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Dynamical Behaviors of Fractional-Order Selkov Model and Its Discretization

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Abstract: In this work, we study the dynamics of a fractional-order Selkov model, which is a classical mathematical model describing glycolysis, and its corresponding discretized version. First, non-negativity, existence and uniqueness of the solution for the model are discussed. We also investigate the local stability and the existence of a Hopf bifurcation. The discrete fractional-order model is shown to exhibit very rich behaviors and when considering the step size as a control parameter, a flip bifurcation, Neimark-Sacker bifurcation and chaos are obtained. Finally, numerical simulations are carried out to verify the correctness of the theoretical results obtained.

Keywords: *fractional order; Selkov model; local stability; bifurcations; discretization.*

Mathematics Subject Classification (2010): 34A08, 34A34, 34C23, 65P20.

1 Introduction

Glycolysis is the first step in the breakdown of glucose to extract energy for cellular metabolism, it is present in nearly all living organisms. After many years of experimental observations, Higgins [1] was the first to use mathematical modelling to understand the process, he presented a model to explain sustained oscillations in the yeast glycolytic system. His model, however, has no limit cycle for those values of its parameters with which self oscillations are observed experimentally. In 1968, Selkov [2] introduced an alternative mathematical model able to well reproduce the glycolytic oscillations in yeast, it was shown that the Selkov model exhibits a Hopf bifurcation and thus there exist parameter values for which it has a periodic solution. This model has sparked a number

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of further complementary studies. Results on the uniqueness and global attractivity of the glycolytic oscillations were obtained by d'Onofrio [3]. Verveyko et al. [4] studied the influence of periodic influx of glycolytic oscillations within the forced Selkov system, in this work the quasi-periodic oscillations and chaos were obtained. In [5], several properties of the dynamics of the solutions of the model were established, an analysis of the Poincaré compactification of the system was given to study unbounded solutions. Artés et al. [6] described the global dynamics in the Poincaré disc of the Selkov and Higgins-Selkov models.

On the other hand, fractional calculus, defined as a generalisation of ordinary differentiation and integration to arbitrary non-integer order, has ganed a considerable importance during the past four decades and longer due to its use in many fields in natural science and engineering applications such as neurons [7], finance systems [8, 9], biological systems [10, 11], chemical systems [12], nuclear magnetic resonance [13] and so on. The main advantage of fractional-order differential equation models is that they are naturally related to systems with memory, which exist in most biological systems, and it allows greater degree of freedom than integer-order systems. Moreover, it was shown that the stability region of the fractional-order models is greater than the stability region of the integer-order ones.

In this paper, we investigate the dynamic behaviors of the fractional-order Selkov model and its corresponding discretized version. First, we prove different mathematical results such as non-negativity, existence and uniqueness of the solution for the model. We also study the local stability of the unique equilibrium point and the existence of a Hopf bifurcation. We implement a predictor-corrector scheme to do numerical simulations and to illustrate our analytical findings. The discrete fractional-order model is shown to exhibit very rich behaviors. A flip bifurcation, Neimark-Sacker bifurcation and chaos are obtained.

2 Description of the Model

In [2], Selkov proposed the following simplified system:

$$\begin{aligned} \frac{dx}{dt} &= 1 - xy^{\gamma}, \\ \frac{dy}{dt} &= \alpha y \left(xy^{\gamma-1} - 1 \right) \end{aligned} \tag{1}$$

with the initial conditions $x(0) > 0,\ y(0) > 0$. The quantities $x(t)$ and $y(t)$ represent dimensionless concentrations of ATP and ADP at time t, respectively, while t is a dimensionless time variable. $\gamma > 1$ is the stoichiometric parameter of the reaction

$$\gamma ADP + FF \leftrightarrow P,$$

the parameter α is a positive real number.

Selkov showed that the above planar system has only one equilibrium $E = (1, 1)$ in a finite part of the (x, y) phase plane, the characteristic equation in the neighbourhood of E is

$$\begin{vmatrix} -1-\lambda & -\gamma \\ \alpha & \alpha(\gamma-1)-\lambda \end{vmatrix} = 0,$$

its roots are

$$\lambda_{1,2} = \frac{1}{2}\{\alpha(\gamma-1) - 1 \pm \sqrt{[\alpha(\gamma-1)-1]^2 - 4\alpha}\}.$$

Then E is a stable node at $0 < \alpha \leq \alpha_1$, a stable focus at $\alpha_1 < \alpha < \alpha_0$, an unstable focus at $\alpha_0 < \alpha < \alpha_2$ and an unstable node at $\alpha_2 \leq \alpha < \infty$,
where

$$\alpha_0 = \frac{1}{\gamma - 1} \qquad \text{and} \qquad \alpha_{1,2} = \left(\frac{\sqrt{\gamma} \pm 1}{\gamma - 1}\right)^2, \quad (\alpha_1 < \alpha_2). \tag{2}$$

Moreover, if we consider the real part of the eigenvalues as a function of α, then it passes through zero when $\alpha = \alpha_0$ and its derivative with respect to α at that point is non-zero. Thus a Hopf bifurcation occurs. The first Lyapunov number σ of the bifurcation is found to be $\sigma = -\frac{3\pi(\gamma-1)^{\frac{1}{2}}}{4}(\gamma^2(\gamma-1)+1) < 0$. Thus the Hopf bifurcation is non-degenerate and supercritical [5].

Remark 2.1 Selkov showed that the investigation of system (1) in infinity by means of the Poincaré transformations [14] provides that this system has, in a positive quadrant of infinity, two more equilibrium states: $E_1 = (\infty, 0)$ and $E_2 = (0, \infty)$, which explains that some orbits may be unbounded. Here, we will consider only the equilibrium $E = (1, 1)$ in a finite part of the (x, y) phase plane.

The system considered in what follows is

$$\begin{aligned} {}_0^cD_t^q x &= 1 - xy^\gamma, \\ {}_0^cD_t^q y &= \alpha y \left(xy^{\gamma-1} - 1\right), \end{aligned} \tag{3}$$

where ${}_0^cD_t^q$ is the Caputo fractional derivative with the fractional order q $(0 < q \leq 1)$. The main advantage of Caputo's approach is that the initial conditions for the fractional differential equations with Caputo derivatives take the similar form as for the integer-order differential equations. We therefore study the system (3) with the same initial conditions $x(0) > 0,\ y(0) > 0$ as in (1).

3 Non-Negativity, Existence and Uniqueness

Since x and y are the concentrations of ATP and ADP in living cells, respectively, we are only interested in solutions that are non-negative. To prove the non-negativity of our system, we shall use the following results.

Lemma 3.1 *[15] Suppose that $x(t) \in C[a, b]$ and $D_a^q x(t) \in C(a, b]$ with $0 < q \leq 1$. The Generalized Mean Value Theorem states that*

$$x(t) = x(a) + \frac{1}{\Gamma(q)}(D_a^q x)(\xi).(t-a)^q,$$

where $a \leq \xi \leq t,\ \forall t \in (a, b]$.

Corollary 3.1 *[15] Suppose that $x(t) \in C[a, b]$ and ${}_{t_0}^cD_a^q x(t) \in C(a, b]$ with $0 < q \leq 1$. Then the following conditions hold:*

1. *If ${}_{t_0}^cD_a^q x(t) \geq 0,\ \forall t \in (a, b)$, then $x(t)$ is a non-decreasing function for each $t \in [a, b]$.*
2. *If ${}_{t_0}^cD_a^q x(t) \leq 0,\ \forall t \in (a, b)$, then $x(t)$ is a non-increasing function for each $t \in [a, b]$.*

We can now state the following theorem.

Theorem 3.1 *All solutions of system (3) with the initial conditions $x(0) \geq 0$ and $y(0) \geq 0$ are non-negative.*

Proof. We will proof that $x(t) \geq 0$ for all $t \geq 0$. Suppose that the statement is not true, then there is a constant $t_1 > 0$ such that

$$\begin{cases} x(t) > 0, \quad 0 \leq t < t_1, \\ x(t_1) = 0, \\ x(t_1^+) < 0. \end{cases} \tag{4}$$

Substituting the second equation of system (4) into the first equation of (3) gives

$${}_0^c D_t^q x(t)|_{t=t_1} = 1. \tag{5}$$

According to Corollary 3.1 we have $x(t_1^+) \geq 0$, which contradicts the fact that $x(t_1^+) < 0$. Therefore, we have $x(t) \geq 0, \ \forall t \geq 0$. Using the same argument we show that $y(t) \geq 0, \ \forall t \geq 0$. □

Now, we study the existence and uniqueness of the solution of our system (3).We have the following lemma due to [16].

Lemma 3.2 *Consider the system*

$${}_{t_0}^c D_t^q x(t) = f(t, x), \quad t > t_0,$$

with the initial condition x_{t_0}, where $0 \leq q \leq 1$, $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$. If $f(t, x)$ satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of the above system on $[t_0, \infty) \times \Omega$.

We also need to use the following technical lemma.

Lemma 3.3 *Let x and y be two positive real numbers and $\gamma \geq 1$, then*

$$|x^\gamma - y^\gamma| \leq \gamma(\sup(x, y))^{\gamma-1}|x - y|.$$

Theorem 3.2 *We consider system (3) with the initial conditions $x(0) = x_0$ and $y(0) = y_0$ in the region $[0, \infty) \times \Omega$, where $\Omega = \{(x, y) \in \mathbb{R}^2 / \max\{|x|, |y|\} \leq M\}$ for sufficiently large M. This initial value problem has a unique solution.*

Proof. Let us define the functions $f_1(x, y) = 1 - xy^\gamma$ and $f_2(x, y) = \alpha y(xy^{\gamma-1} - 1)$. Further, we define $F = (f_1, f_2)^T$ and $X = (x, y)^T$, then the differential equation can be written as ${}_0^c D_t^q X = F(X)$.
We show that the function $F : \Omega \to \mathbb{R}^2$ is locally Lipschitz. Denote $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$. Then for any $X_1, \ X_2 \in \Omega$ we have

$$\begin{aligned} \|F(X_1) - F(X_2)\| &= |f_1(x_1, y_1) - f_1(x_2, y_2)| + |f_2(x_1, y_1) - f_2(x_2, y_2)| \\ &= |x_1 y_1^\gamma - x_2 y_2^\gamma| + \alpha |x_1 y_1^\gamma - x_2 y_2^\gamma - y_1 + y_2| \\ &\leq (1 + \alpha)|x_1 y_1^\gamma - x_2 y_2^\gamma| + \alpha |y_1 - y_2|, \end{aligned}$$

noticing that

$$|x_1 y_1^\gamma - x_2 y_2^\gamma| = \frac{1}{2}\left[(x_1 - x_2)(y_1^\gamma + y_2^\gamma) + (x_1 + x_2)(y_1^\gamma - y_2^\gamma)\right]$$

and using the triangle inequality and Lemma 3.3 we have

$$
\begin{aligned}
\|F(X_1) - F(X_2)\| \leq& \\
\leq& \ \frac{1+\alpha}{2}\left[|x_1 - x_2||y_1^\gamma + y_2^\gamma| + |x_1 + x_2||y_1^\gamma - y_2^\gamma|\right] + \alpha|y_1 - y_2| \\
\leq& \ \frac{1+\alpha}{2}\left[|x_1 - x_2||y_1^\gamma + y_2^\gamma| + \gamma(\sup(y_1, y_2))^{\gamma-1}|x_1 + x_2||y_1 - y_2|\right] + \alpha|y_1 - y_2| \\
\leq& \ (1+\alpha)M^\gamma|x_1 - x_2| + ((1+\alpha)\gamma M^\gamma + \alpha)|y_1 - y_2| \\
\leq& \ L\|X_1 - X_2\|,
\end{aligned}
$$

where $L = (1+\alpha)\gamma M^\gamma + \alpha$. Thus $F(X)$ satisfies the Lipschitz condition with respect to X, following Lemma 3.2 , there exists a unique solution $X(t)$ of system (3) with the initial condition $X(0) = (x(0), y(0))$. □

4 Local Stability and Hopf Bifurcation

The system (3) has only one equilibrium state E with coordinates $\bar{x} = \bar{y} = 1$. Linearising the system about this point leads to the Jacobian matrix

$$
J = \begin{bmatrix} -1 & -\gamma \\ \alpha & \alpha(\gamma - 1) \end{bmatrix},
$$

the characteristic equation is then

$$
\begin{vmatrix} -1-\lambda & -\gamma \\ \alpha & \alpha(\gamma-1) - \lambda \end{vmatrix} = 0,
$$

its roots are

$$
\lambda_{1,2} = \frac{1}{2}\{\alpha(\gamma-1) - 1 \pm \sqrt{[\alpha(\gamma-1)-1]^2 - 4\alpha}\}.
$$

The system (3) is said to be locally asymptotically stable [17] around the equilibrium point $(\bar{x}, \bar{y})$ if

$$
|\arg(\lambda_i)| > \frac{q\pi}{2}, \quad i = 1, 2.
$$

Hence it follows that

- If $0 < \alpha \leq \alpha_1$: $\lambda_{1,2}$ are real negative eigenvalues, E is asymptotically stable for all $q \in (0, 1]$.
- If $\alpha_1 < \alpha < \alpha_2$: $\lambda_{1,2}$ are complex conjugate eigenvalues, E is asymptotically stable if and only if $|\arg(\lambda_{1,2})| > \frac{q\pi}{2}$, that is, $0 < q < q^*$, where

$$
q^* = \frac{2}{\pi}\left|\arctan\sqrt{\frac{4\alpha}{[\alpha(\gamma-1)-1]^2} - 1}\right|.
$$

- If $\alpha \geq \alpha_2$ $\lambda_{1,2}$ are real positive eigenvalues, E is unstable for all $q \in (0, 1]$,

where α_0, α_1 and α_2 are defined as in (2).

Stability region of the fractional-order Selkov model for $\gamma = 2$ in the $(\alpha - q)$ plane is shown in Fig. 1. The stable and unstable regions are separated by the curve of equation

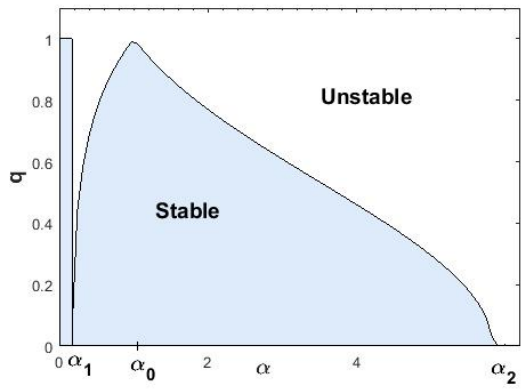


Figure 1: Stability region of the fractional-order Selkov model for $\gamma = 2$.

$q(\alpha) = \frac{2}{\pi}\left|\arctan\sqrt{\frac{4\alpha}{[\alpha(\gamma-1)-1]^2}-1}\right|$. A Hopf bifurcation occurs when the system has a pair of complex conjugate eigenvalues of the Jacobian matrix at an equilibrium point and when the stability of the equilibrium point changes from stable to unstable as the bifurcation parameter crosses a critical value. From the results of local stability, we observe that the order of derivatives has an effect on the stability of model (3). Thus, we can choose the order to be the bifurcation parameter. There are some studies that considered the existence of the Hopf bifurcations in fractional-order systems [18–20]. In this study, we use the conditions for the existence of a Hopf bifurcation which were introduced by Xiang Li and Ranchao Wu [20].

Theorem 4.1 *[20] When the bifurcation parameter q passes through the critical value $q^* \in (0,1)$, fractional-order system (3) undergoes a Hopf bifurcation at the equilibrium point if the following conditions hold:*

(a) the Jacobian matrix of the system (3) at the equilibrium point has a pair of complex conjugate eigenvalues $\lambda_{1,2}$, where $\mathcal{R}e(\lambda_{1,2}) > 0$;

(b) $m(q^*) = 0$, where $m(q) = \frac{q\pi}{2} - \min_{1\le i\le 2} |\arg(\lambda_i)|$;

(c) $\left.\frac{dm(q)}{dq}\right|_{q=q^*} \neq 0.$

If $\alpha_0 < \alpha < \alpha_2$, then the Jacobian matrix of the system (3) at the equilibrium point has a pair of complex conjugate eigenvalues $\lambda_{1,2}$, where $\mathcal{R}e(\lambda_{1,2}) > 0$. Hence, condition (a) in Theorem 4.1 holds. Moreover, when the bifurcation parameter takes the critical value $q = q^*$, we obtain $m(q^*) = 0$ which is the condition (b) of the theorem.

Finally, from the definition of $m(q)$, we have $\left.\frac{dm(q)}{dq}\right|_{q=q^*} = \frac{\pi}{2} \neq 0$. This implies that condition (c) holds. Therefore, from Theorem 4.1 model (3) undergoes a Hopf bifurcation at the equilibrium point $E = (1,1)$ when the bifurcation parameter passes through the critical value q^*, where $\alpha_0 < \alpha < \alpha_2$.

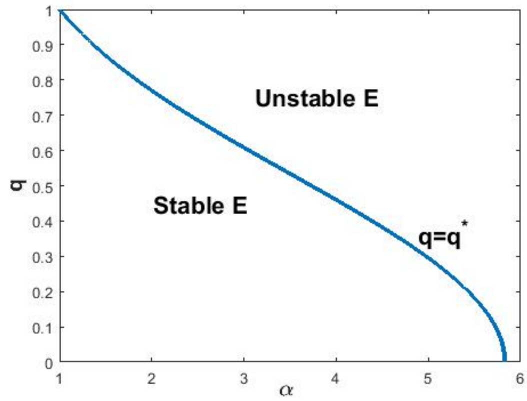


Figure 2: The Hopf bifurcation curve $q = q^*$ separates the stability region of E into the stable and unstable regions for $\gamma = 2$ and $\alpha_0 < \alpha < \alpha_2$.

Remark 4.1 An important difference between the Hopf bifurcation in the integer- and the fractional-order system is that in the integer-order system the limit cycle can be a solution for this system, but in the fractional-order system, the limit cycle can not be a solution of the system and the trajectories approach a limit cycle. In [21], the authors proved that there are no periodic solutions in fractional-order systems. Thus, there is no self-sustained solutions in the fractional-order Selkov model.

5 Discretized Fractional-Order Selkov Model and Its Analysis

In this section, we investigate some dynamical behavior of the discretized fractional-order Selkov system (3). The discretization process can be done in the following manner [22,23]. Assume that $x(0) = x_0$ and $y(0) = y_0$ are the initial conditions of system (3). The discretization with a piecewise constant argument is given as

$$\begin{aligned} &{}_0^cD_t^q x = 1 - x([t/s]s)y^\gamma([t/s]s), \\ &{}_0^cD_t^q y = \alpha y([t/s]s)\left(x([t/s]s)y^{\gamma-1}([t/s]s) - 1\right). \end{aligned} \tag{6}$$

We suppose that $t \in [0, s)$, hence $t/s \in [0, 1)$. So, we have

$$\begin{aligned} &{}_0^cD_t^q x = 1 - x_0 y_0^\gamma, \\ &{}_0^cD_t^q y = \alpha y_0\left(x_0 y_0^{\gamma-1} - 1\right), \end{aligned} \tag{7}$$

whose solution becomes

$$\begin{aligned} x_1(t) &= x_0 + I_0^q\left(1 - x_0 y_0^\gamma\right), \\ &= x_0 + \frac{t^q}{q\Gamma(q)}\left(1 - x_0 y_0^\gamma\right), \\ y_1(t) &= y_0 + I_0^q\left(\alpha y_0\left(x_0 y_0^{\gamma-1} - 1\right)\right), \\ &= y_0 + \frac{t^q}{q\Gamma(q)}\left(\alpha y_0\left(x_0 y_0^{\gamma-1} - 1\right)\right), \end{aligned} \tag{8}$$

where $I_0^q = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-\tau)^{q-1} d\tau, \quad q > 0.$

In the second step, we assume $t \in [s, 2s)$ so that $t/s \in [1, 2)$ and obtain

$$\begin{aligned} {}_0^c D_t^q x &= 1 - x(s) y^{\gamma}(s), \\ {}_0^c D_t^q y &= \alpha y(s) \left(x(s) y^{\gamma-1}(s) - 1 \right). \end{aligned} \tag{9}$$

The solution of this equation is

$$\begin{aligned} x_2(t) \quad &= x_1(s) + I_s^q \left(1 - x_1(s) y_1^{\gamma}(s)\right), \\ &= x_1(s) + \frac{(t-s)^q}{q\Gamma(q)} \left(1 - x_1(s) y_1^{\gamma}(s)\right), \\ y_2(t) \quad &= y_1(s) + I_s^q \left(\alpha y_1(s) \left(x_1(s) y_1^{\gamma-1}(s) - 1\right)\right), \\ &= y_1(s) + \frac{(t-s)^q}{q\Gamma(q)} \left(\alpha y_1(s) \left(x_1(s) y_1^{\gamma-1}(s) - 1\right)\right), \end{aligned} \tag{10}$$

where $I_s^q = \dfrac{1}{\Gamma(q)} \displaystyle\int_s^t (t-\tau)^{q-1} d\tau, \quad q > 0.$

Repeating the discretization process n times, we have

$$\begin{aligned} x_{n+1}(t) \quad &= x_n(ns) + \frac{(t-ns)^q}{q\Gamma(q)} \left(1 - x_n(ns) y_n^{\gamma}(ns)\right), \\ y_{n+1}(t) \quad &= y_n(ns) + \frac{(t-ns)^q}{q\Gamma(q)} \\ &\quad \left(\alpha y_n(ns) \left(x_n(ns) y_n^{\gamma-1}(ns) - 1\right)\right), \end{aligned} \tag{11}$$

where $t \in [ns,\ (n+1)s)$. Making $t \to (n+1)s$, we obtain the corresponding fractional discrete model of the continuous fractional model (3) as

$$\begin{aligned} x_{n+1} \quad &= x_n + \frac{s^q}{q\Gamma(q)} \left(1 - x_n y_n^{\gamma}\right), \\ y_{n+1} \quad &= y_n + \frac{s^q}{q\Gamma(q)} \left(\alpha y_n \left(x_n y_n^{\gamma-1} - 1\right)\right). \end{aligned} \tag{12}$$

It should be noticed that if the fractional parameter q converges to one in equation (12), we have the forward Euler discretization of system (3).

5.1 Stability of the fixed point of the discretized system

In this subsection, we study the asymptotic stability of the fixed points of the model (12). By solving the equations $x_{n+1} = x_n = x$ and $y_{n+1} = y_n = y$, we can easily find that (12) has the same unique fixed point as in the fractional-order system (3) given by $E = (1, 1)$ for every parameter value α and step size s. To investigate the stability of the fixed point of model (12), we need the following lemma [24].

Lemma 5.1 *Let $F(\lambda) = \lambda^2 - Tr\lambda + Det$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the two roots of $F(\lambda)$. Then*

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Det < 1$.

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$.

(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $Det > 1$.

(vi) $\lambda_1 = -1$ *and* $\lambda_2 \neq 1$ *if and only if* $F(-1) = 0$ *and* $Tr \neq 0, 2$.

(v) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2|$ *if and only if* $Tr^2 - 4Det < 0$ *and* $Det = 1$.

To distinguish between the different topological types for the fixed point E, we need the following lemma [25].

Lemma 5.2 *(i) A fixed point* E *is called a sink if* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$*, so the sink is locally asymptotically stable.*

(ii) A fixed point E *is called a source if* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$*, so the source is locally unstable.*

(iii) A fixed point E *is called a saddle if* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*), the saddle is locally unstable.*

(iv) A fixed point E *is called non-hyperbolic if either* $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$.

Theorem 5.1 *For all* $0 < q \leq 1$ *we have the following results:*

- *If* $0 < \alpha \leq \alpha_1$*, then the fixed point* E *is a sink if* $0 < s < s_1$*, a saddle point if* $s_1 < s < s_2$*, and a source if* $s \geq s_2$.
- *If* $\alpha_1 < \alpha < \alpha_0$*, then the fixed point* E *is a sink if* $0 < s < s_0$*, non-hyperbolic if* $s = s_0$*, and a source if* $s > s_0$.
- *If* $\alpha \geq \alpha_0$*, then the fixed point* E *is a source for all* $s > 0$*,*

where

$$s_0 = \left[\frac{q\Gamma(q)}{\alpha}(1 - \alpha(\gamma - 1))\right]^{1/q}$$

$$s_{1,2} = \left[\frac{q\Gamma(q)}{\alpha}\left((1 - \alpha(\gamma - 1)) \pm \sqrt{(\alpha(\gamma - 1) - 1)^2 - 4\alpha}\right)\right]^{1/q}, \ s_1 < s_2.$$

Proof. The Jacobian matrix of system (12) at the fixed point E is

$$J^* = \begin{bmatrix} 1 - \dfrac{s^q}{q\Gamma(q)} & -\gamma\dfrac{s^q}{q\Gamma(q)} \\ \alpha\dfrac{s^q}{q\Gamma(q)} & 1 + \alpha(\gamma - 1)\dfrac{s^q}{q\Gamma(q)} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix can be written as

$$\lambda^2 - Tr\lambda + Det = 0,$$

where Tr is the trace and Det is the determinant of the Jacobian matrix J^*, they are given as

$$Tr = 2 + \frac{s^q}{q\Gamma(q)}\left(\alpha(\gamma - 1) - 1\right),$$

$$Det = 1 + \frac{s^q}{q\Gamma(q)}\left(\alpha(\gamma - 1) - 1\right) + \alpha\left(\frac{s^q}{q\Gamma(q)}\right)^2.$$

<table>
<tr><td>α</td><td colspan="4">0 $\qquad\qquad\qquad\qquad$ α_1 $\qquad\qquad\qquad$ α_0 $\qquad\qquad$ α_2</td></tr>
<tr><td>Selkov model</td><td>Stable Node</td><td>Stable Focus</td><td>Unstable Focus</td><td>Unstable Node</td></tr>
<tr><td>FO Selkov model</td><td>LA Stable $\forall q \in (0, 1]$</td><td colspan="2">LA Stable iff $0 < q < q^*$</td><td>Unstable $\forall q \in (0, 1]$</td></tr>
<tr><td>Discritized FO Selkov model</td><td>a Sink for $0 < s < s_1$
a Saddle for $s_1 < s < s_2$
a Source for $s \geq s_2$
$\forall q \in (0, 1]$</td><td>a Sink for $s < s_0$
a Source for $s > s_0$
$\forall q \in (0, 1]$</td><td colspan="2">a Source $\forall s > 0$
$\forall q \in (0, 1]$</td></tr>
</table>

Table 1: Comparison of dynamical behaviors of the Selkov model (1) with its corresponding fractional-order system (3) and discretized fractional-order system (12).

Hence, the eigenvalues are

$$\lambda_{1,2} = 1 + \frac{s^q}{2q\Gamma(q)} \left(\alpha(\gamma - 1) - 1 \pm \sqrt{(\alpha(\gamma - 1) - 1)^2 - 4\alpha} \right).$$

We have

$$F(1) = \alpha \left(\frac{s^q}{q\Gamma(q)} \right)^2 > 0, \quad \text{for all } \alpha > 0,\ s > 0,$$

and

$$F(-1) = 4 + 2\frac{s^q}{q\Gamma(q)} \left(\alpha(\gamma - 1) - 1 \right) + \alpha \left(\frac{s^q}{q\Gamma(q)} \right)^2.$$

Applying Lemma 5.1 with some algebraic manipulations, one can obtain the results. □ A comparison table on dynamical behaviors of the Selkov model (1) with its corresponding fractional-order system (3) and discretized fractional-order system (12) has been given in Table 1.

5.2 Bifurcation analysis

The necessary and sufficient conditions ensuring that $|\lambda_1| < 1$ and $|\lambda_2| < 2$ are

$$F(1) > 0, \quad F(-1) > 0 \text{ and } Det > 1.$$

The violation of one of these conditions, with the other two being simultaneously fulfilled leads to:

(i) a fold or transcritical bifurcation (a real eigenvalue that passes through +1). This local bifurcation leads to the stability switching between two different steady states;

(ii) a flip bifurcation (a real eigenvalue that passes through -1). This local bifurcation entails the birth of a period 2-cycle;

(iii) a Neimark-Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1). This local bifurcation implies the birth of an invariant curve in the phase plane. The Neimark-Sacker bifurcation is considered to be equivalent to the Hopf bifurcation in continuous time and is indeed the major instrument to prove the existence of quasi-periodic orbits for the map.

In [25], the author presents a complete study of the main types of bifurcations for two-dimensional maps.

Now, we study the bifurcation types of the fixed point E.

Theorem 5.2 *The fixed point E loses its stability*

(i) via a flip bifurcation when $0 < \alpha \leq \alpha_1$ and $s = s_1$.

(ii) via a Neimark-Sacker bifurcation when $\alpha_1 < \alpha < \alpha_0$ and $s = s_0$.

Proof.

(i) When $0 < \alpha \leq \alpha_1$, the Jacobian matrix J^* has a pair of real eigenvalues. At $s = s_1$, we have $\lambda_1 \neq 0$ and $\lambda_2 = -1$, thus the model (12) undergoes a flip bifurcation which entails the birth of a period 2-cycle.

(ii) When $\alpha_1 < \alpha < \alpha_0$, the Jacobian matrix J^* has a pair of complex conjugate eigenvalues. At $s = s_0$, the modulus of $\lambda_{1,2}$ is equal to 1, thus the model (12) undergoes a Neimark-Sacker bifurcation. □

6 Numerical Simulation

In this section, we perform numerical simulation to confirm the above theoretical analysis and to illustrate the dynamics of both the fractional-order Selkov model and its discretization.

There are many numerical methods for solving nonlinear fractional differential equations such as the Adomian decomposition method [26], variational iteration method [27] and homotopy perturbation method [28]. In this study, we use the Adams-type predictor-corrector method [29] for the numerical solution of system (3). This method is a very effective tool to give numerical solutions of fractional-order differential equations.

Without loss of generality, we fix $\gamma = 2$, which gives $\alpha_0 = 1$, $\alpha_1 = 0.17157$ and $\alpha_2 = 5.8284$, we choose $\alpha = 1.2$ (in the interval (α_1, α_2)) and we vary the order q, the step size is considered as 0.01 and the initial point $x(0) = 1.5$, $y(0) = 1.3$. Note that for $\alpha = 1.2$, there exists a critical value $q = q^* = 0.9418$ below which the equilibrium point E is asymptotically stable and above which unstable. The stable behavior of the system is presented in Fig.5(a)-(b) for $q = 0.92$ (< 0.9418) and the unstable behavior of the system for $q = 0.98$ (> 0.9418) is presented in Fig.5(c)-(d). A Hopf bifurcation occurs at $q = q^*$.

To illustrate the corresponding discrete system (12) of the fractional-order Selkov system (3), we consider $q = 0.85$. Stability of the fixed point depends on the step size s. First, we take $\alpha = 0.1$ (in the interval $(0, \alpha_1)$), then the step size s should be less than $s_1 = 2.8775$ for E to be stable and otherwise unstable. The fixed point undergoes

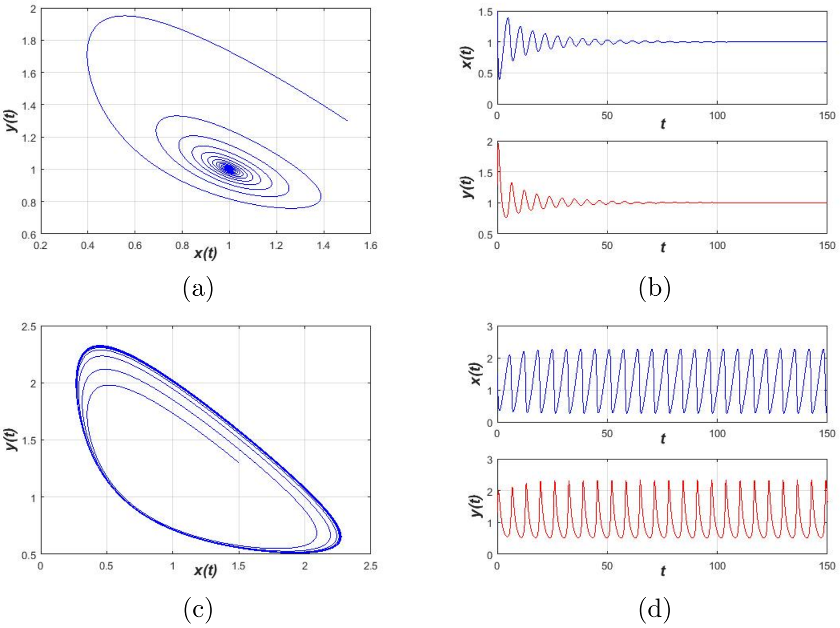
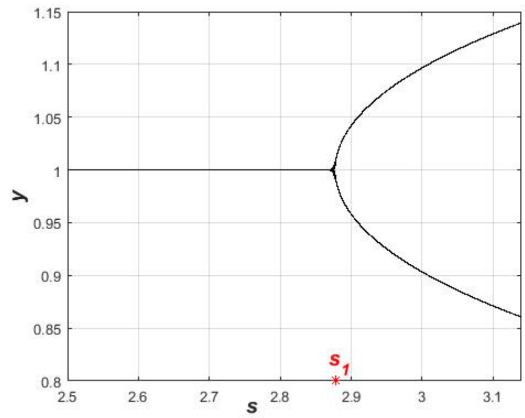


Figure 3: Phase portraits and the time series fractional-order Selkov model for (a),(b) $q = 0.92$, (c), (d) $q = 0.98$.

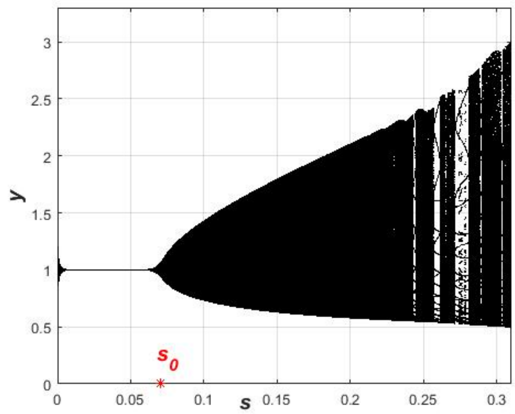
a flip bifurcation (Fig.4(a)) at the critical step size $s = s_1$. Now, we take $\alpha = 0.9$ (in the interval (α_1, α_0)), the Jacobian matrix J^* has a pair of complex conjugate eigenvalues $\lambda_{1,2} = 0.99409+0.11203i$, where $|\lambda_{1,2}| = 1$, for the value of step size $s = s_0 = 0.07597$ this implies that the system (12) undergoes a Neimark-Saker bifurcation at the fixed point E. The bifurcation diagram (Fig.4(b)) represents it succinctly. The maximal Lyapunov exponents corresponding to Fig.4(b) are computed and plotted in Fig.4(c), from which we deduce that chaotic behaviors may arise when the step size s is increasing. Some phase portraits are displayed in Fig.5. It clearly depicts the process of how a smooth invariant closed curve (Fig.5(b)) bifurcates from the stable fixed point E (Fig.5(a)). When s exceeds $s_0 = 0.07597$ (for example, $s = 0.08$), a simple closed curve enclosing the fixed point E appears. When s increases, the simple closed curve disappears. The strange attractor of system (12) for $s = 0.305$ is shown in Fig.5(f).

7 Conclusion

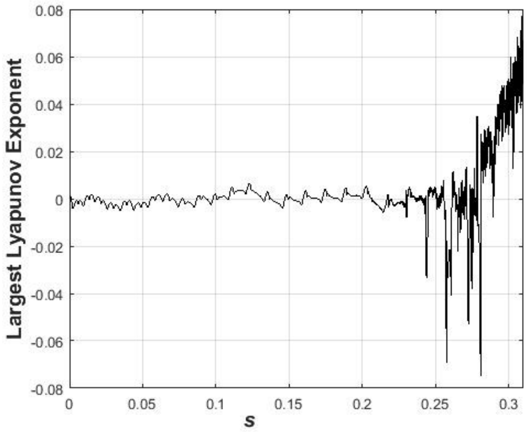
In this work, we have introduced a fractional-order Selkov model and its discretized counterpart. For the fractional-order system we have proved some mathematical results such as non-negativity, existence and uniqueness of the solution. We have also studied the local stability of the equilibrium point and proved the existence of a Hopf bifurcation with respect to the fractional order. Discretization of the fractional-order system was done with piecewise constant arguments and the corresponding dynamics was explored. It is observed that the dynamics of the discrete system depends on both the step size and the fractional order. Existence of the Neimark-Saker and flip bifurcations has been shown both analytically and numerically. It is also observed that the discrete fractional-order



(a)

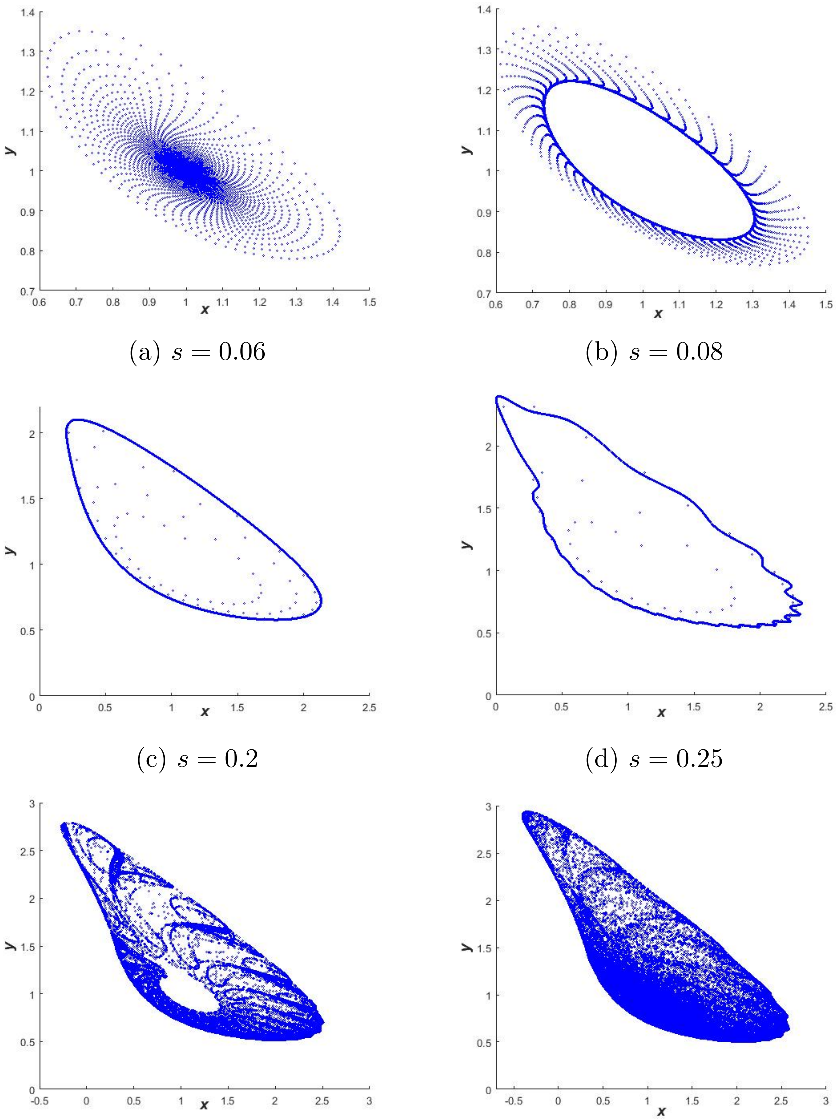


(b)



(c)

Figure 4: (a) Bifurcation diagram of the discrete fractional-order Selkov system (12) for $q = 0.85$ and $\alpha = 0.1$, a flip bifurcation occurs at $s = s_1 = 2.8775$; (b),(c) Bifurcation diagram and the corresponding maximal Lyapunov exponent of system (12) for $q = 0.85$ and $\alpha = 0.9$, a Neimark-Saker bifurcation occurs at $s = s_0 = 0.070591$.



(a) $s = 0.06$ (b) $s = 0.08$

(c) $s = 0.2$ (d) $s = 0.25$

(e) $s = 0.29$ (f) $s = 0.305$

Figure 5: Phase portraits of the discretized fractional-order Selkov model for some values of the step size s.

system shows more complex dynamics as the step size becomes larger. Our simulation results revealed that the discrete system exhibits chaotic dynamics for a larger step size.

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