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Existence Result for Positive Solution of a Degenerate Reaction-Diffusion System via a Method of Upper and Lower Solutions

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Abstract: The aim of this paper is to prove the existence of positive maximal and minimal solutions for a class of degenerate elliptic reaction-diffusion systems, including the uniqueness of the positive solution. To answer these questions, we use a technique described by Pao based on the method of upper and lower solutions, its associated monotone interactions and various comparison principles.

Keywords: *reaction-diffusion systems; degenerate elliptic systems; upper and lower solutions.*

Mathematics Subject Classification (2010): 35J62, 35J70, 35K57.

1 Introduction

Reaction-diffusion systems are widely used in biology, ecology, engineering, physics and chemistry. What we observe in modern scientific studies is the great interest of scientists in studying this type of systems; this confirms once again its importance in the development of applied and technological sciences. Various models and real examples can be found in various scientific fields, see Murray [13, 14]. The propagation of epidemics (Coronavirus, Hepatitis, ...), population dynamics, migration of biological species are among many examples of such phenomena. There are many methods and techniques for studying these issues. The reader can see some of them in the works of Alaa and Mesbahi [2,3,11,12], Abbassi et *al.* [1], Lions [10], Raheem [19] and the references therein.

In recent years, special attention has been paid to degenerate systems. However, most of the discussions relate to systems of two equations of the porous reaction medium type

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diffusion and with diffusion coefficients and specific reaction functions. This is because of their wide applications in various sciences. Among the important works on degenerate systems, we mention, for example, Alaa et *al.* [3], Al-Hdaibat et *al.* [4], Anderson [5], where we find techniques and methods of treatment.

The aim of this paper is to show the existence of positive maximal and minimal solutions for a quasilinear elliptic degenerate system, including the uniqueness of the positive solution. The two elliptic operators of the system under consideration can degenerate in the sense that $D_1(0) = 0$ or $D_2(0) = 0$. To answer these questions, we use a technique described by Pao, based on the upper and lower solutions. For more details on this technique, see Deuel and Hess [6], Pao et *al.* [15]- [18]. So, we need to construct suitable upper and lower solutions. We are therefore interested in studying the following system:

$$\begin{cases} -\mathbf{div}\left(D_1(u)\nabla u\right) = f(x,u,v) & \text{in } \Omega, \\ -\mathbf{div}\left(D_2(v)\nabla v\right) = g(x,u,v) & \text{in } \Omega, \\ u(x) = u_0(x)\ ,\ v(x) = v_0(x) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with the boundary $\partial\Omega$. D_1, D_2, f and g are prescribed functions satisfying the conditions in hypotheses (H_1) and (H_3). We remark that these two functions f and g verify simple properties, this allows us to choose them from a wide range. Below we will denote $C^\alpha(\Omega)$ to the space of Hölder continuous functions in Ω.

The results obtained in this paper can be applied to a large number of reaction-diffusion models, which arise in various fields of the applied science such as theory of shells, Brownian motion theory and many problems of physics and biology. In addition to the classical problems in the fields of mass-heat transfer, chemical reactors, and nuclear reactor dynamics, there are many recently developed models from enzyme kinetics, population growth, nerve axion problems, and others.

The system (1) can model the circulation of an ideal gas in a homogeneous porous medium with an isentropic flow. It can also model the steady state of phenomena such as the heat propagation in a two-components combustible mixture, chemical processes, the interaction of two non-self-limiting biological groups, etc. We send the reader to see many models and applications in Deuel and Hess [6], Friedman [7], Ladyženskaja et *al.* [8], Lei and Zheng [9], especially Pao [16,17] and the references therein. For example, the steady state of the Gas-Liquid Interaction Problem, when considering a dissolved gas **A** and a dissolved reactant **B** that interact in a bounded diffusion medium, is a special case of (1) with the reaction terms $f(u,v) = -\sigma_1 uv$, $g(u,v) = -\sigma_2 uv$, where σ_1 is the rate constant and $\sigma_2 = k_1\sigma_1$. In a more general reaction scheme called the (m,n) order reaction, the resulting equations are given by (1) with $f(x,u,v) = -\sigma_1 u^m v^n + q_1(x)$, $g(x,u,v) = -\sigma_2 u^m v^n + q_2(x)$, $m,n \geq 1$ are constants and $q_1(x), q_2(x) \geq 0$ are possible internal sources.

In the problems of molecular interactions and subsonic flows, a simple model for the density function u is given by (1) with the reaction function $f(u) = \sigma u^p$, with $\sigma > 0$, $p \geq 1$.

This model also describes the temperature in radiating bodies or gases and in nuclear reactors with positive temperature feedback. For more information on this model, and also to see other models, we refer the reader to Pao [17].

The rest of this paper is organized as follows. In the next section, we state our main result. In the third section, we provide some preliminary results on the scalar problem which we need in the proof of the main theorem. Next, we give some results concerning the approximate problem. The fifth section is devoted to proving the main result. Finally, we give an application to the problem under study. The paper ends with a concluding remarks and some perspectives.

2 Statement of the Main Result

In all that follows, we denote $\mathbf{u} \equiv (u, v)$, $\tilde{\mathbf{u}}_s \equiv (\tilde{u}, \tilde{v})$, $\hat{\mathbf{u}}_s \equiv (\hat{u}, \hat{v})$. The inequality $\hat{\mathbf{u}}_s \leq \tilde{\mathbf{u}}_s$ means that $\hat{u} \leq \tilde{u}$ and $\hat{v} \leq \tilde{v}$.

2.1 Assumptions

First, we have to clarify in which sense we want to solve our problem.

Definition 2.1 A pair of functions $\tilde{\mathbf{u}}_s \equiv (\tilde{u}, \tilde{v})$, $\hat{\mathbf{u}}_s \equiv (\hat{u}, \hat{v})$ in $C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ are called ordered upper and lower solutions of (1) if $\hat{\mathbf{u}}_s \leq \tilde{\mathbf{u}}_s$ and

$$\begin{cases} -\mathbf{div}\left(D\left(\hat{u}\right)\nabla \hat{u}\right) \leq f\left(x, \hat{u}, \hat{v}\right) & \text{in } \Omega, \\ -\mathbf{div}\left(D\left(\hat{v}\right)\nabla \hat{v}\right) \leq g\left(x, \hat{u}, \hat{v}\right) & \text{in } \Omega, \\ \hat{u}\left(x\right) \leq u_0\left(x\right) \ , \ \hat{v}\left(x\right) \leq v_0\left(x\right) & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and $\tilde{u}$, $\tilde{v}$ satisfies (2) with inequalities reversed.

For a given pair of ordered upper and lower solutions $\tilde{\mathbf{u}}_s$ and $\hat{\mathbf{u}}_s$, we define

$$\begin{aligned} S_1^* &= \left\{u \in C\left(\bar{\Omega}\right) \mid \hat{u} \leq u \leq \tilde{u}\right\} \quad , \quad S_2^* = \left\{v \in C\left(\bar{\Omega}\right) \mid \hat{v} \leq v \leq \tilde{v}\right\}, \\ S^* &= \left\{\mathbf{u} = (u, v) \in \left(C\left(\bar{\Omega}\right)\right)^2 \mid \hat{\mathbf{u}}_s \leq \mathbf{u} \leq \tilde{\mathbf{u}}_s\right\}. \end{aligned}$$

Now, we make the following assumptions:

(H_1) $f(x, \cdot)$, $g(x, \cdot) \in C^\alpha\left(\bar{\Omega}\right)$ and $u_0(x)$, $v_0(x) \in C^\alpha(\partial\Omega)$.

(H_2) $D_1(u) \in C^2([0, M_1])$, $D_1(u) > 0$ in $(0,\ M_1]$, and $D_1(0) \geq 0$ with $M_1 = \|\tilde{u}\|_{C(\bar{\Omega})}$.
$D_2(v) \in C^2([0, M_2])$, $D_2(v) > 0$ in $(0,\ M_2]$, and $D_2(0) \geq 0$ with $M_2 = \|\tilde{v}\|_{C(\bar{\Omega})}$.

(H_3) $f(\cdot, \mathbf{u})$, $g(\cdot, \mathbf{u}) \in C^1(S^*)$, and

$$\frac{\partial f}{\partial v}(\cdot, \mathbf{u}) \geq 0 \quad \text{and} \quad \frac{\partial g}{\partial u}(\cdot, \mathbf{u}) \geq 0 \quad \text{for all } \mathbf{u} \in S^*.$$

(H_4) There exists a constant $\delta_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a ball $\mathbf{K}$ outside of Ω with radius $r \geq \delta_0$ such that $\mathbf{K} \cap \bar{\Omega} = \{x_0\}$.

In the above system, we further assume $D_1(0) = 0$ or $D_2(0) = 0$.
Let $\gamma_1(x)$ and $\gamma_2(x)$ be smooth positive functions satisfying

$$\gamma_1(x) \geq \max\left\{-\frac{\partial f}{\partial u}(x,\ \mathbf{u}) \ ; \ \mathbf{u} \in S^*\right\} \text{ and } \gamma_1(x) \geq C_1(x) + \delta_1, \tag{3}$$

$$\gamma_2(x) \geq \max\left\{-\frac{\partial g}{\partial v}(x,\ \mathbf{u}) \ ; \ \mathbf{u} \in S^*\right\} \text{ and } \gamma_2(x) \geq C_2(x) + \delta_2 \tag{4}$$

for some constants $\delta_1,\ \delta_2 > 0$, where $C_1(x)$ and $C_2(x)$ are analogous to $C(x)$ defined in Section 3 by the relations (11), i.e.,

$$\begin{aligned} C_1(x) &= -\mathbf{div}\nabla(\tilde{u})\,D_1'(\theta_1) - f_u(x,\theta_2)\,, \\ C_2(x) &= -\mathbf{div}\nabla(\tilde{v})\,D_2'(\bar{\theta}_1) - g_v(x,\bar{\theta}_2)\,. \end{aligned}$$

We define for all $\mathbf{u} \in S^*$

$$F(x,\mathbf{u}) = \gamma_1(x)\,u + f(x,\mathbf{u}) \quad \text{and} \quad G(x,\mathbf{u}) = \gamma_2(x)\,v + g(x,\mathbf{u})\,. \tag{5}$$

A typical example where the result of this paper can be applied is

$$\begin{cases} -\Delta u^{\lambda} = p(x)\,u^{j}v^{k} & \text{in } \Omega, \\ -\Delta v^{\mu} = q(x)\,u^{\ell}v^{m} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\lambda,\ \mu > 1$, $j,\ k,\ \ell,\ m > 0$ and $p(x),\ q(x) > 0$ in Ω.

It is obvious that the problem (6) is a special case of (1) with

$$\begin{aligned} D_1(u) &= \lambda u^{\lambda-1} \quad , \quad D_2(v) = \mu u^{\mu-1} \quad , \quad u_0(x) = v_0(x) = 0, \\ f(x,u,v) &= p(x)\,u^{j}v^{k} \quad , \quad g(x,u,v) = q(x)\,u^{\ell}v^{m}. \end{aligned}$$

Lemma 2.1 *$F(x,\mathbf{u})$ and $G(x,\mathbf{u})$ are nondecreasing functions in $\mathbf{u}$ for all $\mathbf{u} \in S^*$.*

Proof. According to (H_3) and (5), we have for all $\mathbf{u} \in \mathbf{S}^*$

$$\frac{\partial F}{\partial v}(x,\mathbf{u}) = \frac{\partial f}{\partial v}(x,\mathbf{u}) \geq 0 \quad \text{and} \quad \frac{\partial G}{\partial u}(x,\mathbf{u}) = \frac{\partial g}{\partial u}(x,\mathbf{u}) \geq 0.$$

By $(3)-(5)$, we obtain

$$\frac{\partial F}{\partial u}(x,\mathbf{u}) = \gamma_1(x) + \frac{\partial f}{\partial u}(x,\mathbf{u}) \geq 0 \quad \text{and} \quad \frac{\partial G}{\partial v}(x,\mathbf{u}) = \gamma_2(x) + \frac{\partial g}{\partial v}(x,\mathbf{u}) \geq 0,$$

which implies the desired result.

2.2 The main result

Now, we can state the main result of this paper.

Theorem 2.1 *Let $\tilde{\mathbf{u}}_s$, $\hat{\mathbf{u}}_s$ be ordered positive upper and lower solutions of* (1)*, and let hypotheses $(H_1)-(H_4)$ hold. Then problem* (1) *has a minimal solution $\underline{\mathbf{u}}_s$ and a maximal solution $\overline{\mathbf{u}}_s$ such that $\hat{\mathbf{u}}_s \leq \underline{\mathbf{u}}_s \leq \overline{\mathbf{u}}_s \leq \tilde{\mathbf{u}}_s$. If $\underline{\mathbf{u}}_s = \overline{\mathbf{u}}_s\,(\equiv \mathbf{u}_s^*)$, then $\mathbf{u}_s^*$ is the unique positive solution in S^*.*

3 Preliminary Results for the Scalar Problem

To illustrate our basic approach to the coupled system (1), we first consider the following scalar quasilinear elliptic boundary problem:

$$\begin{cases} -\mathbf{div}\,(D(w)\,\nabla w) = h(x,w) & \text{in } \Omega, \\ u(x) = h(x) & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where D and h are prescribed functions satisfying hypotheses $(H_1)-(H_4)$ above.

The following theorem ensures the existence of positive solutions to the scalar problem (7). For the proof, we refer to Friedman [7], Ladyženskaja et *al.* [8], Pao and Ruan [15].

Theorem 3.1 *Let $\tilde{w}_s(x)$, $\hat{w}_s(x)$ be a pair of upper and lower solutions of* (7) *such that $\tilde{w}_s(x) \geq \hat{w}_s(x) > 0$ in Ω, and let hypotheses (H_1) and (H_3) hold. Then problem* (7) *has a classical solution $w_s(x)$ such that $\hat{w}_s(x) \leq w_s(x) \leq \tilde{w}_s(x)$ in $\overline{\Omega}$. Furthermore, there are maximal and minimal solutions $\overline{w}_s(x)$ and $\underline{w}_s(x)$ such that every solution $w_s \in S_0^*$ satisfies $\underline{w}_s(x) \leq w_s(x) \leq \overline{w}_s(x)$.*

Remark 3.1 We consider the scalar problem (7) for w. In this case, we can write

$$-\mathbf{div}\left(D\left(\hat{w}\right)\nabla\hat{w}\right) \leq h\left(x,\hat{w}\right) \text{ in } \Omega, \tag{8}$$

$$-\mathbf{div}\left(D\left(\tilde{w}\right)\nabla\tilde{w}\right) \geq h\left(x,\tilde{w}\right) \text{ in } \Omega. \tag{9}$$

Subtracting (9) from (8), we find

$$\begin{aligned} & -\mathbf{div}\left[D_1\left(\hat{w}\right)\nabla\left(\hat{w}-\tilde{w}\right)+\nabla\tilde{w}\left(\frac{D\left(\hat{w}\right)-D\left(\tilde{w}\right)}{\hat{w}-\tilde{w}}\left(\hat{w}-\tilde{w}\right)\right)\right] \\ \leq \quad & \frac{h\left(x,\hat{w}\right)-h\left(x,\tilde{w}\right)}{\hat{w}-\tilde{w}}\left(\hat{w}-\tilde{w}\right). \end{aligned}$$

According to the mean value theorem, there exist $\theta_1, \theta_2 \in [0, M]$, where $M = \|\tilde{w}\|_{C(\bar{\Omega})}$, such that

$$-\mathbf{div}\left[D\left(\hat{w}\right)\nabla z+\nabla\tilde{w}\left(D'\left(\theta_1\right)z\right)\right] \leq h_w\left(x,\theta_2\right)z$$

with $z = \hat{w} - \tilde{w}$, then

$$\begin{gathered} -\mathbf{div}\left(\nabla z\right)\left(D\left(\hat{w}\right)\right)-\nabla\left(D\left(\hat{w}\right)\right)\nabla z \\ -\mathbf{div}\left(\nabla\tilde{w}\left(D'\left(\theta_1\right)z\right)\right)-\nabla\left(\tilde{w}\right)D'\left(\theta_1\right)\nabla z-h_w\left(x,\theta_2\right)z \leq 0. \end{gathered}$$

We get

$$-D\left(\hat{w}\right)\Delta z+\left[-\nabla D\left(\hat{w}\right)-D'\left(\theta_1\right)\nabla\left(\tilde{w}\right)\right]\nabla z+\left[-\nabla.\nabla\left(\tilde{w}\right)D'\left(\theta_1\right)-h_w\left(x,\theta_2\right)\right]z \leq 0.$$

We denote

$$B\left(x\right) = -\nabla D\left(\hat{w}\right)-D'\left(\theta_1\right)\nabla\left(\tilde{w}\right), \tag{10}$$

$$C\left(x\right) = -\mathbf{div}\nabla\left(\tilde{w}\right)D'\left(\theta_1\right)-h_w\left(x,\theta_2\right). \tag{11}$$

To understand the calculations well, see Deuel and Hess [6], Friedman [7], Ladyženskaja et *al.* [8], Pao and Ruan [15].

Another important result is the following.

Lemma 3.1 *If $\underline{z}$, $\overline{z}$ are in $C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ and satisfy the relation*

$$\begin{cases} -\Gamma\left[\underline{z}\right]+\gamma\underline{z} \leq -\Gamma\left[\overline{z}\right]+\gamma\overline{z} & \text{in } \Omega, \\ \underline{z}\left(x\right) \leq \overline{z}\left(x\right) & \text{on } \partial\Omega \end{cases}$$

with $\Gamma[u] = \mathbf{div}\left(D\left(w\right)\nabla w\right)$, then $\underline{z}(x) \leq \overline{z}(x)$ on $\overline{\Omega}$.

Proof. Let $z(x) = \underline{z}(x) - \overline{z}(x)$. Firstly, we have

$$-\Gamma[\underline{z}] + \gamma\underline{z} \leq -\Gamma[\overline{z}] + \gamma\overline{z} = \gamma\overline{z} + h(x, \overline{z}) \equiv F(x, \overline{z}),$$

then

$$-\Gamma[\underline{z}] + \gamma(\underline{z} - \overline{z}) - h(x, \overline{z}) \leq 0. \tag{12}$$

On the other hand, we have

$$\Gamma[\overline{z}] + \gamma(\underline{z} - \overline{z}) + h(x, \underline{z}) \leq 0. \tag{13}$$

Adding (12) and (13), we obtain

$$\begin{array}{c} -\mathbf{div}\left[D(\underline{z})\nabla(\underline{z} - \overline{z}) + \nabla\overline{z}\left(\frac{D(\underline{z}) - D(\overline{z})}{\underline{z} - \overline{z}}(\underline{z} - \overline{z})\right)\right] \\ +2\gamma(\underline{z} - \overline{z}) + \frac{h(x,\underline{z}) - h(x,\overline{z})}{\underline{z} - \overline{z}}(\underline{z} - \overline{z}) \leq 0. \end{array}$$

According to the mean value theorem, $\exists\ \theta_1, \theta_2 \in [0, M]$ such that

$$-\mathbf{div}\left[D(\underline{z})\nabla z + \nabla\overline{z}(D'(\theta_1)z)\right] + 2\gamma(\underline{z} - \overline{z}) + \frac{h(x, \underline{z}) - h(x, \overline{z})}{\underline{z} - \overline{z}}(\underline{z} - \overline{z}) \leq 0$$

with $z = \underline{z} - \overline{z}$, then we get

$$\begin{array}{c} -\mathbf{div}(\nabla z)(D(\underline{z})) - \nabla(D(\underline{z}))\nabla z \\ -\mathbf{div}(\nabla\overline{z}(D'(\theta_1)z)) - \nabla(\overline{z})D'(\theta_1)\nabla z + h_w(x, \theta_2)z \leq 0. \end{array}$$

We obtain then

$$\begin{array}{c} -D(\underline{z})\Delta z - [\nabla D(\underline{z}) - D'(\theta_1)\nabla(\overline{z})]\nabla z \\ -\mathbf{div}\nabla(\overline{z})D'(\theta_1)z + 2\gamma z + h_w(x, \theta_2)z \leq 0, \\ -D(\underline{z})\Delta z + [-\nabla D(\underline{z}) - D'(\theta_1)\nabla(\overline{z})]\nabla z + \\ [\gamma + \mathbf{div}\nabla(\overline{z})D'(\theta_1) + h_w(x, \theta_2)]z \leq 0. \end{array}$$

We come to

$$-D(\underline{z})\Delta z + (\mathbf{B}(x))\nabla z + (\gamma - \mathbf{C}(x))z \leq 0,$$

where $\mathbf{B}(x)$ and $\mathbf{C}(x)$ are defined in the same way as $B(x)$ and $C(x)$ of relations (10) and (11), i.e.,

$$\begin{array}{rcl} \mathbf{B}(x) & = & -\nabla D(\underline{z}) - D'(\theta_1)\nabla(\overline{z}), \\ \mathbf{C}(x) & = & -\mathbf{div}\nabla(\overline{z})D'(\theta_1) + h_w(x, \theta_2). \end{array}$$

Assume, by contradiction, that $z(x)$ has a positive maximum at some point $x_0 \in \overline{\Omega}$. Then $x_0 \in \Omega$ and $\Delta z(x_0) \leq 0$, $\nabla z(x_0) = 0$. This implies that $(\gamma - C)z(x_0) \leq 0$, which is a contradiction because $\gamma - C = \delta > 0$.

4 Approximating Scheme

To prove the main theorem, we use the method of upper and lower solutions and its associated monotonic iteration. The basic idea of this method is that when using an upper solution or a lower solution as the initial iteration in a suitable iterative process, the resulting sequence of iterations is monotone and converges to a solution of the problem. Using then either $\hat{\mathbf{u}}_s$ or $\tilde{\mathbf{u}}_s$ as the initial iteration, we construct a sequence $\left\{\mathbf{u}_s^{(m)}\right\}$ from the iteration process

$$\begin{cases} -\Phi\left[u^{(m)}\right]+\gamma_1 u^{(m)}=F\left(x,\mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega, \\ -\Psi\left[v^{(m)}\right]+\gamma_2 v^{(m)}=G\left(x,\mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega, \\ u^{(m)}(x)=u_0(x)\ ,\ v^{(m)}(x)=v_0(x) & \text{on } \partial\Omega \end{cases} \tag{14}$$

with

$$\Phi[u]=\mathbf{div}\left(D_1(u)\nabla u\right) \quad , \quad \Psi[v]=\mathbf{div}\left(D_2(v)\nabla v\right).$$

We denote the sequence by $\left\{\underline{\mathbf{u}}_s^{(m)}\right\}$ if $\mathbf{u}_s^{(0)}=\hat{\mathbf{u}}_s$, and by $\left\{\overline{\mathbf{u}}_s^{(m)}\right\}$ if $\mathbf{u}_s^{(0)}=\tilde{\mathbf{u}}_s$. We call them minimal and maximal sequences, respectively. The existence of these sequences is ensured by the previous Lemma 3.1.

Lemma 4.1 *The minimal and maximal sequences* $\left\{\underline{\mathbf{u}}_s^{(m)}\right\}$, $\left\{\overline{\mathbf{u}}_s^{(m)}\right\}$ *exist and possess the monotone property*

$$\hat{\mathbf{u}}_s \le \underline{\mathbf{u}}_s^{(m)} \le \underline{\mathbf{u}}_s^{(m+1)} \le \overline{\mathbf{u}}_s^{(m+1)} \le \overline{\mathbf{u}}_s^{(m)} \le \tilde{\mathbf{u}}_s \quad \textit{for all } m \ge 1. \tag{15}$$

Proof. Firstly, we consider the scalar problem

$$\begin{cases} -\Phi\left[u^{(m)}\right]+\gamma_1 u^{(m)}=F\left(x,\mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega \\ u^{(m)}(x)=u_0(x) & \text{on } \partial\Omega. \end{cases} \tag{16}$$

We prove by induction. Start from $m=1$ and $\mathbf{u}_s^{(0)}=\hat{\mathbf{u}}_s$. By Definition 2.1, the components $\hat{u}$ of $\hat{\mathbf{u}}_s$ satisfy the relation

$$\begin{cases} -\Phi[\hat{u}]+\gamma_1\hat{u}\le F(x,\hat{\mathbf{u}}_s)=F\left(x,\underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \hat{u}(x)\le u_0(x) & \text{on } \partial\Omega \end{cases} \tag{17}$$

and the components $\tilde{u}$ of $\tilde{\mathbf{u}}_s$ satisfy the above inequalities (17) in reverse order, i.e.,

$$\begin{cases} -\Phi[\tilde{u}]+\gamma_1\tilde{u}\ge F(x,\tilde{\mathbf{u}}_s)\ge F\left(x,\underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \tilde{u}(x)\ge u_0(x) & \text{on } \partial\Omega. \end{cases}$$

Similarly, by considering the case $m=1$ and $\mathbf{u}_s^{(0)}=\tilde{\mathbf{u}}_s$, we have

$$\begin{cases} -\Phi[\hat{u}]+\gamma_1\hat{u}\le F(x,\hat{\mathbf{u}}_s)\le F(x,\tilde{\mathbf{u}}_s)=F\left(x,\overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \hat{u}(x)\le u_0(x) & \text{on } \partial\Omega \end{cases} \tag{18}$$

and the components $\tilde{u}$ of $\tilde{\mathbf{u}}_s$ satisfy the above inequalities (18) in reverse order, i.e.,

$$\begin{cases} -\Phi[\tilde{u}]+\gamma_1\tilde{u}\ge F(x,\tilde{\mathbf{u}}_s)=F\left(x,\overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ \tilde{u}(x)\ge u_0(x) & \text{on } \partial\Omega. \end{cases}$$

We see that $\tilde{u}$ and $\hat{u}$ are ordered upper and lower solutions of (16) for the case $m = 1$. By Theorem 3.1, problem (16) has also a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ such that $\hat{u} \leq \underline{u} \leq \overline{u} \leq \tilde{u}$. We choose $\underline{u}$ (or $\overline{u}$) as $\underline{u}^{(1)}$ if $\mathbf{u}_s^{(0)} = \hat{\mathbf{u}}_s$ and $\overline{u}$ (or $\underline{u}$) as $\overline{u}^{(1)}$ if $\mathbf{u}_s^{(0)} = \tilde{\mathbf{u}}_s$. So, we get $\hat{u} \leq \underline{u}^{(1)} \leq \overline{u}^{(1)} \leq \tilde{u}$.

The same works if we consider the problem

$$\begin{cases} -\Psi\left[v^{(m)}\right] + \gamma_2 v^{(m)} = G\left(x, \mathbf{u}_s^{(m-1)}\right) & \text{in } \Omega, \\ v^{(m)}(x) = v_0(x) & \text{on } \partial\Omega, \end{cases}$$

which gives $\hat{v} \leq \underline{v}^{(1)} \leq \overline{v}^{(1)} \leq \tilde{v}$.

This shows that $\underline{\mathbf{u}}_s^{(1)} \equiv \left(\underline{u}^{(1)}, \underline{v}^{(1)}\right)$ and $\overline{\mathbf{u}}_s^{(1)} \equiv \left(\overline{u}^{(1)}, \overline{v}^{(1)}\right)$ are solutions of (14) for $m = 1$ and satisfy $\hat{\mathbf{u}}_s \leq \underline{\mathbf{u}}_s^{(1)} \leq \overline{\mathbf{u}}_s^{(1)} \leq \tilde{\mathbf{u}}_s$.

Assume, by induction, that $\underline{\mathbf{u}}_s^{(m-1)} \leq \underline{\mathbf{u}}_s^{(m)} \leq \overline{\mathbf{u}}_s^{(m)} \leq \overline{\mathbf{u}}_s^{(m-1)}$ for some $m > 1$. Then, by the nondecreasing property of $F(., \mathbf{u})$, for $\mathbf{u} \in S^*$, we have

$$\begin{cases} -\Phi\left[\underline{u}^{(m)}\right] + \gamma_1 \underline{u}^{(m)} = F\left(x, \underline{\mathbf{u}}_s^{(m-1)}\right) \leq F\left(x, \underline{\mathbf{u}}_s^{(m)}\right), \\ -\Phi\left[\overline{u}^{(m)}\right] + \gamma_1 \overline{u}^{(m)} = F\left(x, \overline{\mathbf{u}}_s^{(m-1)}\right) \geq F\left(x, \overline{\mathbf{u}}_s^{(m)}\right), \\ \underline{u}^{(m)} = \overline{u}^{(m)} = u_0(x). \end{cases}$$

This implies that $\overline{u}^{(m)}$, $\underline{u}^{(m)}$ are ordered upper and lower solutions of (16) when $(m-1)$ is replaced by m and $\mathbf{u}_s^{(m)}$ is either $\underline{\mathbf{u}}_s^{(m)}$ or $\overline{\mathbf{u}}_s^{(m)}$. Again, by Theorem 3.1, problem (16) has a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$. We choose $\underline{u}$ (or $\overline{u}$) as $\underline{u}^{(m+1)}$ if $\mathbf{u}_s^{(m)} = \underline{\mathbf{u}}_s^{(m)}$ and $\underline{u}$ (or $\overline{u}$) as $\overline{u}^{(m+1)}$ if $\mathbf{u}_s^{(m)} = \overline{\mathbf{u}}_s^{(m)}$, which gives us $\underline{u}^{(m)} \leq \underline{u}^{(m+1)} \leq \overline{u}^{(m+1)} \leq \overline{u}^{(m)}$.

This choice ensures that $\underline{\mathbf{u}}_s^{(m+1)} \equiv \left(\underline{u}^{(m+1)}, \underline{v}^{(m+1)}\right)$ and $\overline{\mathbf{u}}_s^{(m+1)} \equiv \left(\overline{u}^{(m+1)}, \overline{v}^{(m+1)}\right)$ are solutions of (14) and possess the monotone property (15), which implies, by induction, the truth of the relation (15).

5 Proof of the Main Result

We are now ready to prove the main result of this work.

Proof. [Proof of Theorem 2.1] In view of Lemma 4.1 the pointwise limits

$$\lim_{m\longrightarrow\infty} \underline{\mathbf{u}}_s^{(m)} = \underline{\mathbf{u}}_s, \quad \lim_{m\longrightarrow\infty} \overline{\mathbf{u}}_s^{(m)} = \overline{\mathbf{u}}_s \tag{19}$$

exist and satisfy $\hat{\mathbf{u}}_s \leq \underline{\mathbf{u}}_s \leq \overline{\mathbf{u}}_s \leq \tilde{\mathbf{u}}_s$. To prove that $\underline{\mathbf{u}}_s$ and $\overline{\mathbf{u}}_s$ are, respectively, the minimal and maximal solutions of (1), we first consider the minimal sequence $\left\{\underline{\mathbf{u}}_s^{(m)}\right\} \equiv \left\{\underline{u}^{(m)}, \underline{v}^{(m)}\right\}$. Define for each m

$$\begin{cases} \underline{w}_1^{(m)}(x) = I_1\left(\underline{u}^{(m)}\right) = \int_0^{\underline{u}^{(m)}} D_1(s)\, ds, \\ \underline{Q}_1^{(m)}(x) = -\gamma_1(x)\, \underline{u}^{(m)} + F\left(x, \underline{\mathbf{u}}^{(m-1)}\right), \end{cases}$$

and

$$\begin{cases} \underline{w}_2^{(m)}(x) = I_2\left(\underline{v}^{(m)}\right) = \int_0^{\underline{v}^{(m)}} D_2(s)\, ds, \\ \underline{Q}_2^{(m)}(x) = -\gamma_2(x)\, \underline{v}^{(m)} + F\left(x, \underline{\mathbf{u}}^{(m-1)}\right). \end{cases}$$

We remark that $I_1'(\underline{u}) = D_1(\underline{u})$ and $I_2'(\underline{v}) = D_2(\underline{v})$. The inverse of $I_1(\underline{u})$ and $I_2(\underline{v})$ exist and are denoted, respectively, by $q_1(\underline{w}_1)$ and $q_2(\underline{w}_2)$.

The quasilinear problem (14) may be written as the scalar linear problem

$$\begin{cases} -\nabla^2 \underline{w}_1^{(m)} = \underline{Q}_1^{(m)}(x) & \text{in } \Omega, \\ -\nabla^2 \underline{w}_2^{(m)} = \underline{Q}_2^{(m)}(x) & \text{in } \Omega, \\ \underline{w}_1^{(m)}(x) = u_0^*(x) \ , \ \underline{w}_2^{(m)}(x) = v_0^*(x) & \text{on } \partial\Omega, \end{cases}$$

where $u_0^*(x) = I_1(u_0) \geq 0$ and $v_0^*(x) = I_2(v_0) \geq 0$. It is clear from (19) and (5) that $\underline{w}_1^{(m)} \to \underline{w}_1 \equiv I_1(\underline{u})$, $\underline{w}_2^{(m)} \to \underline{w}_2 \equiv I_2(\underline{v})$ and $\underline{Q}_1^{(m)} \to f(x, \underline{\mathbf{u}}_s)$, $\underline{Q}_2^{(m)} \to g(x, \underline{\mathbf{u}}_s)$ as $m \to \infty$.

By the argument in the proof for the scalar problem (7), $\underline{w}_1$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \underline{w}_1^{(m)}(x) = \underline{Q}_1^{(m)}(x), \\ \underline{w}_1^{(m)}(x) = u_0^*(x) \end{cases}$$

and $\underline{w}_2$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \underline{w}_2^{(m)}(x) = \underline{Q}_2^{(m)}(x), \\ \underline{w}_2^{(m)}(x) = v_0^*(x). \end{cases}$$

This shows that $\underline{\mathbf{u}}_s \equiv (\underline{u}, \underline{v})$, where $\underline{u} = q_1(\underline{w}_1)$ and $\underline{v} = q_2(\underline{w}_2)$ are solutions of (1) and $\underline{\mathbf{u}}_s \in S^*$.

Now, we show that $\overline{\mathbf{u}}_s$ is a solution of (1) in S^*, for this we consider the maximal sequence $\left\{\overline{\mathbf{u}}_s^{(m)}\right\} \equiv \left\{\overline{u}^{(m)}, \overline{v}^{(m)}\right\}$. Define for each m

$$\begin{cases} \overline{w}_1^{(m)}(x) = I_1\left(\overline{u}^{(m)}\right) = \int_0^{\overline{u}^{(m)}} D_1(s)\, ds, \\ \overline{Q}_1^{(m)}(x) = -\gamma_1(x)\, \overline{u}^{(m)} + F\left(x, \overline{\mathbf{u}}^{(m-1)}\right) \end{cases}$$

and

$$\begin{cases} \overline{w}_2^{(m)}(x) = I_2\left(\overline{v}^{(m)}\right) = \int_0^{\overline{v}^{(m)}} D_2(s)\, ds, \\ \overline{Q}_2^{(m)}(x) = -\gamma_2(x)\, \overline{v}^{(m)} + G\left(x, \overline{\mathbf{u}}^{(m-1)}\right). \end{cases}$$

Then the quasilinear problem (14) may be written as the scalar linear problem

$$\begin{cases} -\nabla^2 \overline{w}_1^{(m)} = \overline{Q}_1^{(m)}(x) & \text{in } \Omega, \\ -\nabla^2 \overline{w}_2^{(m)} = \overline{Q}_2^{(m)}(x) & \text{in } \Omega, \\ \overline{w}_1^{(m)}(x) = u_0^*(x) \ , \ \overline{w}_2^{(m)}(x) = v_0^*(x) & \text{on } \partial\Omega. \end{cases}$$

It is clear from (19) and (5) that $\overline{w}_1^{(m)} \to \overline{w}_1 \equiv I_1(\underline{u})$, $\overline{w}_2^{(m)} \to \overline{w}_2 \equiv I_2(\overline{v})$ and $\overline{Q}_1^{(m)} \to f(x, \overline{\mathbf{u}}_s)$, $\overline{Q}_2^{(m)} \to g(x, \overline{\mathbf{u}}_s)$ as $m \to \infty$.

By the argument in the proof for the scalar problem, $\overline{w}_1$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \overline{w}_1^{(m)}(x) = \overline{Q}_1^{(m)}(x), \\ \overline{w}_1^{(m)}(x) = u_0^*(x) \end{cases}$$

and $\overline{w}_2$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \overline{w}_2^{(m)}(x) = \overline{Q}_2^{(m)}(x), \\ \overline{w}_2^{(m)}(x) = v_0^*(x). \end{cases}$$

This shows that $\overline{\mathbf{u}}_s \equiv (\overline{u}, \overline{v})$, where $\overline{u} = q_1(\overline{w}_1)$ and $\overline{v} = q_2(\overline{w}_2)$ are solutions of (1) and $\overline{\mathbf{u}}_s \in S^*$.

To show that $\underline{\mathbf{u}}_s$ and $\overline{\mathbf{u}}_s$ are, respectively, minimal and maximal solutions of (1) in S^*, we observe that every solution $\mathbf{u} = (u, v)$ of (1) in S^* satisfies

$$\begin{cases} -\Phi[u] + \gamma_1 u = F(x, \mathbf{u}_s) \geq F\left(x, \underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ u(x) = u_0(x) & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Psi[v] + \gamma_1 v = G(x, \mathbf{u}_s) \geq G\left(x, \underline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ v(x) = v_0(x) & \text{on } \partial\Omega. \end{cases}$$

By (14) (with $m = 1$ and $u^{(1)} = \underline{u}^{(1)}$ and $v^{(1)} = \underline{v}^{(1)}$) we have

$$\begin{aligned} F\left(x, \underline{\mathbf{u}}_s^{(0)}\right) &= -\Phi\left[\underline{u}^{(1)}\right] + \gamma_1 \underline{u}^{(1)}, \\ G\left(x, \underline{\mathbf{u}}_s^{(0)}\right) &= -\Psi\left[\underline{v}^{(1)}\right] + \gamma_1 \underline{v}^{(1)}, \end{aligned}$$

then

$$\begin{aligned} -\Phi[u] + \gamma_1 u &\geq -\Phi\left[\underline{u}^{(1)}\right] + \gamma_1 \underline{u}^{(1)}, \\ -\Psi[v] + \gamma_1 v &\geq -\Psi\left[\underline{v}^{(1)}\right] + \gamma_1 \underline{v}^{(1)}. \end{aligned}$$

By Lemma 3.1 we have $u \geq \underline{u}^{(1)}$ and $v \geq \underline{v}^{(1)}$, i.e. $\mathbf{u} \geq \underline{\mathbf{u}}_s^{(1)}$. This implies, by Lemma 2.1, that $F(x, \mathbf{u}) \geq F\left(x, \mathbf{u}_s^{(1)}\right)$ and $G(x, \mathbf{u}) \geq G\left(x, \mathbf{u}_s^{(1)}\right)$. It follows by an induction argument that

$$\begin{aligned} F(x, \mathbf{u}) &\geq F\left(x, \underline{\mathbf{u}}_s^{(1)}\right) \geq F\left(x, \underline{\mathbf{u}}_s^{(2)}\right) \geq \ldots \geq F\left(x, \underline{\mathbf{u}}_s^{(m)}\right), \\ G(x, \mathbf{u}) &\geq G\left(x, \underline{\mathbf{u}}_s^{(1)}\right) \geq G\left(x, \underline{\mathbf{u}}_s^{(2)}\right) \geq \ldots \geq G\left(x, \underline{\mathbf{u}}_s^{(m)}\right), \end{aligned}$$

then $\mathbf{u} \geq \underline{\mathbf{u}}_s^{(m)}$, for every $m \geq 1$.

In the same way, we observe that every solution $\mathbf{u} = (u, v)$ of (1) in S^* satisfies

$$\begin{cases} -\Phi[u] + \gamma_1 u = F(x, \mathbf{u}) \leq F\left(x, \overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ u(x) = u_0(x) & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Psi[v] + \gamma_1 v = G(x, \mathbf{u}) \leq G\left(x, \overline{\mathbf{u}}_s^{(0)}\right) & \text{in } \Omega, \\ v(x) = v_0(x) & \text{on } \partial\Omega. \end{cases}$$

By (14) (with $m = 1$ and $u^{(1)} = \overline{u}^{(1)}$ and $v^{(1)} = \overline{v}^{(1)}$) we have

$$\begin{aligned} F\left(x, \overline{\mathbf{u}}_s^{(0)}\right) &= -\Phi\left[\overline{u}^{(1)}\right] + \gamma_1 \overline{u}^{(1)}, \\ G\left(x, \overline{\mathbf{u}}_s^{(0)}\right) &= -\Psi\left[\overline{v}^{(1)}\right] + \gamma_1 \overline{v}^{(1)}, \end{aligned}$$

then

$$\begin{aligned} -\Phi[u] + \gamma_1 u &\leq -\Phi\left[\overline{u}^{(1)}\right] + \gamma_1 \overline{u}^{(1)}, \\ -\Psi[v] + \gamma_1 v &\leq -\Psi\left[\overline{v}^{(1)}\right] + \gamma_1 \overline{v}^{(1)}. \end{aligned}$$

By Lemma 3.1, we have $u \leq \overline{u}^{(1)}$ and $v \leq \overline{v}^{(1)}$, i.e., $u_s \leq \overline{\mathbf{u}}_s^{(1)}$. This implies, by Lemma 2.1, that $F(x, \mathbf{u}) \leq F\left(x, \overline{\mathbf{u}}_s^{(1)}\right)$ and $G(x, \mathbf{u}) \leq G\left(x, \overline{\mathbf{u}}_s^{(1)}\right)$. It follows by an induction argument that

$$\begin{aligned} F(x, \mathbf{u}) &\leq F\left(x, \overline{\mathbf{u}}_s^{(1)}\right) \leq F\left(x, \overline{\mathbf{u}}_s^{(2)}\right) \leq ... \leq F\left(x, \overline{\mathbf{u}}_s^{(m)}\right), \\ G(x, \mathbf{u}) &\leq G\left(x, \overline{\mathbf{u}}_s^{(1)}\right) \leq G\left(x, \overline{\mathbf{u}}_s^{(2)}\right) \leq ... \leq G\left(x, \overline{\mathbf{u}}_s^{(m)}\right), \end{aligned}$$

which implies $u_s \leq \overline{\mathbf{u}}_s^{(m)}$.

Letting $m \to \infty$ and using relation (19) lead to $\underline{\mathbf{u}}_s \leq \mathbf{u} \leq \overline{\mathbf{u}}_s$. This proves the minimal and maximal property of $\underline{\mathbf{u}}_s$and $\overline{\mathbf{u}}_s$. Finally, if $\underline{\mathbf{u}}_s = \overline{\mathbf{u}}_s$ $(\equiv \mathbf{u}_s^*)$, then this maximal-minimal property ensures that $\mathbf{u}_s^*$ is the unique positive solution in S^*.

6 Application

As an application of the obtained theorem, we give a model concerning the type of diffusion in porous media, where the diffusion coefficients are degenerate; it is the following two-species Lotka–Volterra competition steady-state model:

$$\begin{cases} -D_1(x)\nabla^2 u^\alpha = u(a_1 - b_1 u - c_1 v), \\ -D_2(x)\nabla^2 v^\beta = v(a_2 - b_2 u - c_2 v), \qquad t > 0,\ x \in \Omega, \\ u(x) = u_0(x) > 0\ ,\ v(x) = v_0(x) > 0, \end{cases} \tag{20}$$

where for each $i = 1,\ 2,\ \alpha,\ \beta,\ a_i,\ b_i,\ c_i$ are positive constants, and $\alpha > 1$, $\beta > 1$, with $D_i(x) > 0$ on $\bar{\Omega}$. For more details on this model, we refer the reader to Pao in [16, 17].

7 Concluding Remarks and Perspectives

This work has mainly focused on the question of the existence and the uniqueness of positive maximal and minimal solutions for a class of degenerate reaction-diffusion systems. It should be noted that the results obtained can be applied to a number of models arising from biology, ecology and biochemistry as well as to models in several fields of applied sciences and engineering. We have developed original methods to overcome certain difficulties, and despite the complexity of the model studied, we have succeeded in obtaining an existence result.

There are many additional important open problems, which we hope to address in the near future, they are: Numerical simulation, Generalization to the parabolic case, Generalization to the case of a higher order system. This list of questions corresponds to a work in progress or prospective work. Some are a continuation of the work already done, and some are new research projects. This not only makes it possible to delve deeper into the theoretical study, but also goes beyond the theoretical framework by developing models and techniques.

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