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Fractional Discrete Neural Networks with Different Dimensions: Coexistence of Complete Synchronization, Antiphase Synchronization and Full State Hybrid Projective Synchronization

Mohamed Mellah [1], Adel Ouannas [2,*], Amina-Aicha Khennaoui [3] and Giuseppe Grassi [4]

[1] *Laboratory of Mathematics, Informatics and Systems (LAMIS), University of Larbi Tebessi, Tebessa, Algeria.*
[2] *Department of Mathematics and Computer Science, University of Larbi Ben M'hidi, Oum El Bouaghi, Algeria.*
[3] *Laboratory of Dynamical Systems and Control, University of Larbi Ben Mhidi, Oum El Bouaghi, Algeria.*
[4] *University of Salento, Department of Engineering for Innovation, 73100 Lecce, Italy.*

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Abstract: This paper aims to present the coexistence of complete synchronization, antiphase synchronization and full state hybrid projective synchronization in two fractional discrete neural networks with different dimensions. A new theorem is proved, which assures the coexistence of these synchronization types in different dimensional fractional discrete neural networks. Finally, simulation results are reported to confirm the effectiveness of the synchronization approach illustrated herein.

Keywords: *neural networks; synchronization; discrete-fractional calculus.*

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* Corresponding author: mailto:dr.ouannas@gmail.com

1 Introduction

Chaos synchronization is a key issue in the study of nonlinear dynamical systems described by differential equations (continuous-time systems) or difference equations (discrete-time systems). Starting from that concept, in the last three decades, different types of chaos synchronization have been conceived for both integer-order systems and fractional-order systems described by non-integer order derivative [1–6, 8, 9]. For example, when each response system variable synchronizes with a linear combination of drive system variables, full state hybrid projective synchronization is achieved [2]. On the other hand, when each response system variable synchronizes with the opposite of the drive system variable, antiphase synchronization is obtained [4]. These synchronization types have been applied to both integer-order systems [1–4] and fractional systems described by non-integer order derivative [5–7]. In particular, a recent paper has analyzed the coexistence of some synchronization types in two chaotic systems described by fractional derivatives [7].

It should be noted that, differently from fractional systems described by non-integer order derivative, few synchronization types have been introduced for fractional systems described by non-integer order difference operators [10–14]. For example, in [10], the complete synchronization of two chaotic fractional Grass-Miller maps is proved. In [11], full state hybrid projective synchronization is achieved between two fractional discrete systems of different dimensions. In [12], the complete synchronization of two fractional forms of the discrete double scroll is illustrated. In [13], the full state hybrid projective synchronization between an integer-order discrete system and a fractional-order discrete system is achieved. In [14], the complete synchronization between two fractional forms of a novel generalized Hénon map is reported. When considering fractional discrete-time neural networks, which represent a class of discrete systems described by non-integer order difference operators [15–19], it should be noted that very few examples of synchronization have been published to date. For example, in [20], the complete synchronization of fractional discrete neural networks with time delays is discussed. In [21], the complete synchronization of discrete-time fractional-order complex-valued neural networks has been illustrated. To the best of the authors' knowledge, no synchronization method for fractional discrete neural networks, different from the complete synchronization, has been reported to date. Additionally, the topic related to the coexistence of different synchronization types in fractional discrete neural networks is unexplored.

The paper is organized as follows. In Section 2, some basic notions regarding discrete fractional calculus are given. In Section 3, the two-dimensional fractional discrete neural network, considered as a master system, and the three-dimensional fractional discrete neural network, considered as a slave system, are illustrated. In Section 4, a new theorem is proved, which assures the coexistence of complete synchronization, antiphase synchronization and full state hybrid projective synchronization in the two neural networks with different dimensions considered in the previous section. Finally, simulation results are reported to confirm the effectiveness of the synchronization approach illustrated herein.

2 Discrete Fractional Operators

In the following we recall some definitions and preliminaries of the discrete fractional calculus. The notation $\mathbb{N}_a$ denotes the isolated time scale and $\mathbb{N}_a = \{a, a+1, a+2, ...\}$, $(a \in \mathbb{R} \text{ fixed})$.

Definition 2.1 [22] The ν–th fractional sum is defined by

$$\Delta_a^{-\nu} X\left(t\right) = \frac{1}{\Gamma\left(\upsilon\right)} \sum_{s=a}^{t-\nu} \left(t-s-1\right)^{\left(\nu-1\right)} X\left(s\right), \quad t \in \mathbb{N}_{a+\nu},\ \nu > 0, \tag{1}$$

where $X\left(t\right) : \mathbb{N}_a \to \mathbb{R}$, and the term $t^{(\upsilon)}$ denotes the falling function defined in terms of the Gamma function Γ as

$$t^{(\nu)} = \frac{\Gamma\left(t+1\right)}{\Gamma\left(t+1-\nu\right)}. \tag{2}$$

Definition 2.2 [23] For $0 < \nu < 1$, and $X\left(t\right)$ defined on $\mathbb{N}_a$, the Caputo-like delta difference is defined by

$$^{C}\Delta_a^{\nu} X\left(t\right) = \frac{1}{\Gamma\left(n-\nu\right)} \sum_{s=a}^{t-(n-\nu)} \left(t-s-1\right)^{\left(-\nu\right)} \Delta X\left(s\right), \quad t \in \mathbb{N}_{a+1-\nu}. \tag{3}$$

To deal with fractional-order systems in discrete time and to employ our numerical tools, we recall the following result.

Theorem 2.1 *[24] For the delta fractional difference equation*

$$\begin{cases} ^{C}\Delta_a^{\nu} u(t) = f(t+\upsilon-1, u(t+\nu-1)), \\ \Delta^k = u_k,\ n = [\nu]+1,\ k = 0,1,...,n-1, \end{cases} \tag{4}$$

the equivalent discrete integral equation can be obtained as

$$u(t) = u_0(t) + \frac{1}{\Gamma(\nu)} \sum_{s=a+n-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s+\nu-1, u(s+\nu-1)), t \in \mathbb{N}_{\alpha+n}, \tag{5}$$

where

$$u_0(t) = \sum_{k=0}^{m-1} \frac{(t-a)^k}{k} \Delta^k u(a). \tag{6}$$

3 Master and Slave Fractional Discrete Neural Networks

In this section, we will endeavour to show that the two fractional maps, even though they have different dimensions, can still be synchronized within time in the light of an appropriate synchronization scheme.

3.1 The master system

One may consider the two-dimensional fractional discrete neural network introduced in [19] as a master system and distinguish its states by typing the subscript m for each of them. That is,

$$\begin{cases} ^{C}\Delta_a^{\nu} x_{1m}(t) = -0.7 x_{1m}(t+\nu-1) + a_{11}\tanh(x_{1m}(t+\nu-1)) + a_{12}\tanh(x_{2m}(t+\nu-1)), \\ ^{C}\Delta_a^{\nu} x_{2m}(t) = -0.85 x_{2m}(t+\nu-1) + a_{21}\tanh(x_{1m}(t+\nu-1)) + a_{22}\tanh(x_{2m}(t+\nu-1)), \end{cases} \tag{7}$$

where ${}^C\Delta_a^\nu$ denotes the Caputo difference operator, $0 \le \nu \le 1$, $t \in \mathbb{N}_{a+(1-\nu)}$, a is the starting point and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.11 \\ 0.12 & 0.13 \end{pmatrix}.$$

According to Theorem 2.1, the equivalent implicit discrete formula can be written in the form

$$\begin{cases} x_{1m}(n) = x_{1m}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left[-0.7x_{1m}(j) + a_{11}\tanh(x_{1m}(j)) + a_{12}\tanh(x_{2m}(j)\right], \\ x_{2m}(n) = x_{2m}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left[-0.85x_{2m}(j) + a_{21}\tanh(x_{1m}(j)) + a_{22}\tanh(x_{2m}(j))\right]. \end{cases} \tag{8}$$

3.2 The slave system

On the other hand, the three-dimensional fractional discrete neural network, proposed in [16], is considered as slave and all its states are distinguished by typing another subscript, say s, for each of them, i.e.,

$$\begin{cases} {}^C\Delta_a^\nu y_{1s}(t) = -y_{1s}(t+\nu-1) + b_{11}\tanh(y_{1s}(t+\nu-1)) + b_{12}\tanh(y_{2s}(t+\nu-1)) \\ \qquad\qquad + b_{13}\tanh(y_{3s}(t+\nu-1)) + \mathbf{C}_1, \\ {}^C\Delta_a^\nu y_{2s}(t) = -y_{2s}(t+\nu-1) + b_{21}\tanh(y_{1s}(t+\nu-1)) + b_{22}\tanh(y_{2s}(t+\nu-1)) \\ \qquad\qquad + b_{23}\tanh(y_{3s}(t+\nu-1)) + \mathbf{C}_2, \\ {}^C\Delta_a^\nu y_{3s}(t) = -y_{3s}(t+\nu-1) + b_{33}\tanh(y_{1s}(t+\nu-1)) + b_{32}\tanh(y_{2s}(t+\nu-1)) \\ \qquad\qquad + b_{33}\tanh(y_{3s}(t+\nu-1)) + \mathbf{C}_3, \end{cases} \tag{9}$$

where the system parameters are given as

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{31} & b_{33} \end{pmatrix} = \begin{pmatrix} 2 & -1.2 & 0 \\ 2 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix},$$

and $\mathbf{C}_1, \mathbf{C}_2$, $\mathbf{C}_3$ are synchronization controllers that have to be designed. On the other hand, the numerical formulae can be given accordingly

$$\begin{cases} y_{1s}(n) = y_{1s}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} - y_{1s}(j) + 2\tanh(y_{1s}(j)) + 1.2\tanh(y_{2s}(j)), \\ y_{2s}(n) = y_{2s}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} - y_{2s}(j) + 2\tanh(y_{1s}(j)) + 1.71\tanh(y_{2s}(j)) \\ \qquad\qquad +1.15\tanh(y_{3s}(j)), \\ y_{3s}(n) = y_{3s}(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} - y_{3s}(j) - 4.75\tanh(y_{1s}(j)) + 1.1\tanh(y_{3s}(j)). \end{cases} \tag{10}$$

4 Synchronization

In this section, we will endeavor to show that the two fractional discrete neural networks, even though they have different dimensions, can still be synchronized within time in the light of an appropriate synchronization scheme. Actually, the process of picking up an adaptive controll law $(\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3)^T$ aims to compel the following synchronization errors:

$$\begin{cases} e_1 = y_{1s} - x_{1m}, \\ e_2 = y_{2s} + x_{2m}, \\ e_3 = y_{3s} - (x_{1m} + x_{2m}) \end{cases} \tag{11}$$

to be asymptotically tended to the origin, i.e.,

$$\lim_{t \to +\infty} |e_i(t)| = 0, \qquad \text{for } i = 1, 2, 3. \tag{12}$$

Remark 4.1 From the error system (11), it is obvious that states y_{1s} and x_{1m} are complete synchronized, y_{2s} is anti-synchronized with x_{2m}, and y_{3s} is full state synchronized with x_{1m} and x_{2m}. So, complete synchronization, anti-synchronization and full state hybrid projective synchronization co-exist between the fractional discrete neural networks (7) and the slave system (9).

Before stating the proposed control law and establishing its stability, it is important to state the following theorem, which is essential for our proof. Interested readers are referred to [25] for the proof of this result.

Theorem 4.1 *The zero equilibrium of the linear fractional-order discrete–time system*

$${}^{C}\Delta_a^{\nu} X(t) = \mathbf{M} X(t + \nu - 1), \tag{13}$$

where $X(t) = (x_1(t), ..., x_n(t))^T$, $0 < \nu \leq 1$, $\mathbf{M} \in \mathbb{R}^{n \times n}$ *and* $\forall t \in \mathbb{N}_{a+1-\nu}$, *is asymptotically stable if*

$$\lambda \in \left\{ z \in \mathbb{C} : \ |z| < \left(2 \cos \frac{|\arg z| - \pi}{2 - \nu} \right)^{\nu} \ \textit{and} \ |\arg z| > \frac{\nu \pi}{2} \right\} \tag{14}$$

for all the eigenvalues λ *of* $\mathbf{M}$.

With this stability result, we are now ready to state the following theorem which is considered the main result of this work.

Theorem 4.2 *Complete synchronization, antiphase synchronization and full state hybrid projective synchronization coexist in the synchronization of the master system (7) and the slave system (9) if the control law is selected as follows:*

$$\begin{cases} \mathbf{C}_1(t) = 0.3x_{1m}(t) - b_{11} tanh(y_1(t)) - b_{12} tanh(y_2(t)) \\ \qquad\qquad + a_{11} tanh(x_{1m}(t)) + a_{12} tanh(x_{2m}(t)), \\ \mathbf{C}_2(t) = -b_{21} tanh(y_{1s}(t)) - b_{22} tanh(y_{2s}(t)) \\ \qquad\qquad - b_{23} tanh(y_{3s}(t)) - 0.15x_{2m}(t) \\ \qquad\qquad - a_{21} tanh(x_{1m}(t)) - a_{22} tanh(x_{2m}(t)), \\ \mathbf{C}_3(t) = 0.3x_{1m}(t) + 0.15x_{2m}(t) \\ \qquad\qquad - b_{31} tanh(y_{1s}(t)) - b_{33} tanh(y_{3s}(t)) \\ \qquad\qquad + a_{11} tanh(x_{1m}(t)) + a_{12} tanh(x_{2m}(t)) \\ \qquad\qquad + a_{21} tanh(x_{1m}(t)) + a_{22} tanh(x_{2m}(t)), \end{cases} \tag{15}$$

where $t \in \mathbb{N}_{a+1-\nu}$.

Proof. For the purpose of establishing an asymptotic convergence of the synchronization errors, given in (11), to zero, we start applying the Caputo-type fractional-order differences to (11), which yields

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu}e_1(t) = -y_{1s}\left(t+\nu-1\right) + b_{11}\tanh(y_{1s}\left(t+\nu-1\right)) \\ \qquad\qquad + b_{12}\tanh(y_{2s}\left(t+\nu-1\right)) \\ \qquad\qquad + 0.7x_{1m}\left(t+\nu-1\right) - a_{11}\tanh(x_{1m}\left(t+\nu-1\right)) \\ \qquad\qquad - a_{12}\tanh(x_{2m}\left(t+\nu-1\right)) + \mathbf{C}_1, \\ {}_{h}^{C}\Delta_{a}^{\nu}e_2(t) = -y_{2s}\left(t+\nu-1\right) + b_{21}\tanh(y_{1s}\left(t+\nu-1\right)) \\ \qquad\qquad + b_{22}\tanh(y_{2s}\left(t+\nu-1\right)) + b_{23}\tanh(y_{3s}\left(t+\nu-1\right)) \\ \qquad\qquad - 0.85x_{2m}\left(t+\nu-1\right) + a_{21}\tanh(x_{1m}\left(t+\nu-1\right)) \\ \qquad\qquad + a_{22}\tanh(x_{2m}\left(t+\nu-1\right)) + \mathbf{C}_2, \\ {}_{h}^{C}\Delta_{a}^{\nu}e_3(t) = -y_{3s}\left(t+\nu-1\right) + b_{31}\tanh(y_{1s}\left(t+\nu-1\right)) \\ \qquad\qquad + b_{33}\tanh(y_{3s}\left(t+\nu-1\right)) + 0.7x_{1m}\left(t+\nu-1\right) \\ \qquad\qquad - a_{11}\tanh(x_{1m}\left(t+\nu-1\right)) - a_{12}\tanh(x_{2m}\left(t+\nu-1\right)) \\ \qquad\qquad + 0.85x_{2m}\left(t+\nu-1\right) - a_{21}\tanh(x_{1m}\left(t+\nu-1\right)) \\ \qquad\qquad - a_{22}\tanh(x_{2m}\left(t+\nu-1\right)) + \mathbf{C}_3. \end{cases} \tag{16}$$

Substituting the proposed control law given in (15) into (16) leads to the following new discrete system:

$$\begin{cases} {}^{C}\Delta_{a}^{\nu}e_1\left(t\right) = -e_1\left(t+\nu-1\right), \\ {}^{C}\Delta_{a}^{\nu}e_2\left(t\right) = -e_2\left(t+\nu-1\right), \\ {}^{C}\Delta_{a}^{\nu}e_3\left(t\right) = -e_3\left(t+\nu-1\right). \end{cases} \tag{17}$$

It can be described more compactly as

$${}^{C}\Delta_{a}^{\nu}\left(e_1,e_2,e_3\right)\left(t\right)^T = \mathbf{M}\times\left(e_1,e_2,e_3\right)\left(t+\nu-1\right)^T, \tag{18}$$

where

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

We aim to show that the zero solution of (18) is globally asymptotically stable, which guarantees that all states converge towards zero at infinite time. In order to do so, we make use of the stability theory described in Theorem 4.2. Simply, we can show that the eigenvalues of the matrix $\mathbf{M}$ are: $\lambda_1 = \lambda_2 = \lambda_3 = -1$. It is easy to see that all the eigenvalues of the matrix $\mathbf{M}$ satisfy $|\arg\lambda_i| = \pi > \frac{\nu\pi}{2}$ and $|\lambda_i| < \left(2\cos\frac{|\arg\lambda_i|-\pi}{2-\nu}\right)^{\nu}$ for $i = 1, 2, 3$. According to Theorem 4.2, the zero solution of (18) is globally asymptotically stable. Hence, the master system (7) and the slave system (9) has been synchronized by means of control laws (15).

Here, some numerical experiments are considered to verify the effectiveness of the proposed approach. The initial values of the master system (7) are set as $x_{1m}(0) = 0.2$, $x_{2m}(0) = 0.3$, and the initial values of the slave system are considered as $y_{1s} = 0.02, y_{2s} = -0.5$ and $y_{3s} = 0.8$. Figure 1 illustrates the hybrid synchronization error system (14) with $\mu = 0.05$. It is very clear that the error states e_1, e_2 and e_3 can converge to zero when the controller functions C_1, C_2 and C_3 are added to the slave systems, which implies that the synchronization is realized. Furthermore, we plot the evolution of states of the master and slave systems for the fractional-order value $\mu = 0.05$. Figure 2 illustrates the results. Clearly, the state variables of the master and slave systems are synchronized completely. Thus, the numerical results show very well that the two fractional maps achieve hybrid synchronization.

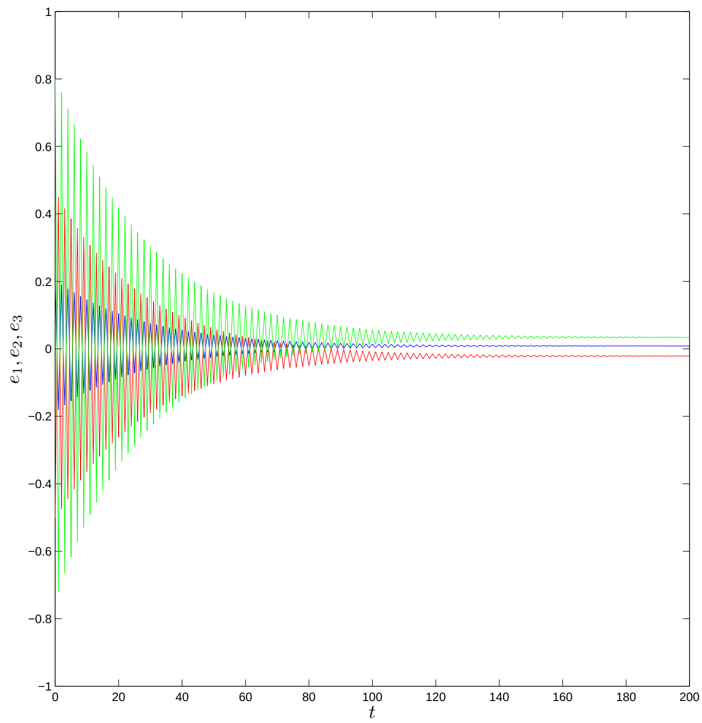

Figure 1: Hybrid synchronization error between the master system (7) and the slave system (9) with $\nu = 0.05$.

5 Conclusion

This paper has shown the coexistence of some synchronization types in two fractional discrete neural networks with different dimensions. At first, a two-dimensional fractional discrete neural network (considered as a master system) and a three-dimensional fractional discrete neural network (considered as a slave system) have been introduced. Then a new theorem has been proved, which assures the coexistence of complete synchronization, antiphase synchronization and full state hybrid projective synchronization in different dimensional fractional discrete neural networks. Finally, simulation results have been obtained to highlight the effectiveness of the conceived approach.

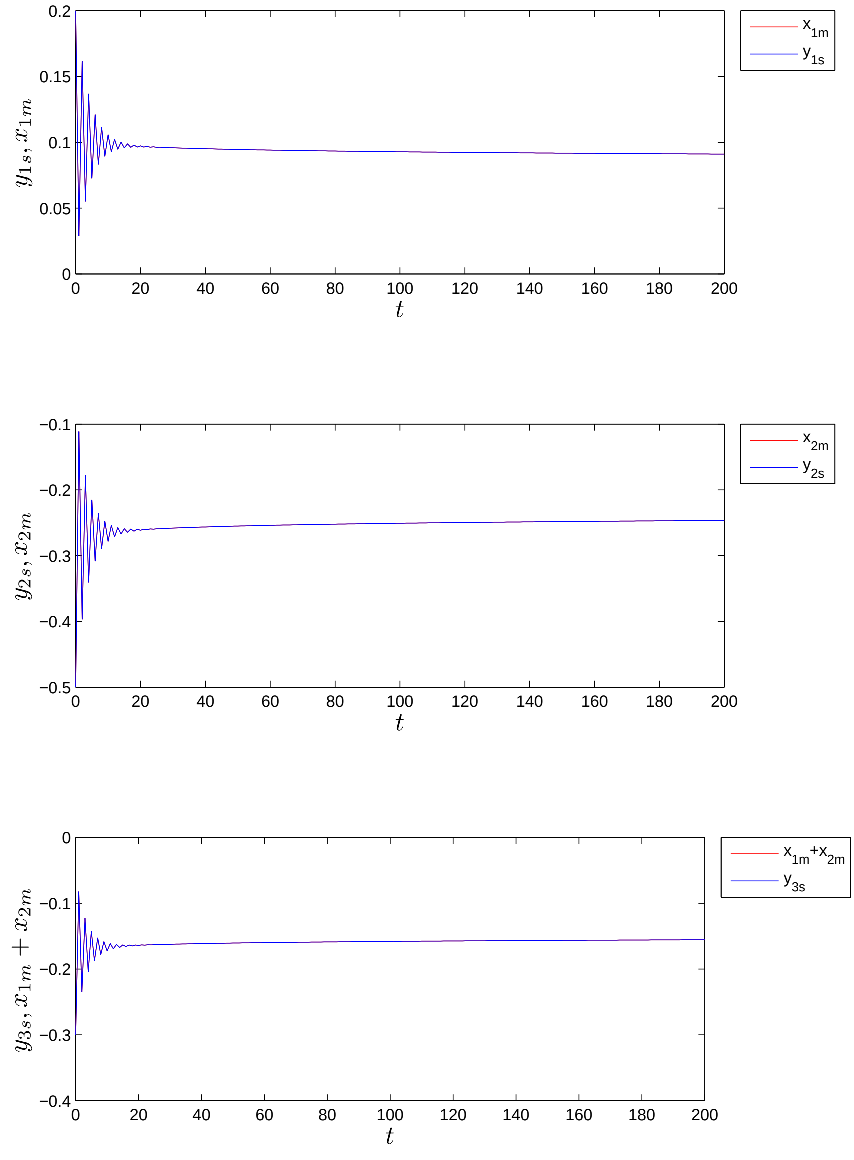

Figure 2: The time history of the master system (7) and the slave system (9) with $\nu = 0.05$.

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