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A New Fractional-Order 3D Chaotic System Analysis and Synchronization

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Abstract: In this work, a fractional-order form of a novel 3D chaotic system is introduced. Firstly, this fractional system can display chaotic behavior for a given minimal commensurate order. Secondly, theoretical and numerical solution representation is given by exploiting the Adams–Bashforth–Moulton algorithm for the presented novel fractional-order system. Thirdly, we have studied full-state hybrid projective synchronization (FSHPS) type the novel 3D fractional-order system and the fractional-order hyper-chaotic Lorenz system based on the definition of this kind of synchronization and the Lyapunov theory of stability of linear fractional-order systems. Finally, numerical simulations are given to show the effectiveness of the proposed controller via the improved Adams–Bashforth–Moulton algorithm.

Keywords: *fractional-order system; chaotic system; FSHPS; Lyapunov theory; synchronization.*

Mathematics Subject Classification (2010): 37B55, 34C28, 34D08, 37B25, 37D45, 93C40, 93D05.

1 Introduction

During these years the synchronization of chaotic dynamical systems has generated a great interest among researchers in nonlinear sciences, in view of its practical applications, many types of synchronization have been reported in the literature; these include complete, generalized, anticipated, lag, measure, projective, phase, reduced order and adaptive synchronizations. These concepts of synchronization have led to the creation of many methods of controlling chaos and synchronization by many researchers, including

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the active control method [2], sliding mode control [3, 40], backstepping control [4], adaptive control [5, 39, 40], fuzzy control [6], passive control [7], projective synchronization method [8], function projective synchronization method [9], etc.

In recent years, fractional-order systems have become a hot research field addressed by many researchers such as [10-16, 25-33]. Study of chaos and synchronization in fractional-order dynamical systems has attracted considerable attention [17-19] due to its powerful potential applications in different fields such as secure communication, telecommunications, cryptography [20-21]. Diverse techniques of different types of synchronization have been proposed and developed for fractional-order systems. These include the sliding mode control [22-23], active control technique [24-25], function projective synchronization [26], and modified projective synchronization [27], hybrid projective synchronization [28], full state hybrid projective synchronization [29], inverse full state hybrid projective synchronization [35-36] and others, see, for example, [30-34, 40-41].

This paper is organized as follows. In Section 2, the Caputo fractional-order derivative definition is given with some notes. In Section3, we described the new form of the three-dimensional system in fractional order, Section 4 is devoted to the study of the FSHP synchronization between the novel 3D fractional-order system and the fractional-order hyper-chaotic Lorenz system based on the definition of this kind of synchronization and the Lyapunov theory of stability. In Section 5, we present the numerical results to verify the effectiveness of the method. Finally, the conclusion is mentioned in Section 6.

2 Preliminaries

In general, for fractional derivatives there are three well known definitions, that is, the Riemann-Liouville, Grünwald-Letnikov and Caputo definitions. The Caputo fractional derivative is much preferred since it is more popular in real application and the initial conditions for fractional-order differential equations with the Caputo derivative are in the same form as for integer-order differential equations.

The Caputo fractional derivative of $f(t)$ is given as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(\tau - n)^{q-n+1}} d\tau \tag{1}$$

for $n-1 < q \leq n, n \in N, t > 0$. $\Gamma(\cdot)$ is the gamma function.

Some other important properties of the fractional derivatives and integrals can be found in several works ([10-15], etc). Geometric and physical interpretation of fractional integration and fractional differentiation were exactly described in [16].

2.1 Problem Formulation

We consider the drive system given by

$$D_t^{q_i} x_i(t) = f_i(X(t)), i = 1, .., n, \tag{2}$$

where $X(t) = (x_1, x_2, ..., x_n)^T$ is the state vector of the system (2), $f_i : R^n \longrightarrow R^n$, for $i = 1, .., n$, are nonlinear functions, and the response system is given by

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{m} b_{ij} y_j(t) + g_i(Y(t)) + u_i, i = 1, .., m, \tag{3}$$

where $Y(t) = (y_1, y_2, ..., y_m)^T$ is the state vector of the system (3), $g_i : R^m \longrightarrow R^m$, for $i = 1, .., m$, are nonlinear functions, $0 < q_i < 1, D_t^{q_i}$ is the Caputo fractional derivative of order q_i. u_i, $i = 1, .., m$, are controllers to be designed so that the system (2) and the system (3) to be synchronized.

Lemma 2.1 *The trivial solution of the following fractional-order system* [35]*:*

$$D_t^q X(t) = F(X(t)), \tag{4}$$

where D_t^q is the Caputo fractional derivative of order q, $0 < q \leq 1$, $F : R^n \longrightarrow R^n$, is asymptotically stable if there exists a positive definite function $V(X(t))$ such that $D_t^q V(X(t)) < 0$ for all $t > 0$.

Lemma 2.2 $\forall X(t) \in R^n, \forall q \in [0,1]$, *and* $\forall\ t > 0$

$$\frac{1}{2} D_t^q \left(X^T(t) X(t)\right) \leq X^T(t) D_t^q (X(t)). \tag{5}$$

Now, we introduce the definition of FSHPS between the master and slave systems.

Definition 2.1 FSHPS occurs between the master and slave systems (2) and (3) when there exist controllers u_i, $i = 1, 2, ..., n$, and given real numbers $(\alpha_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ such that the synchronization errors

$$e_i(t) = y_i(t) - \sum_{j=1}^{n} \alpha_{ij} x_j(t), i = 1, .., m \tag{6}$$

satisfy $\lim_{t\to+\infty} e_i(t) = 0$.

3 Description of the Novel Chaotic System

In this research work, we consider the fractional-order form of the integer-order 3D chaotic system introduced in [38] and given by

$$\begin{cases} D_t^{q_1} x = a(y - x) + byz, \\ D_t^{q_2} y = (c - a)x + cy - xz, \\ D_t^{q_3} z = xy - z, \end{cases} \tag{7}$$

where a, b, c are positive real parameters with $b \neq 1$. In the first part of this paper, we shall show that the system (7) is chaotic when the parameters a, b and c take the values

$$a = 15, b = 8/3, c = 10. \tag{8}$$

3.1 Dynamical behavior

For the values of parameters (8), the system (7) has five equilibrium points

$$\begin{cases} E_0 = (0, 0, 0), \\ E_{1,3} = (\pm 4.049, \mp 3.1659, -12.819), \\ E_{2,4} = (\pm 1.7464, \pm 1.2563, 2.194). \end{cases} \tag{9}$$

For E_0, we obtain the eigenvalues

$$\lambda_1 = -1, \lambda_2 = -11.514, \lambda_3 = 6.5139. \tag{10}$$

This implies that E_0 is an unstable saddle point.

With the same method, the eigenvalues of the Jacobian at E_1 and E_3 are

$$\lambda_1 = 3.3706 - 8.3184i, \lambda_2 = 3.3706 + 8.3184i, \lambda_3 = -12.741 \tag{11}$$

and the eigenvalues of the Jacobian at E_2 and E_4 are

$$\lambda_1 = 1.0824 + 4.5105i, \lambda_2 = 1.0824 - 4.5105i, \lambda_3 = -8.1648, \tag{12}$$

then E_1 and E_3 are two unstable saddle-focus points and E_2 and E_4 are two unstable saddle-focus points because none of the eigenvalues have real part zero and λ_1, λ_2 are complex.

In the case of the comensurate-order system, where $q_1 = q_2 = q_3$, a necessary condition for the fractional-order nonlinear system (7) to be chaotic is $q > \frac{2}{\pi} \arctan\left(\frac{|Im(\lambda)|}{Re(\lambda)}\right)$, where λ are the eigenvalues of the saddle equilibrium point of index two in system (7). From the above eigenvalues we can determine a minimal commensurate order to keep the system (7) chaotic and it is $q > 0.75491$ for E_1 and E_3 and $q > 0.85006$ for E_2 and E_4. Thus, the necessary condition of existence of chaos in fractional-order system (7) is $q > 0.85006$.

The necessary condition for the system (7) to exhibit chaotic oscillations in the incommensurate case is $\frac{\pi}{2M} - min_i(|arg(\lambda_i (J_E))|) > 0$, where $\lambda_i (J_E)$, $i = 1, 2, 3$, are the eigenvalues of the Jacobian matrix J_E of the system (7) at the equilibrium E, M is the LCM of the fractional orders. For example, if $q_1 = 0.9, q_2 = 0.9, q_3 = 0.8$, then $M = 10$. The characteristic equation of the system evaluated at the equilibrium E_i is $det(diag[\lambda^{Mq_1}, \lambda^{Mq_2}, \lambda^{Mq_3}] - J_{E_i}) = 0$, i.e., $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_i}) = 0$, $i = 1, 2, 3, 4$.

We get $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_0}) = 0, det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_{1,3}}) = 0, det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{E_{2,4}}) = 0$. This yields: $\lambda^{26} + \lambda^{18} + 5\lambda^{17} + 5\lambda^9 - 75\lambda^8 - 75 = 0, \lambda^{26} + \lambda^{18} + 5\lambda^{17} - 5.3334\lambda^9 - 3.04 \times 10^{-4}\lambda^8 + 1026.4 = 0$, $\lambda^{26} + \lambda^{18} + 5\lambda^{17} + 3.8412\lambda^9 + 2.094 \times 10^{-3}\lambda^8 + 175.67 = 0$.

From the roots of the above equations, we find $\lambda = 1.2315$ whose argument is zero, which is the minimum argument, and hence the necesary stability condition holds because $\frac{\pi}{2M} - 0 > 0$.

3.2 Application of Adams–Bashforth–Moulton algorithm to the system (7)

By exploiting the Adams–Bashforth–Moulton algorithm [37], the novel fractional-order chaotic system (7) can be written as

$$\begin{cases} x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)} \begin{pmatrix} a\left(y^p_{n+1} - x^p_{n+1}\right) + by^p_{n+1}z^p_{n+1} \\ + \sum_{j=1}^{n} a_{1,j,n+1}\left(a\left(y_j - x_j\right) + by_j z_j\right) \end{pmatrix}, \\ y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)} \begin{pmatrix} (c-a)\, x^p_{n+1} + cy^p_{n+1} - x^p_{n+1}z^p_{n+1} \\ + \sum_{j=1}^{n} a_{2,j,n+1}\left((c-a)\, x_j + cy_j - x_j z_j\right) \end{pmatrix}, \\ z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)} \left(x^p_{n+1}y^p_{n+1} - z^p_{n+1} + \sum_{j=1}^{n} a_{3,j,n+1}\left(x_j y_j - z_j\right)\right) \end{cases} \tag{13}$$

in which

$$\begin{cases} x^p_{n+1} = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=1}^{n} b_{1,j,n+1} \left(a\left(y_j - x_j\right) + b y_j z_j\right), \\ y^p_{n+1} = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=1}^{n} b_{2,j,n+1} \left(\left(c-a\right) x_j + c y_j - x_j z_j\right), \\ z^p_{n+1} = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=1}^{n} b_{3,j,n+1} \left(x_j y_j - z_j\right) \end{cases} \tag{14}$$

with

$$\begin{cases} b_{1,j,n+1} = \frac{h^{q_1}}{q_1} \left((n-j+1)^{q_1} - (n-j)^{q_1}\right), \\ b_{2,j,n+1} = \frac{h^{q_2}}{q_2} \left((n-j+1)^{q_2} - (n-j)^{q_2}\right), \\ b_{3,j,n+1} = \frac{h^{q_3}}{q_3} \left((n-j+1)^{q_3} - (n-j)^{q_3}\right) \end{cases} \tag{15}$$

and

$$a_{i,j,n+1} = \begin{cases} (n)^{q_i+1} - (n-q_i)(n+1)^{q_i}, j = 0, \\ (n-j+2)^{q_i+1} - (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, 1 \le j \le n, \quad i = 1,2,3, \\ 1, j = n+1. \end{cases} \tag{16}$$

Applying the above algorithm, numerical solution of a fractional-order system can be computed.

Fig.1 depicts the simulation result (double scroll-attractor) for the fractional-order system (7) projected onto the $x - z$ plane, computed for the simulation time $Tsim = 100s$ and $q_1 = q_2 = q_3 = 0.88$ and time step $h = 0.005$, $a = 15$, $b = 8/3$, $c = 10$, $(x(0), y(0), z(0)) = (1, -1, 2)$.

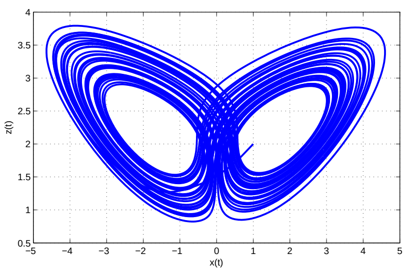

Figure 1: Simulation result for the fractional-order system (7) projected onto the $x - z$ plane.

Also, the simulation result for the fractional-order system (7) projected onto the $z - y$ plane for $a = 15, b = 8/3, c = 10$, $q_1 = q_2 = q_3 = 0.95, (x(0), y(0), z(0)) = (-2, 5, -10)$ is shown in Fig.2.

4 FSHP Synchronization of Fractional-Fractional Order Systems

In this section, the new fractional-order chaotic system (7) and fractional-order hyperchaotic Lorenz system are used to achieve FSHPS between these systems. Thus, as the driving system, we consider the novel fractional-order chaotic system given by

$$\begin{cases} D_t^{q_1} x_1 = a(x_2 - x_1) + b x_2 x_3, \\ D_t^{q_2} x_2 = (c-a) x_1 + c x_2 - x_1 x_3, \\ D_t^{q_3} x_3 = x_1 x_2 - x_3, \end{cases} \tag{17}$$

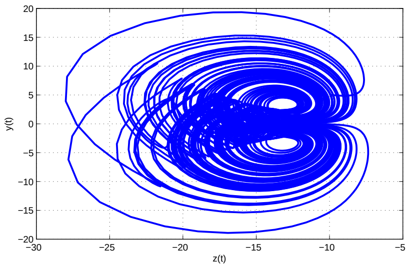

Figure 2: Simulation result for the fractional-order system (7) projected onto the $z-y$ plane.

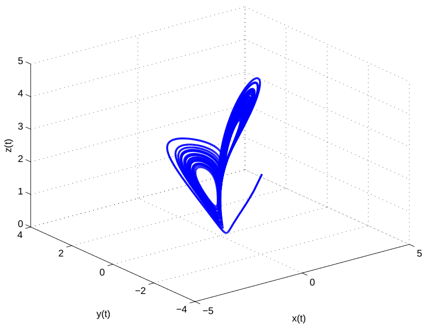

Figure 3: Simulation result for the fractional-order system (7) projected onto the $x-y-z$ plane for $q_1 = q_2 = 0.9, q_3 = 0.8, (x(0), y(0), z(0)) = (1, -1, 2)$.

where $a = 15, b = 8/3, c = 10$, and as the response system, we consider the controlled fractional-order hyper-chaotic Lorenz system given by

$$\begin{cases} D_t^{q_1} y_1 = a\left(y_2 - y_1\right) + y_4 + u_1, \\ D_t^{q_2} y_2 = c y_1 - y_2 - y_1 y_3 + u_2, \\ D_t^{q_3} y_3 = -b y_3 + y_1 y_2 + u_3, \\ D_t^{q_4} y_4 = -y_2 y_3 + d y_4 + u_4, \end{cases} \tag{18}$$

where $a = 10, b = 28, c = 8/3, d = -1$ and with the fractional-orders of the system $(q_1, q_2, q_3, q_4) = (0.98, 0.98, 0.98, 0.98)$.

In view of Definition 2.1, the state errors for (17) and (18) are

$$e_i = y_i - \sum_{j=1}^{3} \alpha_{ij} x_j, i = 1, 2, 3, 4. \tag{19}$$

This gives

$$D_t^{q_i} e_i = D_t^{q_i} y_i - D_t^{q_i} \left(\sum_{j=1}^{3} \alpha_{ij} x_j \right), i = 1, 2, 3, 4. \tag{20}$$

Consequently, the error dynamic system is given by

$$\begin{cases} D_t^{q_1} e_1 = D_t^{q_1} y_1 - D_t^{q_1} \left(\sum_{j=1}^{3} \alpha_{1j} x_j \right), \\ D_t^{q_2} e_2 = D_t^{q_2} y_2 - D_t^{q_2} \left(\sum_{j=1}^{3} \alpha_{2j} x_j \right), \\ D_t^{q_3} e_3 = D_t^{q_3} y_3 - D_t^{q_3} \left(\sum_{j=1}^{3} \alpha_{3j} x_j \right), \\ D_t^{q_4} e_4 = D_t^{q_4} y_4 - D_t^{q_4} \left(\sum_{j=1}^{3} \alpha_{4j} x_j \right), \end{cases} \tag{21}$$

i.e.,

$$\begin{cases} D_t^{q_1} e_1 = a(y_2 - y_1) + y_4 + u_1 - D_t^{q_1} (\alpha_{11} x_1 + \alpha_{12} x_2 + \alpha_{13} x_3), \\ D_t^{q_2} e_2 = c y_1 - y_2 - y_1 y_3 + u_2 - D_t^{q_2} (\alpha_{21} x_1 + \alpha_{22} x_2 + \alpha_{23} x_3), \\ D_t^{q_3} e_3 = -b y_3 + y_1 y_2 + u_3 - D_t^{q_3} (\alpha_{31} x_1 + \alpha_{32} x_2 + \alpha_{33} x_3), \\ D_t^{q_4} e_4 = -y_2 y_3 + d y_4 + u_4 - D_t^{q_4} (\alpha_{41} x_1 + \alpha_{42} x_2 + \alpha_{43} x_3). \end{cases} \tag{22}$$

The system (22) can be described as

$$\begin{cases} D_t^{q_i} e_i = \sum_{j=1}^{4} b_{ij} e_j(t) + R_i + u_i, i = 1, 2, 3, 4, \end{cases} \tag{23}$$

where

$$\begin{cases} R_1 = -D_t^{q_1} (\alpha_{11} x_1 + \alpha_{12} x_2 + \alpha_{13} x_3) + \sum_{j=1}^{4} b_{1j} (y_j(t) - e_j(t)), \\ R_2 = -y_1 y_3 - D_t^{q_2} (\alpha_{21} x_1 + \alpha_{22} x_2 + \alpha_{23} x_3) + \sum_{j=1}^{4} b_{2j} (y_j(t) - e_j(t)), \\ R_3 = y_1 y_2 - D_t^{q_3} (\alpha_{31} x_1 + \alpha_{32} x_2 + \alpha_{33} x_3) + \sum_{j=1}^{4} b_{3j} (y_j(t) - e_j(t)), \\ R_4 = -y_2 y_3 - D_t^{q_4} (\alpha_{41} x_1 + \alpha_{42} x_2 + \alpha_{43} x_3) + \sum_{j=1}^{4} b_{4j} (y_j(t) - e_j(t)). \end{cases} \tag{24}$$

Rewrite error system (23) in the compact form

$$D_t^q e = Be + R + U, \tag{25}$$

where $B = (b_{ij})_{4\times4}$ and $e = (e_1, e_2, e_3, e_4)^T, R = (R_i)_{1\le i\le 4}, U = (u_i)_{1\le i\le 4}$.

Theorem 4.1 *FSHPS between the master system* (17) *and the slave system* (18) *occurs under the following control law:*

$$U = -(R + De), \tag{26}$$

where D is a 4×4 feedback gain matrix selected so that $B - D$ is a negative definite matrix.

Proof. By inserting (26) into (25), we get

$$D_t^q e = (B - D) e, \tag{27}$$

where $B = (b_{ij}), D = (d_{ij})$ are two 4×4 matrices and $e = (e_1, e_2, e_3, e_4)^T$ is the error vector of the system. If we choose matrix D such that $B - D$ is negative, then all the eigenvalues $\lambda_i, i = 1, 2, 3, 4$, of $B - D$ stay in the left-half plane, i.e., $Re(\lambda_i) < 0$, and if a candidate Lyapunov function is chosen as

$$V = \sum_{i=1}^{4} \frac{1}{2} e_i^2, \tag{28}$$

then the time Caputo fractional derivative of order 0.98 of V along the trajectory of system (27) is as follows:

$$D_t^{0.98}V = \sum_{j=1}^{4} D_t^{0.98}\left(\frac{1}{2}e_i^2\right). \tag{29}$$

Using Lemma 2.2, we get

$$\begin{aligned} D_t^{0.98}V &\leq \sum_{j=1}^{4} e_i D_t^{0.98} e_i && (30)\\ &= \lambda_1 e_1^2 + \lambda_2 e_2^2 + \lambda_3 e_3^2 + \lambda_4 e_4^2 < 0, && (31) \end{aligned}$$

which ensures, according to Lemma 2.1, that the trivial solution of the fractional-order system (27) is asymptotically stable. Hence the FSHP synchronization between the system (17) and the system (18) is achieved. This completes the proof.

5 Numerical Simulation

According to the above method, for FSHPS we have

$$B = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 8/3 & -1 & 0 & 0 \\ 0 & 0 & -28 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{32}$$

and

$$\begin{cases} R_1 = 10e_1 - 10e_2 - e_4 + 25x_1 - 35x_2 + 5x_3 - 10y_1 + 10y_2 + y_4 - 5x_1x_2 + 2x_1x_3 - \frac{8}{3}x_2x_3, \\ R_2 = e_2 - \frac{8}{3}e_1 + 35x_1 - 40x_2 - 3x_3 + \frac{8}{3}y_1 - y_2 + 3x_1x_2 + x_1x_3 - \frac{16}{3}x_2x_3 - y_1y_3, \\ R_3 = 28e_3 + 75x_1 - 90x_2 + 2x_3 - 28y_3 - 2x_1x_2 + 3x_1x_3 - \frac{32}{3}x_2x_3 + y_1y_2, \\ R_4 = e_4 + 100x_1 - 110x_2 + 7x_3 - y_4 - 7x_1x_2 + 2x_1x_3 - 16x_2x_3 - y_2y_3, \end{cases} \tag{33}$$

and

$$(u_1, u_2, u_3, u_4)^T = -\left(R + D\,(e_1, e_2, e_3, e_4)^T\right), \tag{34}$$

i.e.,

$$\begin{cases} u_1 = 35x_2 - 25x_1 - 15e_1 - 5x_3 + 10y_1 - 10y_2 - y_4 + 5x_1x_2 - 2x_1x_3 + \frac{8}{3}x_2x_3, \\ u_2 = 40x_2 - 35x_1 - 5e_2 + 3x_3 - \frac{8}{3}y_1 + y_2 - 3x_1x_2 - x_1x_3 + \frac{16}{3}x_2x_3 + y_1y_3, \\ u_3 = 90x_2 - 75x_1 - 30e_3 - 2x_3 + 28y_3 + 2x_1x_2 - 3x_1x_3 + \frac{32}{3}x_2x_3 - y_1y_2, \\ u_4 = 110x_2 - 100x_1 - 10e_4 - 7x_3 + y_4 + 7x_1x_2 - 2x_1x_3 + 16x_2x_3 + y_2y_3 \end{cases} \tag{35}$$

for the chosen $(\alpha_{ij})_{4\times 3} = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & -3 \\ 4 & 3 & 2 \\ 6 & 2 & 7 \end{pmatrix}, D = \begin{pmatrix} 5 & 10 & 0 & 1 \\ 8/3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}.$

Then the error system is given by

$$\begin{pmatrix} D_t^{q_1}e_1 \\ D_t^{q_2}e_2 \\ D_t^{q_3}e_3 \\ D_t^{q_4}e_4 \end{pmatrix} = \begin{pmatrix} -15 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{36}$$

and then the eigenvalues of the matrix $(B - D)$ are given by $\lambda_1 = -15$, $\lambda_2 = -5$, $\lambda_3 = -30$, $\lambda_4 = -10$, which all are negative. Hence the error system is asymptotically stable and the synchronization between the two systems (17) and (18) is achieved.

We used the improved classical Adams–Bashforth–Moulton method to show the effectiveness of the proposed controller by solving the system (36) for $(q_1, q_2, q_3, q_4) = (0.98, 0.98, 0.98, 0.98)$ and with the initial conditions chosen as $(e_1(0), e_2(0), e_3(0), e_4(0)) = (-2, 1, -5, 1)$. In Figs.4-7, the time-history of the synchronization errors $e_1(t); e_2(t); e_3(t);\ e_4(t)$ is depicted.

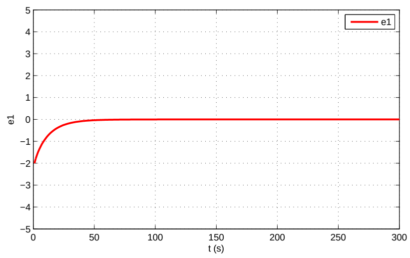

Figure 4: Time-response of the error $e_1(t)$ under the controller (35) $forFSHP$.

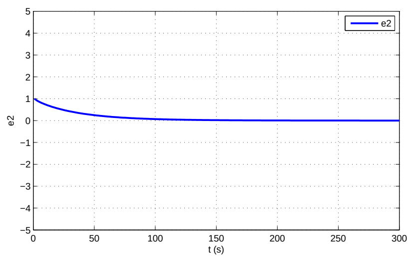

Figure 5: Time-response of the error $e_2(t)$ under the controller (35) $forFSHP$.

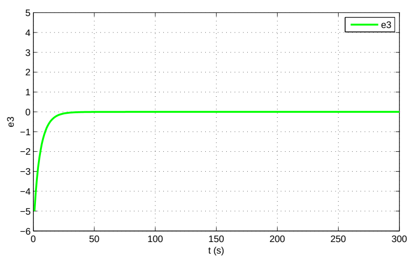

Figure 6: Time-response of the error $e_3(t)$ under the controller (35) $forFSHP$.

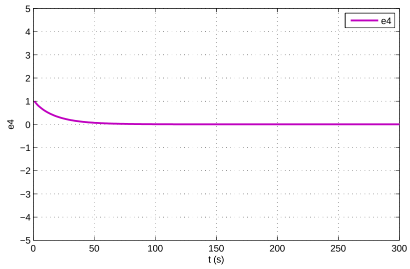

Figure 7: Time-response of the error $e_4(t)$ under the controller (35) $forFSHP$.

6 Conclusion

In this work, we have presented a fractional-order form of a new 3D chaotic system based on the Caputo derivative definition, where we have proven that this system has chaotic behavior starting with a specific value of minimal commensurate order. A theoretical and numerical solution representation was given using the Adams–Bashforth–Moulton algorithm for this system. Also, full-state hybrid projective synchronization (FSHPS) between the novel 3D fractional-order system and the fractional-order hyper-chaotic Lorenz system has been studied based on the definition of this kind of synchronization and the Lyapunov theory of stability for linear fractional-order systems. Numerical simulations are given to validate the effectiveness of the proposed controller via the improved Adams–Bashforth–Moulton algorithm in Matlab. Further studies regarding practical applications of this fractional system will be carried out in our next works.

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