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Study of a Non-Isothermal Hooke Operator in Thin Domain with Friction on the Bottom Surface

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Abstract: This work is focused on the study of the asymptotic behavior of a coupled problem that consists of an elastic body and the change of heat. The friction exerted on the body is nonlinear of Coulomb type in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$. As a first step, we give the variational formulation of the problem and the establishment of the existence and uniqueness results for the weak solution. We proceed to the asymptotic analysis. To do this, we use the scale change following the third component and new unknowns to conduct the study on a domain Ω independent of ε. Then we prove some estimates for the displacement and the temperature. Finally, these estimates allow us to have the limit problem and prove the uniqueness of the solution.

Keywords: *a priori inequalities; boundary conditions; Coulomb law; coupled problem; elastic body; Fourier law.*

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1 Introduction

In solid mechanics, thin structures are widely used in several fields of industry, for example, in underwater industry, aerospace, civil engineering and in common constructions, in the field of energy, industrial design, and even in the living world. We also find the use of thin structures in the metallurgical industry, in particular in the rolling process of thin sheets etc. More details can be seen in [1]. In mathematical literature, the problems in thin areas and especially in the elasticity of thin films, plates and shells have already been studied for more than a century. For example, Ciarlet in [10] and Destuynder in [12] have studied the equilibrium states of a thin plate $\Omega \times (-\varepsilon, +\varepsilon)$ under external forces, where Ω is a smooth domain in $\mathbb{R}^2$ and ε is a small parameter, to justify the

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two-dimensional model of the plates. In recent years, many authors have applied asymptotic methods in three-dimensional or two-dimensional elasticity and viscosity problems to derive new two-dimensional or one-dimensional reduced models. The importance of asymptotic methods is that they can be used in place of full three-dimensional models when the thickness is small enough. In addition, two-dimensional models are simpler than their three-dimensional counterpart, which facilitates their study. They also allow less costly digital simulations than the three-dimensional ones.

Our goal in this paper is to give the asymptotic behavior of a non-isothermal Hooke operator in a thin domain with Coulomb friction on the bottom surface. One of the objectives of this study is to obtain a two-dimensional equation that allows a reasonable description of the phenomenon occurring in a three-dimensional domain by passing the limit to 0 on the small thickness of the domain (3D). The scientific research in mechanics are articulated around two main components: one devoted to the laws of behavior and the other one to the boundary conditions imposed on the body. Here we describe real phenomena which transform into mathematical problems with boundary conditions and with certain types of friction, the type of problem that is presented here is very common in application. The physical domains are defined, where the height is much smaller then the length, as the problem of elasticity and viscoelasticity of a tire. We consider a non isothermal elastic body with Coulomb free boundary friction conditions in the stationary regime occupying a bounded, homogeneous domain $\Omega^\varepsilon \subset \mathbb{R}^3$, where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero. The boundary of Ω^ε will be noted $\Gamma^\varepsilon = \overline{\Gamma_1^\varepsilon} \cup \overline{\Gamma_L^\varepsilon} \cup \overline{w}$ and assumed to be Lipschitz, such that Γ_1^ε is the upper surface of the equation $x_3 = \varepsilon h(\acute{x})$, Γ_L^ε is the lateral surface and w is a fixed bounded domain of $\mathbb{R}^3$ with $x_3 = 0$, which is a bottom of the domain Ω^ε. Several works have been done on the mechanical contact with the various laws of behavior and various boundary conditions of friction close to our problem, yet these items were based only on the existence and uniqueness of the weak solution. Let us mention, for example, in [2], Bayada et al. are engaged in the asymptotic and numerical analysis for the unilateral contact problem with Coulomb friction between two general elastic bodies and a thin elastic soft layer. Paumier in [21] performs an asymptotic modeling of a unilateral problem of a thin plate. He demonstrates that this three-dimensional problem with friction tends towards another two-dimensional one without friction. The justification by the asymptotic analysis for the elastic plates is given by Gilbert in [16] and for the shells is given by Chacha in [9].

More recently, some research papers have been written dealing with the asymptotic analysis of a boundary value problem governed by the elasticity system. For example, the asymptotic behavior of the dynamical problem of isothermal and non-isothermal elasticity with non linear friction of Tresca type was studied in [5, 24]. Also, the authors in [4] carried out the asymptotic analysis of a frictionless contact between two elastic bodies in a stationary regime in a three-dimensional thin domain with friction. The reader can also consult certain works concerning partial differential equations posed in different thin domains, see, for example, [15, 17, 18, 22, 23].

This paper is organized as follows. As a first step, we give the variational formulation of the problem and demonstrate the results of existence and uniqueness for the weak solution, then we move on to the asymptotic analysis. For this, using the change of scale according to the third component we conduct the study on a domain Ω which does not depend on ε. Then, by the use of different inequalities, we prove some estimates for the displacement and the temperature, which allow us to go to the limit when ε tends towards zero in the variational formulation. Finally, our main result is the proof of the existence and uniqueness of the limit of a weak solution to the problem described in the abstract.

2 Statement of the Problem

In this section, we first define the thin domain and some sets necessary to study the asymptotic behavior of the solutions. Next, we introduce the problem considered in the thin domain. We finish this section giving the weak variational formulations of our problem.

2.1 The domain

We consider a mathematical problem governed to the stationary equation for an elasticity system in three dimensional bounded domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with boundary $\Gamma^\varepsilon = \overline{\Gamma}^\varepsilon_L \cup \overline{\Gamma}^\varepsilon_1 \cup \overline{w}$. We denote by $|.|$ the Euclidean norm on $\mathbb{R}^2$. Let w be a fixed, bounded domain of $\mathbb{R}^3$ with $x_3 = 0$. We suppose that w has a Lipschitz continuous boundary and is the bottom of the domain. The upper surface Γ^ε_1 is defined by $x_3 = \varepsilon h(\acute{x}) = \varepsilon h(x_1, x_2)$. We introduce a small parameter ε, that will tend to zero, where h is a function of class C^1 defined on w such that $0 < h_* < h(\acute{x}) < h^*$, for all $(\acute{x}, 0) \in \omega$, with

$$\Omega^\varepsilon = \left\{(\acute{x}, z) \in \mathbb{R}^3\colon\ (\acute{x}, 0) \in \omega, 0 < x_3 < \varepsilon h(\acute{x})\right\}.$$

We introduce the following functional framework:

$$H^1(\Omega^\varepsilon)^3 = \left\{ v \in (L^2(\Omega^\varepsilon))^3; \qquad \frac{\partial v_i}{\partial x_j} \in L^2(\Omega^\varepsilon); \quad \forall i, j = 1, 2, 3 \right\}.$$

We define the closed non-empty convex of $H^1(\Omega^\varepsilon)^3$:

$$V^\varepsilon = \left\{ \varrho \in \left(H^1(\Omega^\varepsilon)\right)^3;\ \ \varrho = G^\varepsilon \text{ on } \Gamma^\varepsilon_L\ ,\ \varrho = 0 \text{ on } \Gamma^\varepsilon_1 \text{ and } \varrho.n = 0 \text{ on } w \right\},$$

where G^ε is defined below. We note by $H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}$ the vector sub-space of $H^1(\Omega^\varepsilon)$:

$$H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon) = \left\{\varrho \in H^1(\Omega^\varepsilon) : \varrho = 0 \text{ on } \Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1\ \right\}.$$

The spaces Ω^ε, $H^1(\Omega^\varepsilon)^3$, $V^{\ \varepsilon}$ and $H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon)$ are the domain in which we study the asymptotic behavior of elasticity, the Sobolev space, the closed convex, and the vectorial sub-space of $H^1(\Omega^\varepsilon)$ are endowed with their natural norms and scalar product.

2.2 The problem

We assume that the deformations of an elastic body are governed by the following equations. The law of conservation of momentum is $div(\sigma^\varepsilon) + f^\varepsilon = 0$, we designate by $\sigma^\varepsilon = \left(\sigma^\varepsilon_{i,j}\right)_{1 \le i,j \le 3}$ the stress tensor and by $D = (d_{i,j})_{1 \le i,j \le 3}$ the tensor of deformation: $d_{i,j}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$; $1 \le i, j \le 3$.

It is supposed that the law of behavior follows the law of Hooke

$$\sigma^\varepsilon_{i,j}(u) = 2\mu(T^\varepsilon) d_{i,j}(u^\varepsilon) + \lambda(T^\varepsilon) d_{kk}(u^\varepsilon)\delta_{ij}.$$

$u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ is the displacement of the elastic body, λ and μ are the coefficients of Lamé with $\lambda + \mu \geq 0$, T^ε is the temperature, $u_\tau^\varepsilon, u_n^\varepsilon$ are the tangential and normal components of u^ε on the boundary w given by $u_n^\varepsilon = u^\varepsilon.n$, $u_{\tau_i}^\varepsilon = u_i^\varepsilon - u_n^\varepsilon.n_i$ and $\sigma_\tau^\varepsilon, \sigma_n^\varepsilon$ are the tangential and normal components of σ^ε given by

$$\sigma_n^\varepsilon = (\sigma^\varepsilon.n_i)\,.n_j, \sigma_{\tau_i}^\varepsilon = \sigma_{ij}^\varepsilon.n_j - \sigma_n^\varepsilon.n_i.$$

The law of conservation of energy is given by

$$\begin{cases} -\nabla(K^\varepsilon \nabla T^\varepsilon) = \sigma^\varepsilon : D(u^\varepsilon) + r^\varepsilon(T^\varepsilon), \\ \sigma^\varepsilon : D(u^\varepsilon) = \sum_{i,j=1}^{3} \sigma_{i,j}^\varepsilon d_{i,j}(u^\varepsilon), \end{cases}$$

where K^ε is the thermal conductivity and $r^\varepsilon(T^\varepsilon)$ is the heat source.

To describe the boundary conditions, let us introduce first a vector function $g = (g_1, g_2, g_3)$ in $H^{1/2}(\Gamma^\varepsilon)$ such that $\int_{\Gamma^\varepsilon} g.nds = 0$, then according to ([6]) there exists a function G^ε:

$$G^\varepsilon \in (H^1(\Omega^\varepsilon))^3 \quad \text{with} \quad G^\varepsilon = g \text{ on } \Gamma^\varepsilon.$$

Also, we suppose that

$$g_3 = u_3 = 0 \text{ and } s = g \text{ on } \omega.$$

• On Γ_1^ε, no slip condition is given. The upper surface is assumed to be fixed so that $u^\varepsilon = 0$.
• On Γ_L^ε, the displacement is unknown and parallel to the $w-$plane: $u^\varepsilon = g$ with $g_3 = 0$.
• On w, there is no flux condition across w so that $u^\varepsilon.n = 0$.
• The tangential velocity on w is unknown and satisfies the Coulomb friction law:

$$\begin{cases} |\sigma_\tau^\varepsilon| < F^\varepsilon\, |\sigma_n^\varepsilon| \Rightarrow u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = F^\varepsilon\, |\sigma_n^\varepsilon| \Rightarrow \exists \beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon, \end{cases}$$

where $F^\varepsilon \geq 0$ is the coefficient of friction.

For the temperature, we assume that

$$\begin{cases} T^\varepsilon = 0 \quad \text{on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \\ \frac{\partial T^\varepsilon}{\partial n} = 0 \text{ on } w. \end{cases}$$

The complete problem consists of finding the displacement field u^ε and the temperature T^ε which satisfy the following equations and boundary conditions:

$$div(\sigma^\varepsilon) + f^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \tag{2.1}$$

$$\sigma_{i,j}^\varepsilon(u^\varepsilon) = 2\mu^\varepsilon(T^\varepsilon) d_{i,j}(u^\varepsilon) + \lambda^\varepsilon(T^\varepsilon) d_{kk}(u^\varepsilon)\delta_{ij}, \text{ in } \Omega^\varepsilon, \tag{2.2}$$

$$-\nabla(K^\varepsilon \nabla T^\varepsilon) = \sigma^\varepsilon : D(u^\varepsilon) + r^\varepsilon(T^\varepsilon) \qquad \text{in } \Omega^\varepsilon, \tag{2.3}$$

$$u^\varepsilon = 0 \qquad \text{on } \Gamma_1^\varepsilon, \tag{2.4}$$

$$u^\varepsilon = g \quad \text{with } g_3 = 0 \qquad \text{on } \Gamma_L^\varepsilon, \tag{2.5}$$

$$\left\{ \begin{array}{c} |\sigma_\tau^\varepsilon| < F^\varepsilon\, |\sigma_n^\varepsilon| \Rightarrow u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = F^\varepsilon\, |\sigma_n^\varepsilon| \Rightarrow \exists\beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon \end{array} \right. \quad \text{on } w. \tag{2.6}$$

$$T^\varepsilon = 0 \qquad \text{on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \tag{2.7}$$

$$\frac{\partial T^\varepsilon}{\partial n} = 0 \qquad \text{on } w. \tag{2.8}$$

2.3 Weak variational formulations

We finish this section by giving the equivalent weak variational formulation of problem (2.1) – (2.8) which will be useful in the next sections. By standard calculations, the variational formulation of the problem (2.1) – (2.8) is given as follows.

Problem $(\mathbf{P}_v)$ Find a displacement field $u^\varepsilon \in V^\varepsilon(\Omega^\varepsilon)$ and a temperature $T^\varepsilon \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon)$ such that

$$a(T^\varepsilon, u^{\ \varepsilon}, \varrho - u^\varepsilon) + j^\varepsilon(\varrho) - j^\varepsilon(u^\varepsilon) \geq (f^{\ \varepsilon}, \varrho - u^{\ \varepsilon}), \quad \forall \varrho \in V^{\ \varepsilon}. \tag{2.9}$$

$$b(T^\varepsilon, \psi) = c(u^\varepsilon, T^\varepsilon, \psi), \quad \forall \psi \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}, \tag{2.10}$$

where

$$a(T^\varepsilon, u^\varepsilon, v) = \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) d_{ij}(u^\varepsilon) d_{ij}(v) d\acute{x} dx_3 + \int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon)\ div(u^\varepsilon)\ div(v) d\acute{x} dx_3, \tag{2.11}$$

$$(f^\varepsilon, v) = \int_{\Omega^\varepsilon} f^{\ \varepsilon} v d\acute{x} dx_3 = \sum_{i=1}^{3} \int_{\Omega^\varepsilon} f_i^{\ \varepsilon} v_i d\acute{x} dx_3, \tag{2.12}$$

$$j^\varepsilon(v) = \int_w F^\varepsilon \left|\sigma_n^\varepsilon\right| \left|v_T - s\right| d\acute{x} \text{ with } \ S(\sigma_n^\varepsilon) = \left|\sigma_n^\varepsilon\right|, (S \text{ is given below}) \tag{2.13}$$

$$b(T^\varepsilon, \psi) = \int_{\Omega^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial x_i} \frac{\partial \psi}{\partial x_i} d\acute{x} dx_3, \tag{2.14}$$

$$\begin{gathered} c(u^\varepsilon, T^\varepsilon, \psi) = \textstyle\sum_{i=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) d_{ij}^2(u^\varepsilon) \psi d\acute{x} dx_3 + \\ \int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon)\ div(u^\varepsilon)\ div(u^\varepsilon) \psi d\acute{x} dx_3 + \int_{\Omega^\varepsilon} r^\varepsilon(T^\varepsilon) \psi d\acute{x} dx_3. \end{gathered} \tag{2.15}$$

Remark 2.1 ([13, 14]) If we have only $u^\varepsilon \in V^\varepsilon$ and σ_n^ε is defined by duality as an element of $H^{-\frac{1}{2}}(w)$ has no sense, then the integral $j^\varepsilon(v)$ has no meaning. Then from the mathematical point of view it is necessary that $S(\sigma_n^\varepsilon) = |\sigma_n^\varepsilon|$ with S being a regularization operator from $H^{-\frac{1}{2}}(w)$ into $L^2_+(w)$ defined by

$$S(\tau)(x) = \left| \langle \tau, \varphi(x-\tau) \rangle_{H^{-\frac{1}{2}}(w), H^{\frac{1}{2}}_{00}(w)} \right|, \text{ for all} \in H^{-\frac{1}{2}}(w) \text{ and } S(\tau) \in L^2_+(w),$$

where φ is a given positive function of class C^∞ with support in w, and $H^{-\frac{1}{2}}(w)$ is the dual space to

$$H^{\frac{1}{2}}_{00}(w) = \left\{ \varphi_{|w} : \varphi \in H^1(\Omega^\varepsilon);\ \varphi = 0 \quad \text{on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon \right\}.$$

$L^2_+(w)$ is the subspace of $L^2(w)$ of non-negative functions.

Lemma 2.1 *If u^ε and T^ε are solutions of the problem* (2.1) – (2.8), *then they satisfy the variational problem* $(\mathbf{P}_v)$.

Proof. Multiply the equation (2.1) by $(\varrho - u^\varepsilon)$, where $\varrho \in V^\varepsilon$. By performing an integration by parts on Ω^ε, using the Green formula and (2.4)-(2.8), we obtain the variational problem (2.9). For the proof of (2.10), multiplying the equation (2.3) by ψ, where $\psi \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon)$ and using the Green formula, we find

$$\sum_{i=1}^{3} \int_{\Omega^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial x_i} \frac{\partial \psi}{\partial x_i} d\acute{x} dx_3 = \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon (T^\varepsilon) d^2_{ij}(u^\varepsilon) \psi d\acute{x} dx_3 +$$

$$\int_{\Omega^\varepsilon} \lambda^\varepsilon (T^\varepsilon)\ div(u^\varepsilon)\ div(u^\varepsilon) \psi d\acute{x} dx_3 + \int_{\Gamma^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial n_i} \psi ds + \int_{\Omega^\varepsilon} r^\varepsilon (T^\varepsilon) \psi d\acute{x} dx_3.$$

Now, for the boundary condition (2.8), we get $b(T^\varepsilon, \psi) = c(u^\varepsilon, T^\varepsilon, \psi), \quad \forall \psi \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}$.

Theorem 2.1 *If* $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$ *and the friction coefficient* $\digamma^\varepsilon$ *is a non-negative function in* $L^\infty(w)$, *then there exists* $u^\varepsilon \in V^\varepsilon(\Omega^\varepsilon)$ *which is a solution to the problem* (2.9)-(2.10). *Moreover, for small* $\digamma^\varepsilon$, *the solution is unique.*

Proof. The proof is similar to that in [2], and we shall not reproduce it in full giving only a sketch here. Firstly, for the existence of solution u^ε we apply Tichonov's fixed point theorem (the proof can be found in [11]), then to prove the uniqueness of u^ε we use the same procedure as in [2, 20].

3 Problem in Transpose Form and Variational Problem

We shall now focus our attention on the asymptotic analysis of problem (2.1) – (2.8). For this analysis, we use the change of variable $z = \dfrac{x_3}{\varepsilon}$ to transform the initial problem in Ω^ε into a new problem posed in the fixed domain Ω which does not depend on ε:

$$\Omega = \left\{ (\acute{x}, z) \in \mathbb{R}^3 : (\acute{x}, 0) \in \omega, 0 < z < h(\acute{x}) \right\}$$

and we denote by $\Gamma = \overline{\Gamma}_L \cup \overline{\Gamma}_1 \cup \overline{w}$ its boundary. In addition, we define the following functions on Ω:

$$\begin{cases} u^\varepsilon_i(\acute{x}, x_3) = \hat{u}^\varepsilon_i(\acute{x}, z), \qquad i = 1, 2, \\ \varepsilon^{-1} u^\varepsilon_3(\acute{x}, x_3) = \hat{u}^\varepsilon_3(\acute{x}, z), \quad T^{\ \varepsilon}(\acute{x}, x_3) = \hat{T}^{\ \varepsilon}(\acute{x}, z), \end{cases} \tag{3.1}$$

$$\hat{f}^{\ \varepsilon}(\acute{x}, z) = \varepsilon^2 f^{\ \varepsilon}(\acute{x}, x_3), \quad \hat{g}\ (\acute{x}, z) = g\ (\acute{x}, x_3), \tag{3.2}$$

$$\hat{K} = K^{\ \varepsilon}, \quad \hat{r} = \varepsilon\ r^\varepsilon, \quad \hat{\lambda} = \lambda^\varepsilon, \quad \hat{\mu} = \mu^\varepsilon, \ \hat{\digamma} = \varepsilon^{-1} \digamma^\varepsilon, \tag{3.3}$$

with $\hat{\mu}, \hat{\lambda}, \hat{f}, \hat{K}, \hat{\digamma}$ and $\hat{g}$ independent of ε. So, the revaluation G^ε of g is defined by

$$\begin{cases} \varepsilon \hat{G}_3(\acute{x}, z) = G^\varepsilon_3(\acute{x}, x_3), \\ \hat{G}_i(\acute{x}, z) = G^\varepsilon_i(\acute{x}, x_3), \quad i = 1, 2. \end{cases} \tag{3.4}$$

We introduce the functional framework in Ω:

$$\begin{aligned} V &= \left\{ \varrho \in \left(H^1(\Omega)\right)^3; \qquad \varrho = \hat{G} \text{ on } \Gamma_L\ ,\ \varrho = 0 \text{ on } \Gamma^\varepsilon_1 \text{ and } \varrho.n = 0 \text{ on } w \right\}, \\ \Pi(V) &= \left\{ \varrho \in H^1(\Omega)^2 : \varrho = (\varrho_1, \varrho_2)\,, \quad \varrho_i = g \text{ on } \Gamma_L \text{ and } \varrho_i = 0 \text{ on } \Gamma_1, i = 1, 2 \right\}, \\ V_z &= \left\{ v = (v_1, v_2) \in L^2\left(\Omega\right)^2; \quad \frac{\partial v_i}{\partial z} \in L^2\left(\Omega\right), i = 1, 2; v = 0 \text{ on } \Gamma_1 \right\}, \\ H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega) &= \left\{ \varrho \in H^1(\Omega) : \varrho = 0 \text{ on } \Gamma_L \cup \Gamma_1\ \right\}. \end{aligned}$$

It is clear that V_z is a Banach space with the norm

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2} \|v\|^2_{L^2(\Omega)} + \left\| \frac{\partial v_i}{\partial z} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

With the change of scale defined in (3.1) – (3.3), the problem ($\mathbf{P}_v$) becomes as follows.

Find the displacement $\hat{u}^\varepsilon \in V$ and the temperature $\hat{T}^\varepsilon \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega)$ such that

$$a(\hat{T}^\varepsilon, \hat{u}^{\ \varepsilon}, \hat{\varrho} - \hat{u}^\varepsilon) + j(\hat{\varrho}) - j(\hat{u}^\varepsilon) \geq (\hat{f}^{\ \varepsilon}, \hat{\varrho} - \hat{u}^{\ \varepsilon}), \quad \forall \hat{\varrho} \in V, \tag{3.5}$$

$$b(\hat{T}^\varepsilon, \hat{\psi}) = c(\hat{u}^\varepsilon, \hat{T}^\varepsilon, \hat{\psi}), \quad \forall \hat{\psi} \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega), \tag{3.6}$$

where

$$\begin{aligned}
a(\hat{T}^\varepsilon, \hat{u}^\varepsilon, \hat{\varrho} - \hat{u}^\varepsilon) \quad = \quad & \varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial x_j} + \frac{\partial \hat{u}^\varepsilon_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} (\hat{\varrho}_i - \hat{u}^\varepsilon_i) d\acute{x} dz \\
& + \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}^\varepsilon_3}{\partial x_i} \right) \frac{\partial}{\partial z} (\hat{\varrho}_i - \hat{u}^\varepsilon_i) \\
& + \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}^\varepsilon_3}{\partial x_i} \right) \varepsilon^2 \frac{\partial}{\partial x_i} (\hat{\varrho}_i - \hat{u}^\varepsilon_3) d\acute{x} dz \\
& + \varepsilon^2 \int_\Omega 2 \hat{\mu}(\hat{T}^\varepsilon) \frac{\partial \hat{u}^\varepsilon_3}{\partial z} \frac{\partial}{\partial z} (\hat{\varrho}_3 - \hat{u}^\varepsilon_3) d\acute{x} dz \\
& + \varepsilon^2 \int_\Omega \hat{\lambda}(\hat{T}^{\ \varepsilon}) \ div(\hat{u}^\varepsilon) \ div(\hat{\varrho} - \hat{u}^\varepsilon) d\acute{x} dz,
\end{aligned}$$

$$\begin{aligned}
(\hat{f}^{\ \varepsilon}, \hat{\varrho} - \hat{u}^\varepsilon) \quad &= \quad \sum_{i=1}^{2} \int_\Omega \hat{f_i}^{\ \varepsilon} (\hat{\varrho}_i - \hat{u}^\varepsilon_i) \, d\acute{x} dz + \varepsilon \int_\Omega \hat{f_3}^{\ \varepsilon} (\hat{\varrho}_3 - \hat{u}^\varepsilon_3) \, d\acute{x} dz \\
b(\hat{T}^\varepsilon, \hat{\psi}) \quad &= \quad \sum_{i=1}^{2} \int_\Omega \hat{K} \varepsilon^2 \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \frac{\partial \hat{\psi}}{\partial x_i} d\acute{x} dz + \int_\Omega \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial z} \frac{\partial \hat{\psi}}{\partial z} d\acute{x} dz \\
j(\hat{\varrho}) \quad &= \quad \int_w \hat{F} S(\sigma^\varepsilon_n) \, |\hat{\varrho}_T - s| \, d\acute{x},
\end{aligned}$$

$$\begin{aligned}
c(\hat{u}^\varepsilon, \hat{T}^\varepsilon, \hat{\psi}) \quad = \quad & \sum_{i,j=1}^{2} \frac{1}{2} \int_\Omega \varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial x_j} + \frac{\partial \hat{u}^\varepsilon_j}{\partial x_i} \right)^2 \hat{\psi} d\acute{x} dz \\
& + \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}^\varepsilon_3}{\partial x_i} \right)^2 \hat{\psi} d\acute{x} dz + \int_\Omega 2 \varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}^\varepsilon_3}{\partial z} \right)^2 \hat{\psi} d\acute{x} dz \\
& + \int_\Omega \varepsilon^2 \hat{\lambda}(\hat{T}^\varepsilon) \ div(\hat{u}^\varepsilon) \ div(\hat{u}^\varepsilon) \hat{\psi} d\acute{x} dz + \int_\Omega \hat{r}(\hat{T}^\varepsilon) \hat{\psi} d\acute{x} dz.
\end{aligned}$$

In the next subsection, we will do the estimates of $(u^\varepsilon, T^\varepsilon)$ solution of our variational problem in fixed domain.

3.1 A priori estimates of the displacement

It is enough to prove the following essential result.

Lemma 3.1 *Assume that* $f \in \left(L^2(\Omega)\right)^3$, *the coefficient of friction* $F^\varepsilon > 0$ *in* $L^\infty(w)$ *and there are strictly positive constants* $\mu_*, \mu^*, \lambda_*, \lambda^*$ *such that*

$$0 < \mu_* \leq \mu(a) \leq \mu^* \text{ and } 0 < \lambda_* \leq \lambda(b) \leq \lambda^*, \quad \forall a, b \in \mathbb{R}. \tag{3.7}$$

Then there is a strictly positive constant C *independent of* ε *such that*

$$\sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right) \leq C. \tag{3.8}$$

Proof. Let u^ε be the solution of the problem $(\mathbf{P}_v)$ so that

$$a(T^\varepsilon, u^\varepsilon, u^\varepsilon) \leq a(T^\varepsilon, u^\varepsilon, \varrho) + (f^{\,\varepsilon}, u^\varepsilon) + j^\varepsilon(\varrho) - (f^{\,\varepsilon}, \varrho). \tag{3.9}$$

Because $j^\varepsilon(u^\varepsilon)$ is positive and as $\sum_{i,j=1}^{2} |d_{ij}(u^\varepsilon)|^2 \leq |\nabla u^\varepsilon|^2$, $|\ div(u^\varepsilon)|^2 \leq |\nabla u^\varepsilon|^2$, so, according to the inequality of Korn (from [19]), there exists C_K independent of ε such that

$$a(T^\varepsilon, u^\varepsilon, u^\varepsilon) \geq 2\mu_* C_K \left\| \nabla u^\varepsilon \right\|_{L^2(\Omega^\varepsilon)}^2. \tag{3.10}$$

Applying the Hölder and Young inequalities, we find the following:

$$a(T^\varepsilon, u^\varepsilon, \varrho) \leq \frac{3\ \mu_* C_K}{8} \left\| \nabla u^\varepsilon \right\|_{L^2(\Omega^\varepsilon)}^2 + \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K}\right) \left\| \nabla \varrho \right\|_{L^2(\Omega^\varepsilon)}^2. \tag{3.11}$$

Then

$$(f^{\,\varepsilon}, u^\varepsilon) \leq \frac{(\varepsilon h^*)^2}{2\ \mu_* C_K} \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^\varepsilon)}^2 + \frac{\mu_* C_K}{2} \left\| \nabla u^\varepsilon \right\|_{L^2(\Omega^\varepsilon)}^2. \tag{3.12}$$

$$(f^{\,\varepsilon}, \varrho) \leq \frac{(\varepsilon h^*)^2}{2\ \mu_* C_K} \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^\varepsilon)}^2 + \frac{\mu_* C_K}{2} \left\| \nabla \varrho \right\|_{L^2(\Omega^\varepsilon)}^2. \tag{3.13}$$

By $(3.10) - (3.13)$ and choosing $\varrho = G^\varepsilon$, we get the variational equation

$$\begin{aligned} \frac{9}{8} \mu_* C_K \left\| \nabla u^\varepsilon \right\|_{L^2(\Omega^\varepsilon)}^2 \quad \leq \quad & \frac{(\varepsilon h^*)^2}{\mu_* C_K} \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^\varepsilon)}^2 + \\ & \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \left\| \nabla G^\varepsilon \right\|_{L^2(\Omega^\varepsilon)}^2. \end{aligned}$$

As $\varepsilon^2 \left\| \nabla f^{\,\varepsilon} \right\|_{L^2(\Omega^\varepsilon)}^2 = \varepsilon^{-1} \left\| \nabla \hat{f}^{\,\varepsilon} \right\|_{L^2(\Omega^\varepsilon)}^2$ and $\varepsilon \left\| \nabla G^\varepsilon \right\|_{L^2(\Omega^\varepsilon)}^2 = \left\| \nabla \hat{G} \right\|_{L^2(\Omega^\varepsilon)}^2$, then

$$\begin{aligned} \varepsilon \left\| \nabla u^\varepsilon \right\|_{L^2(\Omega^\varepsilon)}^2 = \varepsilon^2 \textstyle\sum_{i,j=1}^{2} & \left\| \dfrac{\partial \hat{u}_i^\varepsilon}{\partial x_J} \right\|_{L^2(\Omega)}^2 + \varepsilon^2 \left\| \dfrac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \\ + \textstyle\sum_{i=1}^{2} & \left(\left\| \dfrac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \varepsilon^4 \left\| \dfrac{\partial \hat{u}_i^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right) \leq C \end{aligned}$$

with

$$C = \frac{8}{9\mu_* C_K} c_0 \text{ and } c_0 = \frac{(h^*)^2}{\mu_* C_K} \left\| \nabla \hat{f}^{\,\varepsilon} \right\|_{L^2(\Omega^\varepsilon)}^2 + \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \left\| \nabla \hat{G} \right\|_{L^2(\Omega^\varepsilon)}^2.$$

3.2 A priori estimates of the temperature

In this subsection, we look for an a priori estimate of the temperature $\hat{T}^{\varepsilon}$, for this we need to establish the following lemma which is a direct consequence of the Poincaré inequality.

Lemma 3.2 *The temperature $\hat{T}^{\varepsilon}$ is increased by*

$$\left\|\hat{T}^{\varepsilon}\right\|_{L^2(\Omega)} \leq h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}. \tag{3.14}$$

Lemma 3.3 *Suppose that the hypotheses of Lemma 3.1 are verified. Furthermore, suppose there are*

two strictly positive constants K_* and K^* such that

$$0 \leq K_* \leq K(\acute{x}, z) \leq K^*, \forall (\acute{x}, z) \in \Omega, \tag{3.15}$$

a positive constant $\hat{r}^*$ such that

$$\hat{r}(a) \leq \hat{r}^*, \tag{3.16}$$

then there exists a positive constant C independent of ε such that

$$\varepsilon^2 \sum_{i=1}^{2} \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial x_i}\right\|_{L^2(\Omega)} + \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)} \leq C. \tag{3.17}$$

Proof. In the variational equation (2.17), we choose $\psi = \hat{T}^{\varepsilon}$, we get

$$\sum_{i=1}^{3} I_i = \sum_{i=1}^{2} \int_{\Omega} \varepsilon^2 \hat{K} \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i} \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i} d\acute{x}dz + \int_{\Omega} \frac{\partial \hat{T}^{\varepsilon}}{\partial z} \frac{\partial \hat{T}^{\varepsilon}}{\partial z} d\acute{x}dz$$

with

$$\begin{aligned} I_1 \;=\; & \sum_{i,j=1}^{2} \frac{1}{2} \int_{\Omega} \varepsilon^2 \hat{\mu}(\hat{T}^{\varepsilon}) \left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j} + \frac{\partial \hat{u}_j^{\varepsilon}}{\partial x_i}\right)^2 \hat{T}^{\varepsilon} d\acute{x}dz + \sum_{i=1}^{2} \int_{\Omega} \left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i}\right)^2 \hat{T}^{\varepsilon} d\acute{x}dz \\ & + \int_{\Omega} 2\varepsilon^2 \hat{\mu}(\hat{T}^{\varepsilon}) \left(\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}\right)^2 \hat{T}^{\varepsilon} d\acute{x}dz, \end{aligned}$$

$$I_2 = \int_{\Omega} \hat{r}(\hat{T}^{\varepsilon}) \hat{T}^{\varepsilon} d\acute{x}dz, \quad I_3 = \int_{\Omega} \varepsilon^2 \hat{\lambda}(\hat{T}^{\varepsilon})\; div(\hat{u}^{\varepsilon})\; div(\hat{u}^{\varepsilon}) \hat{T}^{\varepsilon} d\acute{x}dz.$$

By the Cauchy-Schwartz and the Young inequalities and Lemma 3.2, we find

$$|I_1| \leq 2\hat{\mu}^* C \left\|\hat{T}^{\varepsilon}\right\|^2_{L^2(\Omega)} \leq 2\hat{\mu}^* C h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}. \tag{3.18}$$

The analogue of I_1 gives

$$|I_2| \leq \hat{r}^* \left\|\hat{T}^{\varepsilon}\right\|^2_{L^2(\Omega)} \leq \hat{r}^* h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}, \tag{3.19}$$

$$|I_3| \leq \hat{\lambda}^* C \left\| \hat{T}^\varepsilon \right\|^2_{L^2(\Omega)}. \tag{3.20}$$

On the other hand, by the use of (3.14)-(3.15), we find

$$b(\hat{T}^\varepsilon, \hat{T}^\varepsilon) = \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{K} \left| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right|^2 d\acute{x}dz + \int_\Omega \hat{K} \left| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right|^2 d\acute{x}dz.$$

This implies

$$\hat{K}_* \varepsilon^2 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} + \hat{K}_* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \leq b(\hat{T}^\varepsilon, \hat{T}^\varepsilon) \leq C_1 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)}, \tag{3.21}$$

where C_1 is a constant independent of ε given by $C_1 = 2\hat{\mu}^* h^* C + \hat{r}^* h^* + \hat{\lambda}^* C h^*$, thus

$$\left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \leq \hat{K}_*^{-1} C_1. \tag{3.22}$$

By injecting this last estimate in (3.21), we deduce (3.17).

3.3 Convergence results

In this part, we will establish the following theorem.

Theorem 3.1 *Under the same assumptions as in Lemmas* 3.1 *and* 3.3 *there exist* $u^* = (u_1^*, u_2^*)$ *in* V_z *and* T^* *in* V_z *such that for sub suites of* $\hat{u}^\varepsilon$*(resp* $\hat{T}^\varepsilon$*) noted again* $\hat{u}^\varepsilon$*(resp* $\hat{T}^\varepsilon$*), we have the following convergence results*:

$$\hat{u}_i^\varepsilon \rightharpoonup u_i^* \quad \textit{weakly in} \quad V_z(\Omega), 1 \leq i \leq 2, \tag{3.23}$$

$$\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad \textit{weakly in} \quad L^2(\Omega), 1 \leq i, j \leq 2, \tag{3.24}$$

$$\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \quad \textit{weakly in} \quad L^2(\Omega), \tag{3.25}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \ \textit{weakly in} \quad L^2(\Omega), 1 \leq i \leq 2, \tag{3.26}$$

$$\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0 \quad \textit{weakly in} \quad L^2(\Omega), \tag{3.27}$$

$$\hat{T}^\varepsilon \rightharpoonup 0 \quad \textit{weakly in} \quad V_z(\Omega), \tag{3.28}$$

$$\varepsilon \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \rightharpoonup 0 \ \textit{weakly in} \quad L^2(\Omega), 1 \leq i \leq 2. \tag{3.29}$$

Proof. The convergences of (3.23) to (3.27) are a direct result from the inequality (3.8). By using the estimate (3.17), we deduce that $\left\| \hat{T}^{\ \varepsilon} \right\| \leq h^* \left\| \frac{\partial \hat{T}^{\ \varepsilon}}{\partial z} \right\| \leq h^* C_2$. So, $\hat{T}^{\ \varepsilon}$ is bounded in $V_z(\Omega)$, which shows the existence of $T^{\ *}$ in $V_z(\Omega)$. In addition, $\varepsilon \left\| \frac{\partial \hat{T}^{\ \varepsilon}}{\partial x_i} \right\| \leq C_2$, thus $\left(\varepsilon \ \frac{\partial \hat{T}^{\ \varepsilon}}{\partial x_i} \right)$ converges to $\frac{\partial T^{\ *}}{\partial x_i}$ and $\hat{T}^\varepsilon$ converges to T^* in $V_z(\Omega)$, then $\varepsilon \ \frac{\partial \hat{T}^{\ \varepsilon}}{\partial x_i}$ weakly converges to 0 in $V_z(\Omega)$.

4 Study of the Limit Problem

To reach the desired goal, we need in the rest of this paragraph the results of previous convergences.

Lemma 4.1 *There exists a subsequence of $S(\sigma_n^\varepsilon(u^\varepsilon))$ converging strongly towards $S(\sigma_n^*(u^*))$ in $L^2(w)$*

Proof. To prove this lemma, we use the same technique as in [2] (Lemma 5.1) and in [6] (Lemma 5.2).

Theorem 4.1 $u_i^\varepsilon \to u_i^*$ *strongly in* $V_z(\Omega)$, $i = 1, 2,$ *and with the same assumptions as in Theorem 3.1, the solution* (u^*, T^*) *satisfies*

$$\sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^*) \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z}(\hat{\varrho}_i - u_i^*) d\acute{x}dz + j(\hat{\varrho}) - j(u^*) \geq \sum_{i=1}^{2} \left(\hat{f_i}^\varepsilon, \hat{\varrho}_i - u_i^*\right), \quad \forall \hat{\varrho} \in \Pi(V), \tag{4.1}$$

$$-\frac{\partial}{\partial z}\left(\hat{\mu}(T^*)\frac{\partial u_i^*}{\partial z}\right) = \hat{f_i}^\varepsilon, \quad \textit{for } i = 1, 2 \text{ in } L^2(\Omega), \tag{4.2}$$

$$-\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right) = \sum_{i=1}^{2} \hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2 + \hat{r}(T^*) \textit{ in } L^2(\Omega). \tag{4.3}$$

Proof. For $u_i^\varepsilon \to u_i^*$ strongly in V_z, we use the same methods as in [6] (proof of Theorem 4.2). By applying the convergence results of Theorem 3.1 to the variational equality (3.5) and using the fact that j is convex and lower semi-continuous, we obtain

$$\sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^*) \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z}(\hat{\varrho}_i - u_i^*) d\acute{x}dz + j(\hat{\varrho}) - j(u^*) \geq \sum_{i=1}^{2} \left(\hat{f_i}^\varepsilon, \hat{\varrho}_i - u_i^*\right). \tag{4.4}$$

From [7] (Lemma 5.3), we can choose in (4.4)

$$\hat{\varrho}_i = u_i^* \pm \psi_i, \psi_i \in H_0^1(\Omega) \text{ for } i = 1, 2 \text{ and } \hat{\varrho}_3 = u_3^*,$$

then we get

$$\sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^*) \frac{\partial u_i^*}{\partial z} \frac{\partial \psi_i}{\partial z} d\acute{x}dz = \sum_{i=1}^{2} \left(\hat{f_i}^\varepsilon, \psi_i\right).$$

Using Green's formula and choosing $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Omega)$, then $\psi_2 = 0$ and $\psi_1 \in H_0^1(\Omega)$, we obtain

$$-\int_\Omega \hat{\mu}(T^*) \frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial z}\right) d\acute{x}dz = \int_\Omega \hat{f_i}^\varepsilon \psi_i d\acute{x}dz,$$

thus

$$-\hat{\mu}(T^*)\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial z}\right) = \hat{f_i}^\varepsilon, \text{for } i = 1, 2 \text{ in } H^{-1}(\Omega), \tag{4.5}$$

and as $\hat{f_i}^\varepsilon \in L^2(\Omega)$, then (4.5) is true in $L^2(\Omega)$.

On the other hand, going to the limit in (3.6) and using (3.28)-(3.29), we find

$$\int_\Omega \hat{K} \frac{\partial T^*}{\partial z} \frac{\partial \psi}{\partial z} d\acute{x}dz = \sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^*) \left(\frac{\partial u_i^*}{\partial z}\right)^2 \psi d\acute{x}dz + \int_\Omega \hat{r}(T^*)\psi d\acute{x}dz, \forall \psi \in H^1_{\Gamma_L \cup \Gamma_1}(\Omega).$$

Now, by the formula of Green, we get

$$\int_\Omega \frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right)\psi d\acute{x}dz = \sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2 \psi d\acute{x}dz + \int_\Omega \hat{r}(T^*)\psi d\acute{x}dz, \forall\psi \in H^1_{\Gamma_L\cup\Gamma_1}(\Omega).$$

Consequently,

$$\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right)\psi d\acute{x}dz = \sum_{i=1}^{2}\hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2 \psi d\acute{x}dz + \hat{r}(T^*)\psi d\acute{x}dz, \text{ in } H^1_{\Gamma_L\cup\Gamma_1}(\Omega). \quad (4.6)$$

The formula (4.6) is valid in $L^2(\Omega)$ since $\hat{\mu}$ and $\hat{r}$ are two bounded functions in $\mathbb{R}$ and $\left(\frac{\partial u_i^*}{\partial z}\right)^2$ is an element of $L^2(\Omega)$.

Theorem 4.2 *Under the same assumptions as in the previous theorem, we have*

$$\int_w \hat{F}\,|S(\sigma_n^*(s^*))|\,(|\psi + s^* - s| - |s^* - s|)d\acute{x} - \int_w \hat{\mu}(\varsigma^*)\hat{\xi}.\psi d\acute{x} \geq 0, \forall\psi \in L^2(w) \quad (4.7)$$

$$\left\{\begin{array}{c} \hat{\mu}(\varsigma^*)\hat{\xi}^* < \hat{F}(S(\sigma_n^*(s^*)) \Rightarrow s^* = s, \\ \hat{\mu}(\varsigma^*)\hat{\xi}^* = \hat{F}(S(\sigma_n^*(s^*) \Rightarrow \exists\beta \geq 0 \textit{ such that } s^* = s - \beta\hat{\xi}^* \end{array}\right. \quad on\ w \quad (4.8)$$

with

$$s^*(\acute{x}) = u^*(\acute{x},0), \varsigma^* = T^*(\acute{x},0) \textit{ and } \hat{\xi}^* = \frac{\partial u^*}{\partial z}(\acute{x},0).$$

Another u^* and T^* satisfy the following weak form:

$$\begin{array}{c} \int_w \left(\int_0^h F(\acute{x},z)dz - hs^*(\acute{x},0) - \hat{\mu}(\varsigma^*)\hat{\xi}^*\int_0^h A(\acute{x},z)dz\right)\nabla\psi d\acute{x} \\ + \int_w\int_0^h u^*(\acute{x},z)\nabla\psi dz d\acute{x} = 0, \forall\psi \in H^1(w), \end{array} \quad (4.9)$$

where

$$A(\acute{x},z) = \int_0^z \frac{d\gamma}{\hat{\mu}\left(T^*(\acute{x},\gamma)\right)}, \quad F(\acute{x},z) = \int_0^z\int_0^\gamma \frac{\hat{f}_i^{\,\varepsilon}(\acute{x},\eta)}{\hat{\mu}\left(T^*(\acute{x},\gamma)\right)}d\eta d\gamma.$$

Proof. The proofs of (4.7)-(4.8) are similar to those given in the case of the problem fluids, see [3]. To demonstrate (4.9) by integrating twice the equation (4.2) from 0 to z, we find

$$u^*(\acute{x},z) = s^*(\acute{x},0) + \hat{\mu}(\varsigma^*)\hat{\xi}^* A(\acute{x},z) - F(\acute{x},z), \quad (4.10)$$

and as $u^*(\acute{x},h(x)) = 0$, we have

$$s^*(\acute{x},0) + \hat{\mu}(\varsigma^*)\hat{\xi}^* A(\acute{x},z) = F(\acute{x},z). \quad (4.11)$$

By integrating (4.10) from 0 to h, we obtain

$$\int_0^h u^*(\acute{x},z)dz = s^*(\acute{x},0)h + \hat{\mu}(\varsigma^*)\hat{\xi}^*\int_0^h A(\acute{x},z)dz - \int_0^h F(\acute{x},z)dz. \quad (4.12)$$

So,

$$\int_0^h u^*(\acute{x},z)dz - s^*(\acute{x},0)h - \hat{\mu}(\varsigma^*)\hat{\xi}^*\int_0^h A(\acute{x},z)dz + \int_0^h F(\acute{x},z)dz = 0.$$

From (4.11) and (4.12), we deduce (4.9). This ends the proof requested.
Before studying the existence and uniqueness of the solution, we need the following sets:

$$W_z = \left\{ v \in V_z; \frac{\partial^2 v}{\partial z^2} \in L^2(\Omega) \right\} \text{ and } B_c = \left\{ v \in W_z \times W_z; \left\| \frac{\partial v}{\partial z} \right\|_{V_z} \le c \right\}.$$

Theorem 4.3 *Under the assumptions of Theorem 3.1 and if there exists a positive sufficiently small constant $\digamma^*$ such that $\|\hat{\digamma}\|_{L^\infty(\omega)} \le \digamma^*$, then the solution (u^*, T^*) of the limit problem (4.1)-(4.3) is unique in $B_c \times W_z$.*

Proof. For the uniqueness of solution, we follow the same steps and results as in [2,8]. Suppose there are solutions (u^1, T^1) and (u^2, T^2) to the problem limit (4.1) and (4.3) for every $\psi \in H^1_{\Gamma_L \cup \Gamma_l}(\Omega)$, we have

$$\int_\Omega -\hat{K} \frac{\partial T^1}{\partial z} \frac{\partial \psi}{\partial z} d\acute{x}dz = \sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^1) \left(\frac{\partial u_i^1}{\partial z} \right)^2 \psi d\acute{x}dz + \int_\Omega \hat{r}(T^1)\psi d\acute{x}dz, \tag{4.12}$$

$$\int_\Omega -\hat{K} \frac{\partial T^2}{\partial z} \frac{\partial \psi}{\partial z} d\acute{x}dz = \sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^2) \left(\frac{\partial u_i^2}{\partial z} \right)^2 \psi d\acute{x}dz + \int_\Omega \hat{r}(T^2)\psi d\acute{x}dz. \tag{4.13}$$

By subtracting (4.12) and (4.13), we get

$$\begin{aligned} \textstyle\int_\Omega -\hat{K} \frac{\partial}{\partial z} \left(T^1 - T^2\right) \frac{\partial \psi}{\partial z} d\acute{x}dz = \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) \left(\frac{\partial u_i^1}{\partial z} \right)^2 - \hat{\mu}(T^2) \left(\frac{\partial u_i^2}{\partial z} \right)^2 \right] \psi d\acute{x}dz \\ + \textstyle\int_\Omega \left[\hat{r}(T^1) - \hat{r}(T^2) \right] \psi d\acute{x}dz. \end{aligned} \tag{4.14}$$

In (4.14), we add and subtract the term $\hat{\mu}(T^1) \left(\frac{\partial u_i^2}{\partial z} \right)^2$, we find

$$\begin{aligned} \int_\Omega \hat{K} \frac{\partial}{\partial z} \left(T^1 - T^2\right) \frac{\partial \psi}{\partial z} d\acute{x}dz &= \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) \frac{\partial}{\partial z} \left(u_i^1 + u_i^2\right) \frac{\partial}{\partial z} \left(u_i^1 - u_i^2\right) \right] \psi d\acute{x}dz + \\ & \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) - \hat{\mu}(T^2) \right] \left(\frac{\partial u_i^2}{\partial z} \right)^2 \psi d\acute{x}dz + \\ & \int_\Omega \left[\hat{r}(T^1) - \hat{r}(T^2) \right] \psi d\acute{x}dz. \end{aligned}$$

By choosing $\psi = \left(T^1 - T^2\right) \in H^1_{\Gamma_L \cup \Gamma_l}(\Omega)$, we get

$$\int_\Omega \hat{K} \frac{\partial}{\partial z} \left| T^1 - T^2 \right|^2 d\acute{x}dz = \sum_{i=1}^{3} R_k \tag{4.15}$$

with

$$\begin{aligned} R_1 &= \sum_{i=1}^{2} R_1^i = \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) \frac{\partial}{\partial z} \left(u_i^1 + u_i^2\right) \frac{\partial}{\partial z} \left(u_i^1 - u_i^2\right) \right] \left(T^1 - T^2\right) d\acute{x}dz, \\ R_2 &= \sum_{i=1}^{2} R_2^i = \sum_{i=1}^{2} \int_\Omega \left[\hat{\mu}(T^1) - \hat{\mu}(T^2) \right] \left(\frac{\partial u_i^2}{\partial z} \right)^2 \left(T^1 - T^2\right) d\acute{x}dz, \\ R_3 &= \int_\Omega \left[\hat{r}(T^1) - \hat{r}(T^2) \right] \left(T^1 - T^2\right) d\acute{x}dz, \end{aligned}$$

and similarly,

$$\int_\Omega \hat{K} \frac{\partial}{\partial z} \left| T^1 - T^2 \right|^2 d\acute{x}dz \geq K_* \left[1 + (h^*)^2 \right]^{-1} \left\| T^1 - T^2 \right\|_{V_z}. \tag{4.16}$$

Now, by the Hölder inequality we get

$$\left| R_1^i \right| \leq \mu^* \left\| \frac{\partial}{\partial z} \left(u_i^1 + u_i^2 \right) \right\|_{L^4(\Omega)} \left\| \frac{\partial}{\partial z} \left(u_i^1 - u_i^2 \right) \right\|_{L^2(\Omega)} \left\| T^1 - T^2 \right\|_{L^4(\Omega)},$$

as the compact injection of $V_z(\Omega)$ in $L^4(\Omega)$ is continuous, then there is a constant $\alpha > 0$ such that

$$\left| R_1^i \right| \leq \mu^* \alpha^2 \left\| \frac{\partial}{\partial z} \left(u_i^1 + u_i^2 \right) \right\|_{V_z} \left\| \left(u_i^1 - u_i^2 \right) \right\|_{V_z} \left\| T^1 - T^2 \right\|_{V_z}.$$

And since u_i^1 and u_i^2 are two elements of B_c, we get

$$\left| R_1^i \right| \leq 2\mu^* \alpha^2 c \left\| \left(u_i^1 - u_i^2 \right) \right\|_{V_z} \left\| T^1 - T^2 \right\|_{V_z},$$

using the Young inequality ($\alpha_1 + \alpha_2 \leq \sqrt{2} \left(\alpha_1 + \alpha_2 \right)^{\frac{1}{2}}$ for $\alpha_1, \alpha_2 \geq 0$), we have

$$\left| R_1^i \right| \leq 2\mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \sum_{i=1}^{2} \left\| \left(u_i^1 - u_i^2 \right) \right\|_{V_z} \leq 2\sqrt{2} \mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \left\| \left(u_i^1 - u_i^2 \right) \right\|_{V_z \times V_z},$$

then

$$\left| R_1 \right| \leq 2\sqrt{2} \mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \left\| \left(u^1 - u^2 \right) \right\|_{V_z \times V_z}. \tag{4.17}$$

And

$$\left| R_2^i \right| \leq C_\mu \int_\Omega \left| T^1 - T^2 \right|^2 \left| \frac{\partial u_i^2}{\partial z} \right| d\acute{x}dz \leq C_\mu \alpha^4 c^2 \left\| T^1 - T^2 \right\|_{V_z}^2,$$

thus

$$\left| R_2 \right| \leq 2C_\mu \alpha^4 c^2 \left\| T^1 - T^2 \right\|_{V_z}^2, \tag{4.18}$$

as the function $\hat{r}$ is Lipschitz on $\mathbb{R}$, there exists a constant $C_{\hat{r}}$ such that

$$\left| R_3 \right| \leq C_{\hat{r}} \left\| T^1 - T^2 \right\|_{V_z}^2. \tag{4.19}$$

By injecting (4.14) – (4.19) in (4.13) we have

$$\begin{aligned} K_* \left[1 + (h^*)^2 \right]^{-1} \left\| T^1 - T^2 \right\|_{V_z}^2 \leq & \left(2C_{\hat{\mu}} \alpha^4 c^2 + C_{\hat{r}} \right) \left\| T^1 - T^2 \right\|_{V_z}^2 + \\ & 2\sqrt{2} \mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \left\| \left(u^1 - u^2 \right) \right\|_{V_z \times V_z}, \end{aligned}$$

we suppose that $c < c_0 = \left[2C_{\hat{\mu}} \alpha^4 \right]^{\frac{-1}{2}} \left(K_* \left[1 + (h^*)^2 \right]^{-1} - C_{\hat{r}} \right)^{\frac{1}{2}}$, provided that $K_* > \left[1 + (h^*)^2 \right] C_{\hat{r}}$, then

$$\left\| T^1 - T^2 \right\|_{V_z}^2 \leq 2\sqrt{2} \mu^* \alpha^{-2} C_{\hat{\mu}}^{-1} c (c_0^2 - c^2)^{-1} \left\| \left(u^1 - u^2 \right) \right\|_{V_z \times V_z}. \tag{4.20}$$

We also have the following two inequalities:

$$\begin{aligned}&\textstyle\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varrho}_i^1 - u_i^1\right)d\acute{x}dz + \int_w \hat{F}S(\sigma_n^*(u_i^1))\left|\hat{\varrho}_i^1 - s\right|d\acute{x}\\ &\quad -\textstyle\int_w \hat{F}S(\sigma_n^*(u_i^1))\left|u_i^1 - s\right|d\acute{x} \geq \sum_{i=1}^{2}\left(\hat{f}_i^{\,\varepsilon}, \hat{\varrho}_i^1 - u_i^1\right),\end{aligned} \tag{4.21}$$

$$\begin{aligned}&\textstyle\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varrho}_i^2 - u_i^2\right)d\acute{x}dz + \int_w \hat{F}S(\sigma_n^*(u_i^2))\left|\hat{\varrho}_i^2 - s\right|d\acute{x}\\ &\quad -\textstyle\int_w \hat{F}S(\sigma_n^*(u_i^2))\left|u_i^2 - s\right|d\acute{x} \geq \sum_{i=1}^{2}\left(\hat{f}_i^{\,\varepsilon}, \hat{\varrho}_i^2 - u_i^2\right).\end{aligned} \tag{4.22}$$

We choose $\hat{\varrho}_i^1 = u_i^2$ in (4.21) and $\hat{\varrho}_i^2 = u_i^1$ in (4.22), and after summing up the two inequalities, it comes to $W = u_i^2 - u_i^1$

$$\sum_{i=1}^{2}\int_\Omega\left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z} - \hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz + \int_w \hat{F}S(\sigma_n^*(u_i^1)\left(\left|u_i^2 - s\right| - \left|u_i^1 - s\right|\right)d\acute{x}$$

$$-\int_w \hat{F}S(\sigma_n^*(u_i^2)\left(\left|u_i^2 - s\right| - \left|u_i^1 - s\right|\right)d\acute{x} \geq 0,$$

so, for the next term, we have

$$\begin{aligned}&\int_w \hat{F}S(\sigma_n^*(u_i^1)\left(\left|u_i^2 - s\right| - \left|u_i^1 - s\right|\right)d\acute{x} - \int_w \hat{F}S(\sigma_n^*(u_i^2)\left(\left|u_i^2 - s\right| - \left|u_i^1 - s\right|\right)d\acute{x}\\ \leq\ &\int_w \left|\hat{F}\left(S(\sigma_n^*(u_i^1) - S(\sigma_n^*(u_i^2))\right|\left|u_i^2 - u_i^1\right|d\acute{x}.\end{aligned}$$

According to the inequality of Cauchy-Schwartz we obtain

$$\int_w \left|\hat{F}\left(S(\sigma_n^*(u_i^1) - S(\sigma_n^*(u_i^2))\right|\left|u_i^2 - u_i^1\right|d\acute{x} \leq \left\|\hat{F}\right\|_{L^\infty(w)} C\left\|u_i^2 - u_i^1\right\|_{V_z}^2 \leq F^* C\left\|u_i^2 - u_i^1\right\|_{V_z}^2.$$

By the previous theorem the term $F^* C\left\|u_i^2 - u_i^1\right\|_{V_z}^2$ tends to 0, then we have

$$\sum_{i=1}^{2}\int_\Omega\left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z} - \hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz \geq 0, \tag{4.23}$$

we add and subtract the term $\hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}$ from the equation (4.23), we get

$$\sum_{i=1}^{2}\int_\Omega\left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z} - \hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz + \sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz$$

$$-\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz \geq 0.$$

So,

$$\sum_{i=1}^{2}\int_\Omega -\hat{\mu}(T^1)\frac{\partial W}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz + \sum_{i=1}^{2}\int_\Omega(\hat{\mu}(T^1) - \hat{\mu}(T^2))\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz \geq 0, \tag{4.24}$$

and

$$\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial W}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz \geq \frac{\mu_*}{2}\left\|W\right\|_{V_z}^2. \tag{4.25}$$

Using the Hölder inequality, and the results of [8], we find

$$\left| \sum_{i=1}^{2} \int_{\Omega} (\hat{\mu}(T^1) - \hat{\mu}(T^2)) \frac{\partial u_i^2}{\partial z} \frac{\partial W}{\partial z} d\acute{x} dz \right| \leq \sqrt{2} c \alpha^2 C_{\hat{\mu}} \left\| T^1 - T^2 \right\|_{V_z} \|W\|_{V_z}. \tag{4.26}$$

By injecting (4.25) into (4.24), we get

$$\frac{\mu_*}{2} \|W\|_{V_z} \leq \sqrt{2} c \alpha^2 C_{\hat{\mu}} \left\| T^1 - T^2 \right\|_{V_z}. \tag{4.27}$$

Returning to (4.20), we obtain

$$\begin{aligned} \left\| T^1 - T^2 \right\|_{V_z} &\leq 2\sqrt{2} \mu^* c \alpha^{-2} (c_0^2 - c^2)^{-1} \left\| u^2 - u^1 \right\|_{V_z \times V_z} \\ &\leq 8 \mu_*^{-1} \mu^* C_{\hat{\mu}}^{-1} c^2 (c_0^2 - c^2)^{-1} \left\| T^1 - T^2 \right\|_{V_z} \leq 0, \end{aligned}$$

provided that $0 < c < c_1 = (1 + 8\mu_*^{-1}\mu^*)^{-\frac{1}{2}} c_0$. Therefore, $\left\| T^1 - T^2 \right\|_{V_z} = 0$.

So, there exists $T^1 = T^2$ almost everywhere in V_z. According to (4.27), we deduce that $u^1 = u^2$ almost everywhere in V_z.

5 Conclusion

The purpose of this paper is to study the asymptotic behavior of a non-isothermal elasticity system in a thin domain with Coulomb friction on the bottom surface. One of the objectives of this study is to obtain a two-dimensional equation that allows a reasonable description of the phenomenon occurring in the three-dimensional domain by passing the limit to 0 on the small thickness of the domain (3D). As a first step, we gave the variational formulation of the problem and demonstrate the results of existence and uniqueness of the weak solution, then we moved on to the asymptotic analysis. For this, by using the change of scale according to the third component we conduct the study on a domain Ω which does not depend on ε. Then, by different inequalities, we proved some estimates for the displacement and the temperature, which allow us to go to the limit when ε tends towards zero in the variational formulation. Finally, we have reached our main result concerning the proof of the convergence results and uniqueness of the limit problem.

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