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System Reliability of Ailamujia Model and Additive Failure Rate Models

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Abstract: Dynamic and non-dynamic reliability systems play an important role in industry, manufacturing, safety engineering and quality. The most commonly used models in the parametric statistical reliability analysis are the exponential, Weibull, inverted Weibull, lognormal, Lindley and Raleigh ones as well as their generalizations. In certain engineering applications such as the distribution of repair time and the distribution of delay time, it is found that the Ailamujia model is a suitable alternative compared to other models. This work considers system reliability analysis of the Ailamujia model, in which different reliability measures were computed. The combinations of additive failure rate models associated with the Ailamujia distribution were derived, they include the exponential, Weibull, Frechet and Raleigh distributions.

Keywords: *Ailamujia distribution; stress strength model; reliability; additive rate model.*

Mathematics Subject Classification (2010): 62N05, 68M15, 90B25.

1 Introduction

The lifetime of equipment or apparatus is a random time from the beginning of the operation until the appearance of a complete failure. Reliability is the ability of a system to perform its stated purpose adequately for a specified period of time under specified operational conditions. The system defined here could be an electronic or mechanical hardware product, a software product, a manufacturing process or even a service. For example, in case of a mechanical system, a failure is a breakdown of some of its parts or an increase in vibration above the permitted level. One of the most dangerous failures

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of a nuclear reactor is a leak of radioactive material. The reliability characteristics are usually expressed in terms of the lifetime.

Modeling and analyzing lifetime data are important issues in many disciplines including medicine, engineering, industry, quality control and finance, etc. Different lifetime data can be represented by several well-known continuous probability distributions such as exponential, Lindley, Weibull, lognormal, and Frechet as well as their generalizations. The Ailamujia distribution is a newly proposed lifetime model that has many engineering applications [1]. In some practical applications such as the distribution of repair time and the distribution of delay time, it is found that the Ailamujia model is a convenient one compared to other models. Lv *et. al.* [2] studied the different properties including mean, variance, and median and maximum likelihood estimators. This distribution has also been investigated for the interval estimation and the hypothesis [3]. The minimax estimation of the Ailamujia model parameter has been discussed under a non-informative prior using three loss functions [4].

The probability density function of the Ailamujia distribution is given by

$$f(x,\theta) = 4\theta^2 x e^{-2\theta x};\ x \geq 0,\ \theta > 0, \tag{1}$$

while the corresponding cumulative distribution function is given as

$$F(x,\theta) = 1 - (1 + 2\theta x)e^{-2\theta x};\ x \geq 0,\ \theta > 0, \tag{2}$$

where θ is the unknown parameter. It can be easily concluded that

$$E(X) = \frac{1}{\theta}\ \ and\ \ \sigma^2 = \frac{1}{2\theta}.$$

The maximum likelihood estimator for θ is given by

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i}. \tag{3}$$

The survival function and failure rate are, respectively, given by

$$r(x) = (1 + 2\theta x)e^{-2\theta x}, \tag{4}$$

$$h(x) = \frac{4\theta^2 x}{1 + 2\theta x}. \tag{5}$$

The reliability of the system is given by

$$R(t) = \exp\Big\{ - \int_0^t h(x)dx \Big\}. \tag{6}$$

Having in mind that

$$\int_0^t h(x)dx = \int_0^t \frac{4\theta^2 x}{1 + 2\theta x}dx = 2\theta t - \ln(2\theta t + 1), \tag{7}$$

the reliability can be expressed as

$$R(t) = e^{-\int_0^t \frac{4\theta^2 x}{1+2\theta x}dx} = (2\theta t + 1)e^{-2\theta t}. \tag{8}$$

The time to failure can be expressed as

$$F(t) = 1 - (1 + 2\theta x)e^{2\theta x}. \tag{9}$$

There is a wide application of the mean residual life function in reliability and survival analysis (see [5–8]). The mean residual life function for the Aijamujia distribution is given by

$$e(t) = \frac{\int_t^{\infty} R(x)dx}{R(t)} \tag{10}$$

$$= \frac{\int_t^{\infty} (1 + 2\theta x)e^{-2\theta x}dx}{R(t)} \tag{11}$$

$$= \frac{te^{-2\theta t} + \frac{(1+\theta t)}{\theta} e^{-2\theta t}}{(1 + 2\theta t)e^{-2\theta t}} = \frac{1 + \theta t}{\theta(1 + 2\theta t)}. \tag{12}$$

The following section explains the stress-strength model using the Ailamujia model, while the derivation of additive failure rate models is followed, where the Ailamujia failure rate model is combined with every one of the Ailamujia, exponential, Weibull, Frechet, and Raleigh distributions.

2 Stress-Strength Reliability

The stress-strength reliability describes the life of a component which has a random strength subjected to a random stress. When the stress applied to the component exceeds the strength, the component fails instantly and the component will not function satisfactorily. Therefore, there is a measure of component reliability known as a stress-strength parameter. The stress-strength reliability has wide applications in almost all areas, especially in engineering including structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc. Beg and Singh [9] gave estimation of $P(X > Y)$ for the Pareto distribution. Maroof and Islam [10] studied the Bayesian estimation of a system reliability when the stress and strength follow the Lomax distribution. Nandi and Aich [11] have shown that Reliability (R) can be obtained as the Laplace transform of the stress. Also, Kotz *et. al.* [1] investigated the generalization of the stress-strength model. Their main findings are summarized as follows.

Let X and Y be two non-negative and continuous random variables having densities $f(x)$ and $g(y)$, respectively. If X and Y are independent, then the probability that Y exceeds X is given as [1]

$$R = P(Y > X) = \int_0^{\infty} xf(x)\Big[\int_1^{\infty} g(vx)dv\Big]dx. \tag{13}$$

Theorem 2.1 *Let the random stress X and the random strength Y be two independent Ailamujia distributions with probability density functions given by*

$$f(x, \theta) = 4\theta_1^2 xe^{-2\theta_1 x}; x \geq 0, \theta_1 > 0,$$

$$f(y, \theta) = 4\theta_2^2 ye^{-2\theta_2 x}; y \geq 0, \theta_2 > 0,$$

then the system reliability, $R = P(Y > X)$, is

$$R = P(Y > X) = \frac{\theta_1^2(\theta_1 + 3\theta_2)}{(\theta_1 + \theta_2)}.$$

Proof: It is given that

$$R = P(Y > X) = \int_0^\infty xf(x)\Big[\int_1^\infty 4x\theta_2^2 ve^{-2\theta_2 xv}dv\Big]dx \tag{14}$$

$$= P(Y > X) = \int_0^\infty xf(x)[I]dx, \tag{15}$$

where

$$I = \int_1^\infty 4x\theta_2^2 ve^{-2\theta_2 xv}dv.$$

Then we evaluate the integral I:

$$I = 2\theta_2 \int_1^\infty 2\theta_2 xve^{-2\theta_2 xv}dv.$$

Integration by parts can be used which gives

$$I = 2\theta_2 \int_0^\infty 2\theta_2 xve^{-2\theta_2 xv}dx \tag{16}$$

$$= 2\theta_2\Big\{\Big[-ve^{-2\theta_2 xv}\Big]_1^\infty - \int_1^\infty e^{-2\theta_2 xv}dv\Big\} \tag{17}$$

$$= 2\theta_2\Big\{\Big[-ve^{-2\theta_2 xv}\Big]_1^\infty - \Big[\frac{1}{-2\theta_2 x}e^{-2\theta_2 xv}\Big]_1^\infty\Big\} \tag{18}$$

$$= 2\theta_2\Big\{e^{-2\theta_2 x} + \frac{1}{2\theta_2 x}e^{-2\theta_2 x}\Big\} \tag{19}$$

$$= \frac{1+2\theta_2 x}{x}e^{-2\theta_2 x}. \tag{20}$$

Substitute (20) into (15):

$$R = P(Y > X) = \int_0^\infty xf(x)\Big[\frac{(1+2\theta_2 x)e^{-2\theta_2 x}}{x}\Big]dx$$

$$= \int_0^\infty x(4\theta_1^2 xe^{-2\theta_1 x})\Big[\frac{(1+2\theta_2 x)e^{-2\theta_2 x}}{x}\Big]dx$$

$$= 4\theta_1^2 \int_0^\infty xe^{-2\theta_1 x}\Big((1+2\theta_2 x)e^{-2\theta_2 x}\Big)dx$$

$$= 4\theta_1^2\Big[\int_0^\infty xe^{-2(\theta_1+\theta_2)x}dx + 2\theta_2\int_0^\infty x^2e^{-2(\theta_1+\theta_2)x}dx\Big]$$

$$= 4\theta_1^2\Big[\frac{1}{2(\theta_1+\theta_2)}\int_0^\infty 2(\theta_1+\theta_2)xe^{-2(\theta_1+\theta_2)x}dx$$

$$+ \frac{2\theta_2}{2(\theta_1+\theta_2)}\int_0^\infty 2(\theta_1+\theta_2)x^2e^{-2(\theta_1+\theta_2)x}dx\Big]$$

$$= 4\theta_1^2\Big[\frac{1}{2(\theta_1+\theta_2)}\frac{1}{2(\theta_1+\theta_2)} + \frac{2\theta_2}{(\theta_1+\theta_2)}\frac{2}{4(\theta_1+\theta_2)^2}\Big]$$

$$= \theta_1^2\Big[\frac{1}{(\theta_1+\theta_2)^2} + \frac{2\theta_2}{(\theta_1+\theta_2)^3}\Big] = \frac{\theta_1^2(\theta_1+\theta_3)}{(\theta_1+\theta_2)^3}.$$

3 Additive Failure Rate Models

More attention is given to the reliability of a combination of two failure rate models for a system with two components that function independently. Assume X_1 and X_2 with respective failure densities, failure probabilities and failure rates being $f_1(x), f_2(x); F_1(x), F_2(x); h_1(x), h_2(x)$, then the system reliability is given by

$$R(t) = Exp\left\{ -\int_0^t [h_1(x) + h_2(x]dx \right\}.$$

It is then possible to obtain the failure density and the failure rate of the series system whose reliability is given by (1). Different options have been considered in the literature regarding $h_1(x)$ and $h_2(x)$ [12–15].

The following subsections describe derivation of the additive failure rate models as related to the Ailamujia distribution.

3.1 Ailamujia-Ailamujia failure rate model

The Ailamujia distribution with parameter θ_1 for $h_1(x)$ and the Ailamujia distribution with parameter θ_2 for $h_2(x)$ are selected.

$$\begin{aligned}\int_0^t h(x)dx &= \int_0^t [h_1(x) + h_2(x)]dx \\ &= \int_0^t \frac{4\theta_1^2 x}{1+2\theta_1 x}dx + \int_0^t \frac{4\theta_2^2 x}{1+2\theta_2 x}dx \\ &= 2\theta_1 t - \ln(2\theta_1 t + 1) + 2\theta_2 t - \ln(2\theta_2 t + 1) \\ &= 2(\theta_1 + \theta_2)t - \ln\frac{2\theta_1 t + 1}{2\theta_2 t + 1}.\end{aligned}$$

Then the reliability function of the system can be written as

$$R(t) = e^{-\left((2(\theta_1+\theta_2)t - \ln\frac{2\theta_1 t+1}{2\theta_2 t+1}\right)} = \frac{2\theta_1 t + 1}{2\theta_2 t + 1} e^{-2(\theta_1+\theta_2)t}$$

and the probability density of the Ailamujia-Ailamujia failure rate model **(AAFRM)** is given by

$$f(t) = -\frac{d}{dt}R(t) = 2(\theta_1+\theta_2)\frac{2\theta_1 t + 1}{2\theta_2 t + 1}e^{-2(\theta_1+\theta_2)t} - \frac{2\theta_1 + \theta_2}{(2\theta_2 t + 1)^2}e^{-2(\theta_1+\theta_2)t}.$$

3.2 Ailamujia-exponential failure rate model

The probability density, cumulative distribution, and hazard functions of the exponential distribution are respectively given by

$$\begin{aligned}f(x) &= \lambda e^{-\lambda x}\ ;\ x > 0,\ \lambda > 0, \\ F(x) &= 1 - e^{-\lambda x}\ ;\ x > 0,\ \lambda > 0, \\ R(x) &= e^{-\lambda x}\ and\ h(x) = \lambda.\end{aligned}$$

The Ailamujia distribution is selected with parameter θ for $h_1(x)$ and the exponential distribution for $h_2(x)$.

$$\begin{aligned}\int_0^t h(x)dx &= \int_0^t [h_1(x) + h_2(x)]dx \\ &= \int_0^1 \frac{4\theta^2 x}{1+2\theta x}dx + \int_0^t \lambda dx \\ &= (2\theta + \lambda)t + \ln(2\theta t + 1)\end{aligned}$$

and the reliability function of the system can be written as

$$R(t) = e^{-[(2\theta+\lambda)t+\ln 2\theta t+1]} = (2\theta t + 1)e^{-(2\theta t+\lambda)t}$$

and the probability density of the Ailamujia-Ailamujia failure rate model **(AAFRM)** is given by

$$f(t) = -\frac{d}{dt}R(t) = (4\theta^2 t + 2\theta\lambda t + \lambda)e^{-(2\theta t+\lambda)t}.$$

3.3 Ailamujia-Weibull failure rate model

The probability density, cumulative distribution, and hazard functions of the Weibull distribution are respectively given by

$$\begin{aligned}f(x) &= \lambda\alpha x^{\alpha-1}\ ;\ x > 0,\ \lambda > 0,\ \alpha > 0, \\ F(x) &= 1 - e^{-\lambda x^{\alpha}}\ ;\ x > 0,\ \lambda > 0,\ \alpha > 0, \\ R(x) &= e^{-\lambda x^{\alpha}}\ \textit{and}\ h(x) = \lambda\alpha x^{\alpha-1}.\end{aligned}$$

The Ailamujia distribution was selected for $h_1(x)$ and the Weibull distribution for $h_2(x)$.

$$\begin{aligned}\int_0^t h(x)dx &= \int_0^t [h_1(x) + h_2(x)]dx \\ &= \int_0^1 \frac{4\theta^2 x}{1+2\theta x}dx + \int_0^t \lambda\alpha x^{\alpha-1}dx \\ &= 2\theta t - \ln(2\theta t + 1) + \lambda t^{\alpha}.\end{aligned}$$

Then the reliability function of the system can be written as

$$R(t) = e^{-(2\theta t-\ln 2\theta t+1+\lambda t^{\alpha})} = (2\theta t + 1)e^{-(2\theta t+\lambda t^{\alpha})}$$

and the probability density of the Ailamujia-Weibull failure rate model **(AWFRM)** is given by

$$f(t) = -\frac{d}{dt}R(t) = (4\theta^2 t + 2\theta\lambda\alpha t^{\alpha} + \lambda\alpha t^{\alpha-1})e^{-(2\theta t+\lambda t^{\alpha})}.$$

3.4 Ailamujia-Frechet failure rate model

The probability density, cumulative distribution, and hazard functions of the Frechet distribution are respectively given by

$$\begin{aligned} f(x) &= \lambda\alpha x^{-(\alpha+1)}\ ;\ x>0,\ \lambda>0,\ \alpha>0,\\ F(x) &= 1-e^{-\lambda x^{-\alpha}}\ ;\ x>0,\ \lambda>0,\ \alpha>0,\\ R(x) &= e^{-\lambda x^{-\alpha}}\ \textit{and}\ h(x)=\lambda\alpha x^{-(\alpha+1)}. \end{aligned}$$

The Ailamujia distribution was selected for $h_1(x)$ and the inverted Weibull distribution for $h_2(x)$.

$$\begin{aligned} \int_0^t h(x)dx &= \int_0^t [h_1(x)+h_2(x)]dx\\ &= \int_0^1 \frac{4\theta^2 x}{1+2\theta x}dx + \int_0^t \lambda\alpha x^{-(\alpha+1)}dx\\ &= 2\theta t - \ln(2\theta t+1) - \lambda t^{-\alpha}\\ &= (2\theta t - \lambda t^{-\alpha}) - \ln(2\theta t+1). \end{aligned}$$

Then the reliability function of the system can be written as

$$R(t) = e^{-((2\theta t-\lambda t^{-\alpha})-\ln(2\theta t+1)} = (2\theta t+1)e^{-(2\theta t+\lambda t^{-\alpha})}$$

and the probability density of the Ailamujia-Frechet failure rate model **(AFFRM)** is given by

$$f(t) = -\frac{d}{dt}R(t) = (4\theta^2 t + 2\theta\lambda\alpha t^{\alpha-1} + \lambda\alpha t^{\alpha-1})e^{-(2\theta t+\lambda t^{\alpha})}.$$

3.5 Ailamujia-Raleigh failure rate model

The probability density, cumulative distribution, and hazard functions of the Raleigh distribution are respectively given by

$$\begin{aligned} f(x) &= 2\beta^2 x e^{-(\beta x)^2}\ ;\ x>0,\ \beta>0,\\ F(x) &= 1-e^{-(\beta x)^2}\ ;\ x>0,\ \beta>0,\\ R(x) &= e^{-(\beta x)^2}\ \textit{and}\ h(x)=\beta^2 x. \end{aligned}$$

The Ailamujia distribution was selected for $h_1(x)$ and the Raleigh distribution for $h_2(x)$.

$$\begin{aligned} \int_0^t h(x)dx &= \int_0^t [h_1(x)+h_2(x)]dx\\ &= \int_0^1 \frac{4\theta^2 x}{1+2\theta x}dx + \int_0^t 2\beta^2 x dx\\ &= 2\theta t - \ln(2\theta t+1) + \beta t^2. \end{aligned}$$

Then the reliability function of the system can be written as

$$R(t) = e^{-(2\theta t - \ln(2\theta t+1) + \beta^2 t^2)} = (2\theta t + 1)e^{-(2\theta t + \beta^2 t^2)}$$

and the probability density of the Ailamujia-Raleigh failure rate model **(ARFRM)** is given by

$$f(t) = -\frac{d}{dt}R(t) = 2(2\theta\beta^2 t^2 + 2\theta^2 t^2 + \beta^2 t)e^{-(2\theta t + \beta^2 t^2)}.$$

These findings suggest further research involving estimation of parameters of the failure rate distributions, testing of hypothesis and the power likelihood ratio criterion for the proposed models and apply the proposed failure rate models to certain real lifetime data sets.

4 Conclusion

System reliability measures were derived where the failure data follow the Ailamujia distribution. The system reliability is estimated at the conditions where the applied stress and strength follow the Ailamujia distribution. The combinations of the Ailamujia distribution and every one of well-known reliability distributions are developed. The additive failure rate model of the Ailamujia distribution and every one of the Ailamujia, exponential, Weibull, Frechet, and Raleigh distributions were derived.

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