$$w'(x) = \log^2 x - \log x + 1 + \frac{2(x-3)}{x} \log x > 0.$$

So, $w(x) \geq w(3) > 0$ and we can deduce that u is strictly increasing on $[3, \infty)$.
This fact added to the continuity of v implies that there exists a unique $\lambda \geq 3$ satisfying $v(\lambda) = 0$. The boundedness of λ is easy to check.

Theorem 3.8 *Let A and B be two positive definite matrices such that $\lambda A \leq B$. We have for all $p \in (0, 1]$ the following inequality:*

$$d_l(A\sharp B, T_p(A|B)) \leq \frac{1}{2} d_l(A, B), \tag{17}$$

where λ is the constant defined in Lemma 3.1.

Proof. The inequality

$$d_l(A\sharp B, T_p(A|B)) \leq \frac{1}{2} d_l(A, B)$$

is equivalent to

$$\log\det\left(\frac{\left(\dfrac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p}\right)^{\frac{1}{2}} + \left(\dfrac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p}\right)^{\frac{-1}{2}}}{2}\right) \leq \log\det\left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right)^{\frac{1}{2}}.$$

So, let us prove that if $C \geq \lambda\, I$, then we have

$$\frac{\left(\dfrac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p}\right)^{\frac{1}{2}} + \left(\dfrac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p}\right)^{\frac{-1}{2}}}{2} \leq \left(\frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2}\right)^{\frac{1}{2}},$$

which is equivalent to the inequality

$$\frac{(\dfrac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p}) + (\dfrac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p})^{-1} + 2I}{4} \leq \frac{C^{\frac{1}{2}} + C^{\frac{-1}{2}}}{2},$$

or

$$\left(\frac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p}\right) + \left(\frac{C^{p-\frac{1}{2}} - C^{\frac{-1}{2}}}{p}\right)^{-1} + 2I \leq 2.\left(C^{\frac{1}{2}} + C^{\frac{-1}{2}}\right). \tag{18}$$

Using $i)$ in Lemma 2.1 and since the map $p \mapsto \dfrac{x^p - 1}{p}$ is increasing on $(0, 1]$, we have for any $x \geq \lambda$ that

$$\log x \leq \frac{x^p - 1}{p} \leq x - 1,$$

and we deduce

$$x^{\frac{-1}{2}} \log x \leq \frac{x^{p-\frac{1}{2}} - x^{\frac{-1}{2}}}{p} \leq x^{\frac{1}{2}} - x^{\frac{-1}{2}}. \tag{19}$$

So, by taking into account the result of Lemma 3.1, we get the following inequalities:

$$\left(\frac{x^{p-\frac{1}{2}} - x^{\frac{-1}{2}}}{p}\right)^{-1} \leq \left(x^{\frac{-1}{2}} \log x\right)^{-1} \leq x^{\frac{1}{2}} + 3x^{\frac{-1}{2}} - 2. \tag{20}$$

From (19) and (20), we can deduce

$$\frac{x^{p-\frac{1}{2}} - x^{\frac{-1}{2}}}{p} + \left(\frac{x^{p-\frac{1}{2}} - x^{\frac{-1}{2}}}{p}\right)^{-1} + 2 \leq 2(x^{\frac{1}{2}} + x^{\frac{-1}{2}}).$$

Finally, we can confirm that if $\lambda\, A \leq B$, which means that $\lambda I \leq C$, the desired inequality (18) is satisfied. With this, the proof is achieved.

We end this paper by stating the following remark.

Remark 3.3 $i)$ Thanks to Theorems 3.7 and 3.8, we deduce that for convenient values of the parameter p, the operators $S_p(A|B)$ and $T_p(A|B)$ lie inside the sphere centered at the geometric mean of A and B with the radius equal to half the log-determinant distance between A and B.
$ii)$ If the conditions stated for the matrices A and B from Theorem 3.4 to Theorem 3.8 are not fulfilled, then the related results are no longer true. This fact is ensured by counterexamples, that we omit here for this paper not to become heavier.

4 Conclusion

In this work, we established some properties of some classes of operator entropies by employing the log-determinant distance. In particular, some geometrical aspects have been highlighted such as the localization of the entropies of two positive matrices with respect to their geometric mean.

References

[1] J. Anandan and Y. Aharonov. Geometry of quantum evolution. *Phys. Rev. Lett* **65** (14) (1990) 1697–1700.

[2] K. M. R. Audenaert. In-betweenness, a geometrical monotonicity property for operator means. *Linear Algebra Appl., 16th ILAS Conference Proceedings*, **438** (4) (2013) 1769–1778.

[3] F. Derouiche and T. Hamaizia. An Enhanced Bi-Directional Chaotic Optimization Algorithm. *Nonlinear Dynamics and Systems Theory* **20** (4) (2020) 365–373.

[4] T. H. Dinh, B. K. T. Vo and T. Y. Tam. In-sphere property and reverse inequalities for matrix means. *Electron. J. Linear Algebra* **35** (1) (2019) 35–41.

[5] T. H. Dinh, R. Dumitru and J. A. Franco. Some geometric properties of matrix means with respect to different metrics. *Positivity* (2020). https://doi.org/10.1007/s11117-020-00738-w.

[6] S. S. Dragomir. Inequalities for relative operator entropy in terms of Tsallis entropy. *Asian-European Journal of Mathematics* **12**(4) (2019) 1950063.

[7] J. I. Fujii and E. Kamei. Relative operator entropy in noncommutative information theory. *Math. Japon* **34** (3) (1989) 341–348.

[8] J. I. Fujii and E. Kamei. Uhlmann's interpolational method for operator means. *Math. Japon* **34** (4) (1989) 541–547.

[9] N. Furuichi and N. Minculete. Inequalities for Relative Operator Entropies and Operator Means. *Acta Math. Vietnam* **43** (2018) 607–618. https://doi.org/10.1007/s40306-018-0250-7.

[10] T. Furuta. Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators. *Linear Algebra Appl.* **381** (2004) 219–235.

[11] A. Khernane. Numerical Approximation of the Exact Control for the Vibrating Rod with Improvement of the Final Error by Particle Swarm Optimization. *Nonlinear Dynamics and Systems Theory* **20** (2) (2020) 179–190.

[12] F. Kubo and T. Ando. Means of positive linear operators. *Math. Ann.* **246** (3) (1980) 205–224.

[13] A.R. Kuzmak. Measuring distance between quantum states on a quantum computer. *Quantum Inf. Process* **20**, 269 (2021). https://doi.org/10.1007/s11128-021-03196-9.

[14] H. P. Laba and V. M. Tkachuk. Geometric characteristics of quantum evolution: curvature and torsion. *Cond. Matt. Phy.* **20**, 13003 (2017).

[15] I. Nikoufar. On operator inequalities of some relative operator entropies. *Advances in Mathematics* **259** (2014) 376–383.

[16] S. Kim. Operator entropy and fidelity associated with the geometric mean. *Linear Algebra and its Applications* **438** (2013) 2475–2483.

[17] S. Sra. A new metric on the manifold of kernel matrices with application to matrix geometric means. *Advances in Neural Information Processing Systems (NIPS)* **1** (2012) 144–152.

[18] M. Nakamura and H. Umegaki. A note on the entropy for operator algebras. *Proc. Japan Acad.* **37** (1967) 149–154.

[19] S. H. Strogatz. *Nonlinear Dynamics and Chaos*. Massachussets: Perseus Publishing, 2000.

[20] K. Yanagi, K. Kuriyama and S. Furuichi. Generalized Shannon inequalities based on Tsallis relative operator entropy. *Linear Algebra Appl.* **394** (2005) 109–118.

[21] H. Umegaki. Conditional expectation in operator algebra IV(entropy and information). *Kodai Math. Sem. Rep.* **14** (1962) 59–85.