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Linear Chaos Control of Fractional Generalized Hénon Map

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Abstract: This paper is concerned with the topic of chaos control in fractional maps. It presents two linear control laws to stabilize the dynamics of a new three-dimensional fractional Hénon map. The chaos control has been achieved by proving a new theorem, based on a suitable Lyapunov function and a linear method. Finally, numerical simulations have been carried out to highlight the effectiveness of the proposed control method.

Keywords: *discrete fractional calculus; fractional generalized Hénon map; linear control; Lyapunov method.*

Mathematics Subject Classification (2010): 34H10, 34H15.

1 Introduction

Recently, researchers have diverted their attention to the discrete-time case of fractional calculus and attempted to put together a complete theoretical framework for the subject [1]. Perhaps one of the earliest works is that of Diaz and Olser [2]. Successively, several types of discrete operators have been proposed, including some fractional *h-difference* operators, which represent further generalizations of the fractional difference operators [3–5]. Furthermore, numerical formulas and stability conditions corresponding to fractional difference systems can be found in [6, 7]. Most recently, some advances have been made in the applications of discrete fractional calculus [8]. The introduction of different discrete fractional operators has led to the publication of several papers regarding the chaotic behaviors of fractional nonlinear maps [9–17].

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Dynamics and control of fractional-order chaotic systems have received considerable attention over the last few years [18,19]. So far, nonlinear control laws have been mainly used for controlling at zero the chaotic dynamics of new three-dimensional fractional maps. Some interesting results have been recently published regarding this challenging topic [20–24]. For example, in [20], nonlinear control methods for some three-dimensional fractional chaotic maps (i.e., the Stefanski map, the Rössler map and the Wang map) have been studied. In [21], a novel control law for stabilizing a new three-dimensional fractional Hénon map has been proposed. In [22], the fractional-order Grassi–Miller chaotic map has been stabilized via nonlinear controllers. In [23], a control scheme to control hidden chaotic attractors in a new fractional map has been illustrated. In [24], the chaotic behavior of a new three-dimensional fractional map with no equilibrium has been studied along with a control method that exploits the stability properties of linear fractional discrete systems.

It is worth noting that all the control methods developed so far for fractional chaotic maps have exploited *nonlinear* control laws. This work aims to provide a contribution to the topic by presenting a very simple *linear* control law to control chaotic dynamics of the well-known fractional generalized Hénon map. This map is defined via the Caputo h-difference operator. The asymptotic convergence of the states is established using the Lyapunov method. The paper is organized as follows. In Section 2, some basic notions of the Caputo h-difference operator and discrete fractional calculus are introduced. In Section 3, a novel control result is proved which enables the dynamics of the three dimensional fractional Hénon map to be controlled by a *two-dimensional linear* control. Finally, simulation results are reported through the paper, with the aim to show the effectiveness of the proposed approach.

2 Basic Tools

In this section, some basic concepts related to the Caputo h-difference operator are briefly summarized.

Definition 2.1 [4] Let $X : (h\mathbb{N})_a \to \mathbb{R}$ and $0 < \nu$ be given. a is a starting point. The ν–th order h–sum is given by

$$_h\Delta_a^{-\nu} X(t) = \frac{h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\nu} (t-\sigma(sh))_h^{(\nu-1)} X(sh)\,, \sigma(sh) = (s+1)h, a \in \mathbb{R}, t \in (h\mathbb{N})_{a+\nu h}\,, \tag{1}$$

where the h–falling factorial function is defined as

$$t_h^{(\nu)} = h^\nu \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\nu\right)}, \quad t,\ \nu \in \mathbb{R},$$

where $(h\mathbb{N})_{a+(1-\nu)h} = \{a+(1-\nu)h, a+(2-\nu)h, ...\}$.

Definition 2.2 [5] For $X(t)$ defined on $(h\mathbb{N})_a$ a and $0 < \nu$, $\nu \notin \mathbb{N}$, the Caputo–like difference is defined by

$$_h^C\Delta_a^\nu X(t) = \Delta_a^{-(n-\nu)} \Delta^n X(t)\,, \quad t \in (h\mathbb{N})_{a+(n-\nu)h}\,, \tag{2}$$

where $\Delta X(t) = \frac{X(t+h)-X(t)}{h}$ and $n = \lceil \nu \rceil + 1$.

Now a theorem reported in [25] is briefly illustrated, with the aim to identify the stability conditions of the zero equilibrium point for the fractional nonlinear difference system written in the form

$$ {}_{h}^{C}\Delta_{a}^{\nu} X\left(t\right) = f\left(t+\nu h, X\left(t+\nu h\right)\right), \tag{3} $$

where $X\left(t\right) = \left(x_1\left(t\right), x_2\left(t\right), ..., x_n\left(t\right)\right)^T$, $t \in \left(h\mathbb{N}\right)_{a+(1-\nu)h}$ and f is a nonlinear function.

Theorem 2.1 *Let $x = 0$ be an equilibrium point of the nonlinear discrete fractional system (3). If there exists a positive definite and decrescent scalar function $V\left(t, X\left(t\right)\right)$ such that ${}_{h}^{C}\Delta_{a}^{\nu} V\left(t, X\left(t\right)\right) \leq 0, \;\; t \in \left(h\mathbb{N}\right)_{a+(1-\nu)h}$, then the equilibrium point is asymptotically stable.*

In the following, a useful inequality for Lyapunov functions is introduced.

Lemma 2.1 *[25] For any discrete time $t \in \left(h\mathbb{N}\right)_{a+(1-\nu)h}$, $0 < \nu \leq 1$, the following inequality holds*

$$ {}_{h}^{C}\Delta_{a}^{\nu}\left(X^T\left(t\right)X\left(t\right)\right) \leq 2X^T\left(t+\nu h\right){}_{h}^{C}\Delta_{a}^{\nu} X\left(t\right). \tag{4} $$

3 The Three-Dimensional Fractional Generalized Hénon Map

Recently, a new three-dimensional fractional Hénon map with Lorenz-like attractors has been proposed in [26]. This map is an example of a fractional discrete–time system with the ν Caputo–like operator that can display chaotic behavior. In the following, by adopting the Caputo h-difference operator we described the fractional generalized Hénon map as

$$ \begin{cases} {}_{h}^{C}\Delta_{a}^{\nu} x\left(t\right) = M_1 + Bz\left(t+\nu h\right) + M_2 y\left(t+\nu h\right) - x^2\left(t+\nu h\right) - x\left(t+\nu h\right), \\ {}_{h}^{C}\Delta_{a}^{\nu} y\left(t\right) = x\left(t+\nu h\right) - y\left(t+\nu h\right), \\ {}_{h}^{C}\Delta_{a}^{\nu} z\left(t\right) = y\left(t+\nu h\right) - z\left(t+\nu h\right), \end{cases} \tag{5} $$

where x, y, z are the states of the fractional map (5) and M_1, M_2 and B are parameter values, with $t \in \left(h\mathbb{N}\right)_{a+(1-\nu)h}$.

To study the properties of the fractional generalized Hénon map (5), the following discrete numerical solution is defined based on the h-fractional sum (1) as

$$ \begin{cases} x\left(n+1\right) = x_0 + \frac{h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}\left(M_1 + Bz\left(j+1\right) + M_2 y\left(j+1\right) - x^2\left(j+1\right)\right. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - x\left(j+1\right)\right), \\ y\left(n+1\right) = y_0 + \frac{h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x\left(j+1\right) - y\left(j+1\right)), \\ z\left(n+1\right) = z_0 + \frac{h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(y\left(j+1\right) - z\left(j+1\right)), \end{cases} \tag{6} $$

where x_0, y_0 and z_0 are initial states. According to the discrete equation (6), the fractional generalized Hénon map (5) has memory effects, which means that the implicit solution is determined by all the previous states with the state $x(n+1), y\left(n+1\right)$ and $z\left(n+1\right)$. Considering parameter values $M_1 = 1.4$, $M_2 = 0.2$ and varying B from 0 to 0.3, the resulting bifurcation diagram and the largest Lyapunov exponents are depicted in Figure 1 with fractional order $\nu = 0.98$. Different dynamic behaviors including chaos periodic windows are observed in the fractional generalized Hénon map (5). From this, it can be seen that the system has the positive largest Lyapunov exponent when B takes the smallest values, indicating that the system has indeed a chaotic attractor, as shown in Figure 1**(a)** for $B = 0.1$.

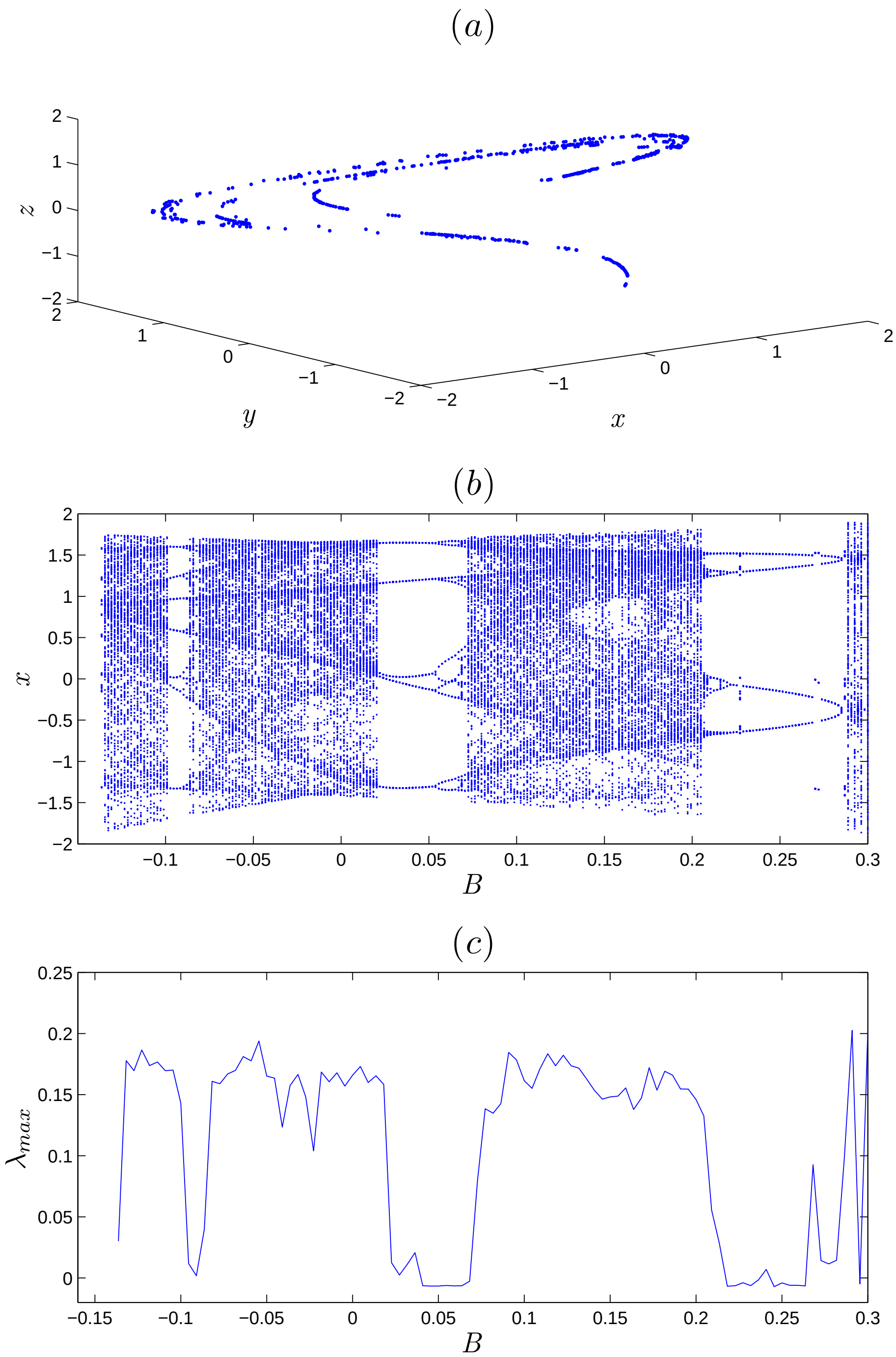

Figure 1: Numerical simulation of the fractional generalized Hénon map for the fractional order value $\nu = 0.98$. (a) The chaotic attractor for $M_1 = 1.4$, $B = 0.1$. (b) The bifurcation diagram versus B. (c) The corresponding largest Lyapunov exponents diagram.

4 Two-Dimensional Chaos Control Law

In this section, by exploiting a novel theorem based on a suitable Lyapunov function, a two-dimensional linear control law is illustrated, with the aim to control the chaotic dynamics of the three-dimensional fractional Hénon map. To obtain our results, the following theorem is presented.

Theorem 4.1 *The three-dimensional fractional Hénon chaotic map is controlled under the following linear two-dimensional control law:*

$$\left\{ \begin{array}{l} \mathbf{C}_1 = -lx\left(t\right) - \left(M_2+1\right)y\left(t\right) - Bz\left(t\right) - M_1, \\ \mathbf{C}_2 = -z\left(t\right), \end{array} \right. \tag{7}$$

where $|x\left(t\right)| \leq l$, *for* $t \in \left(h\mathbb{N}\right)_{a+(1-\nu)h}$.

Proof. The controlled fractional Hénon chaotic map involves the time–varying control law $\left(\mathbf{C}_1, \mathbf{C}_2\right)^T$ and is given by

$$\left\{ \begin{array}{l} {}_{h}^{C}\Delta_a^{\nu}x\left(t\right) = M_1 + Bz\left(t+\nu h\right) + M_2 y\left(t+\nu h\right) - x^2\left(t+\nu h\right) - x\left(t+\nu h\right) + \mathbf{C}_1\left(t+\nu h\right), \\ {}_{h}^{C}\Delta_a^{\nu}y\left(t\right) = x\left(t+\nu h\right) - y\left(t+\nu h\right) + \mathbf{C}_2\left(t+\nu h\right), \\ {}_{h}^{C}\Delta_a^{\nu}z\left(t\right) = y\left(t+\nu h\right) - z\left(t+\nu h\right). \end{array} \right. \tag{8}$$

Substituting the proposed control law (7) into (8) yields the simplified dynamics

$$\left\{ \begin{array}{l} {}_{h}^{C}\Delta_a^{\nu}x\left(t\right) = -y\left(t+\nu h\right) - x^2\left(t+\nu h\right) - \left(l+1\right)x\left(t+\nu h\right), \\ {}_{h}^{C}\Delta_a^{\nu}y\left(t\right) = x\left(t+\nu h\right) - y\left(t+\nu h\right) - z\left(t+\nu h\right), \\ {}_{h}^{C}\Delta_a^{\nu}z\left(t\right) = y\left(t+\nu h\right) - z\left(t+\nu h\right). \end{array} \right. \tag{9}$$

Now, by taking a Lyapunov function in the form

$$V = \frac{1}{2}\left(x^2(t) + y^2(t) + z^2\left(t\right)\right), \tag{10}$$

it follows that

$${}_{h}^{C}\Delta_a^{\nu}V = \frac{1}{2}{}_{h}^{C}\Delta_a^{\nu}x^2(t) + \frac{1}{2}{}_{h}^{C}\Delta_a^{\nu}y^2(t) + \frac{1}{2}{}_{h}^{C}\Delta_a^{\nu}z^2(t), \tag{11}$$

and by exploiting Lemma 1, we get

$$\begin{array}{rcl} {}_{h}^{C}\Delta_a^{\nu}V & \leq & x(t+\nu h){}_{h}^{C}\Delta_a^{\nu}x(t) + y(t+\nu h){}_{h}^{C}\Delta_a^{\nu}y(t) + z(t+\nu h){}_{h}^{C}\Delta_a^{\nu}z(t) \\ & = & -x\left(t+\nu h\right)y\left(t+\nu h\right) - x^3\left(t+\nu h\right) - \left(l+1\right)x^2\left(t+\nu h\right) \\ & & +y\left(t+\nu h\right)x\left(t+\nu h\right) - y^2\left(t+\nu h\right) - y\left(t+\nu h\right)z\left(t+\nu h\right) \\ & & +z\left(t+\nu h\right)y\left(t+\nu h\right) - z^2\left(t+\nu h\right) \\ & = & -\left(l+1\right)x^2(t+\nu h) - y^2(t+\nu h) - z^2(t+\nu h) - x^3\left(t+\nu h\right) \\ & \leq & -\left(l+1\right)x^2(t+\nu h) - y^2(t+\nu h) - z^2(t+\nu h) + |x|\,x^2\left(t+\nu h\right) \\ & \leq & -\left(l+1\right)x^2(t+\nu h) - y^2(t+\nu h) - z^2(t+\nu h) + lx^2\left(t+\nu h\right) \\ & = & -x^2(t+\nu h) - y^2(t+\nu h) - z^2(t+\nu h) < 0. \end{array}$$

It can be concluded that the controlled states of the fractional Hénon chaotic map (5) are stabilized at the origin by the two-dimensional linear control law (7).

To verify the theoretical results obtained above, numerical simulations are preformed using Matlab. We start by employing h-fractional sum (1) to obtain the numerical formula of the controlled dynamical system (5). The parameter values are taken as $M_1 = 1.4$ and $M_2 = 0.2$, $B = 0.1$ to ensure the existence of chaos. Figures 2 and 3 show the states trajectories and the phase portrait, respectively, of the controlled fractional map (5) when the fractional order value is taken as $\nu = 0.98$. These plots clearly show that the chaotic dynamics of the fractional map (5) are controlled to equilibrium point $(0, 0, 0)$ by control law (7).

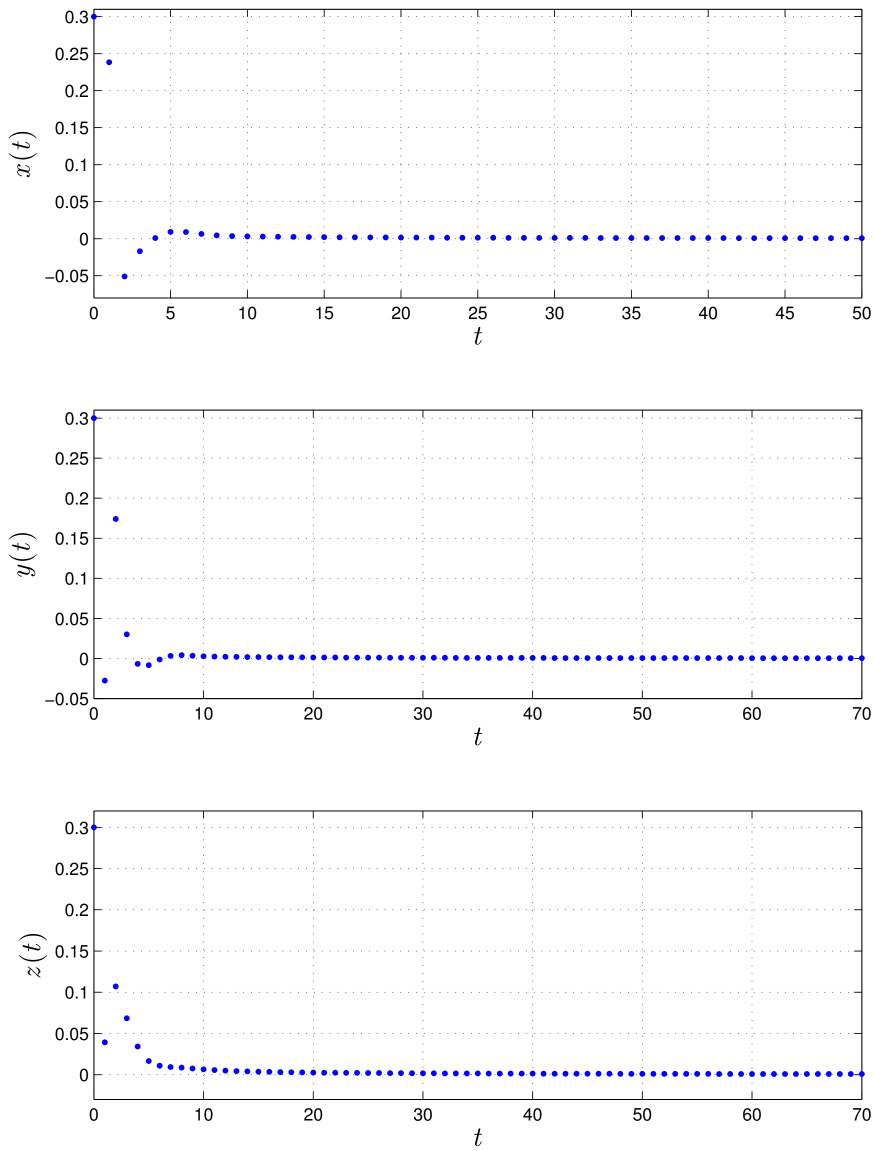

Figure 2: Phase portrait of the controlled fractional generalized Hénon map with $\nu = 0.98$.

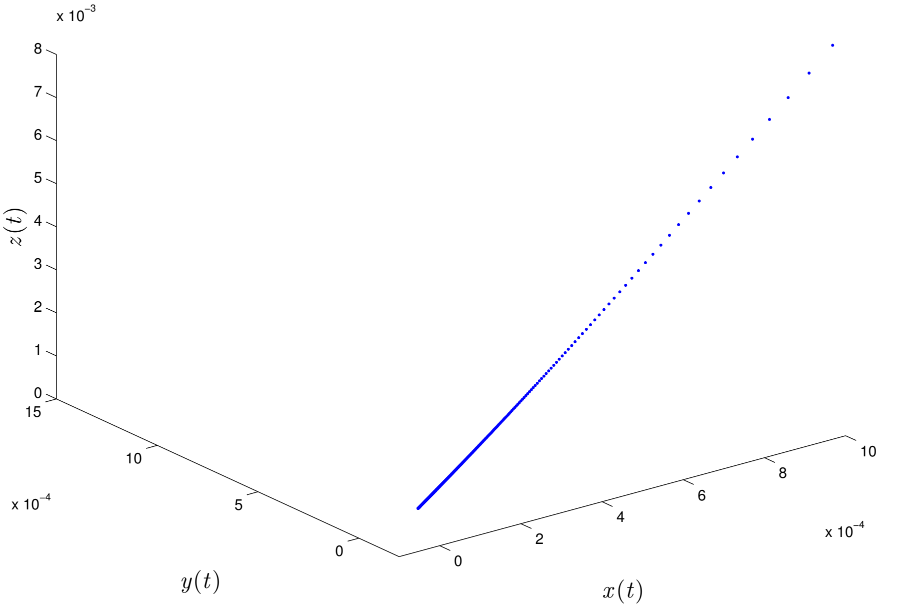

Figure 3: Evolution of the controlled fractional generalized Hénon map with $\nu = 0.98$.

5 Conclusion

Using linear control laws, this paper has studied the control of a new fractional chaotic map. Specifically, the three-dimensional fractional Hénon map has been controlled by a two-dimensional control law. All the results have been achieved by exploiting a new linear control law based on the Lyapunov method as well as on the properties of the Caputo h-difference operator. Note that, by virtue of the linearity of the control law proposed herein, the conceived method for controlling the chaotic dynamics requires less control effort with respect to the nonlinear techniques developed in literature to date. Finally, simulation results have been presented to highlight the effectiveness of the proposed approach.

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