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Analysis of Dynamic Frictional Contact Problem for Electro-Elastic Materials

O. Baiz [1*] and H. Benaissa [2]

[1] *Ibn Zohr University, FP of Ouarzazate, Morocco.*
[2] *Sultan Moulay Slimane University, FP of Khouribga, Morocco.*

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Abstract: In this paper, we study a dynamic frictional contact problem for a piezoelectric body and an electrically conductive foundation. The frictional contact is modeled by a normal compliance condition that depends on both the interpenetrations and the electrical potential difference between the body and the foundation on the contact interface, coupled with a version of Coulomb's law of dry friction with a slip dependent friction coefficient and regularized normal stress, and with an electrical contact condition in which the electrical conductivity coefficient depends on the normal velocity. First, we consider our frictional electro-elastic model and after introducing a convenable functional framework, we derive its weak formulation. Next, we establish the existence and uniqueness result for the weak solution to the problem. Finally, we study the continuous dependence of the weak solution on the data and prove a first convergence result.

Keywords: *piezoelectric material; dynamic frictional contact; variational inequality; history dependent variational inequality; fixed point arguments; existence and uniqueness result; dependence and convergence results.*

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* Corresponding author: mailto:othman.baiz@gmail.com,

1 Introduction

Due to their intrinsic coupling between mechanical and electrical properties, the piezoelectric materials remain an active area of research and engineering applications. In fact, these materials can serve as sensors, actuators or transducers, and their ability is used in various industrial devices such as medical equipment, fuel injection pistons or piezoelectric composites. Motivated by their importance in various engineering devices, the study of frictional contact phenomena involving piezoelectric materials is still relevant, both in modeling and in analysis, and the literature on this topic is still growing. General models using materials with piezoelectric effects can be found, for example, in [7, 21, 23] and the references therein. The static frictional contact problem for electro-elastic materials was considered in [12, 16–18] under the assumption that the foundation is insulated, and in [19, 20] under the assumption that the foundation is electrically conductive. For quasistatic and dynamic models dealing with electro-elastic or electro-viscoelastic materials, we can see [3, 4, 24] and the references therein.

The present paper is devoted to the variational analysis of a dynamic frictional piezoelectric contact problem under small deformations hypothesis. The material's behavior is described by a nonlinear visco-electro-elastic constitutive law and the contact is modeled with a normal compliance condition that depends on both the interpenetrations and the electrical potential difference between the body and the foundation, coupled with an electrical contact condition in which the electrical conductivity coefficient depends on the normal velocity. The friction is described by a version of Coulomb's law of dry friction in which the slip is supposed to depend on the friction coefficient and the nonlocal regularized normal contact stress. To the best of our knowledge, such piezoelectric model, coupling the electrical potential dependent compliance contact condition and the velocity dependent electrical contact condition, has not been studied so far. The variational formulation of this problem is different from that studied previously, particularly in [1, 13, 14], and hence it leads to a new mathematical model, which is a system coupling a nonlinear variational inequality for the displacement field and a nonlinear variational equation for the electric potential. Our goal is to prove the unique solvability of this model and to establish some related dependence and convergence results.

The rest of this paper is structured as follows. In Section 2, we introduce some notations and we present our frictional contact model for an electro-elastic body and an electrically conductive foundation. In Section 3, we list assumptions on the data, we derive the weak formulation of the model and we provide a result on its unique weak solvability, stated in Theorem 3.1. The proof of this theorem is given in Section 4 and it is based on the arguments of variational inequalities and the Banach fixed point theorem. Finally, in Section 5, we state and prove our convergence result which states the continuous dependence of the solution on the data.

2 Problem Statement

We consider a piezoelectric body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a sufficiently regular boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that Γ_1 is of non-zero measure. The body is clamped on Γ_1, a volume force f_0 and volume electric charges q_0 act in Ω and a surface traction f_2 acts on Γ_2. To describe the electric constraints, we consider a partition of $\Gamma_1 \cup \Gamma_2$ into two disjoint parts Γ_a and Γ_b such that Γ_a is of non-zero measure. We assume the electrical potential vanishes on

Γ_a and a surface electrical charge q_2 is prescribed on Γ_b. In the initial configuration, the body may come in contact over Γ_3 with an electrically conductive foundation. Finally, we suppose that the process is dynamic, and it is studied in a time interval $[0,T]$, where T is a positive finite constant.

To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \overline{\Omega}$. The indices i, j, k, l run between 1 and d, the summation convention over repeated indices is used, the index that follows the comma indicates the partial derivative with respect to the corresponding component of the independent variable, e.g., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, and the dot above the variable represents the derivative with respect to time, e.g., $\dot{u} = \frac{du}{dt}$. Moreover, we denote by $\mathbb{S}^d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by $u \cdot v = u_i v_i$ and $\sigma \cdot \tau = \sigma_{ij}\tau_{ij}$, respectively. Throughout the paper, we adopt the notation: $u = (u_i) : \Omega \times (0,T) \to \mathbb{R}^d$ for the displacement field, $\sigma = (\sigma_{ij}) : \Omega \times (0,T) \to \mathbb{S}^d$ for the stress tensor, $D = (D_i) : \Omega \times (0,T) \to \mathbb{R}^d$ for the electric displacement field and $E(\varphi) = (E_i(\varphi)) = -\nabla\varphi$ for the electric vector field, where $\varphi : \Omega \times (0,T) \to \mathbb{R}$ is the electric potential field. In addition, let ν be the unit outward normal vector on Γ, then the normal and tangential components for a vector field v and stress tensor σ on Γ are given by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$, $\sigma_\nu = (\sigma\nu) \cdot \nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

Then the classical formulation of our frictional contact problem is as follows.

Problem (P). *Find a displacement field* $u : \Omega \times (0,T) \to \mathbb{R}^d$ *and an electric potential field* $\varphi : \Omega \times (0,T) \to \mathbb{R}$ *such that*

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathfrak{F}\varepsilon(u) - \mathcal{E}^* E(\varphi) \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$D = \mathcal{E}\varepsilon(u) + \beta E(\varphi) \quad \text{in} \quad \Omega \times (0,T), \tag{2}$$

$$\rho\ddot{u} = \operatorname{Div}\sigma + f_0 \quad \text{in} \quad \Omega \times (0,T), \tag{3}$$

$$\operatorname{div} D = q_0 \quad \text{in} \quad \Omega \times (0,T), \tag{4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{5}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \tag{6}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a \times (0,T), \tag{7}$$

$$D \cdot \nu = q_2 \quad \text{on} \quad \Gamma_b \times (0,T), \tag{8}$$

$$\sigma_\nu = -h_\nu(\varphi - \varphi_F)\, p_\nu(u_\nu - g) \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{9}$$

$$\left.\begin{array}{l} \|\sigma_\tau\| \le \mu\,|R\sigma_\nu(u,\varphi)| \\ \|\sigma_\tau\| < \mu\,|R\sigma_\nu(u,\varphi)| \Rightarrow \dot{u}_\tau = 0 \\ \|\sigma_\tau\| = \mu\,|R\sigma_\nu(u,\varphi)| \Rightarrow \exists\,\lambda \in \mathbb{R}_+,\ \sigma_\tau = -\lambda\,\dot{u}_\tau \end{array}\right\} \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{10}$$

$$D \cdot \nu = p_e(\dot{u}_\nu)\, h_e(\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3 \times [0,T], \tag{11}$$

$$u(0) = 0, \quad \dot{u}(0) = 0 \quad \text{in} \quad \Omega. \tag{12}$$

Equations (1)–(2) represent the electro-visco-elastic constitutive law of the material. Here, $\varepsilon(u) = (\nabla u + (\nabla u)^\perp)/2$ stands for the linearized strain tensor, $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a nonlinear viscosity tensor, $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a nonlinear elasticity tensor, $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ is a linear piezoelectric tensor, $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear electric permittivity tensor and $\mathcal{E}^*$ denotes the transpose tensor of $\mathcal{E}$ defined as follows:

$$\mathcal{E}\sigma \cdot v = \sigma \cdot \mathcal{E}^* v, \quad \forall\, \sigma \in \mathbb{S}^d, \forall\, v \in \mathbb{R}^d. \tag{13}$$

Equations (3)–(4) are the equilibrium equations where the mass density ρ is chosen to be normalized $\rho = 1$. Relations (5)-(8) represent the displacement, the traction and the electric boundary conditions. Condition (9) represents the normal compliance contact condition in which p_ν is a prescribed nonnegative function which vanishes when its argument is negative, h_ν is a given positive function, g represents the maximum interpenetration of body's and foundation's asperities and φ_F denotes the electric potential of the foundation. Relations (10) represent Coulomb's friction law written in terms of the tangential components of the velocity $\dot{u}_\tau$ and the stress σ_τ, the coefficient of friction μ and the regularized normal stress $R\sigma_\nu$. The normal regularization operator R is introduced in (10) for mathematical considerations since σ_ν is only square-integrable on Ω and hence its trace on a contact surface Γ_3 is not a well-defined function, see [9, 22]. For some examples of such operator, we refer to [7, 9, 22]. Equation (11) is a regularized electrical contact condition where p_e represents the electrical conductivity coefficient which vanishes when its argument is nonnegative and h_e is a given function, see [15]. Finally, conditions (12) represents the initial displacement and the initial velocity.

The variational analysis of Problem (P) will be presented in the next sections, where we give our main existence and uniqueness result for the weak solution of Problem (P).

3 Variational Formulation and Main Result

In this section, we state hypotheses and derive a weak formulation of Problem (P). First, we introduce the following real Hilbert spaces:

$$H = L^2(\Omega)^d \ , \quad \mathcal{H} = \{\sigma = (\sigma_{ij})\,;\, \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = H^1(\Omega)^d \ , \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H}\,;\, \mathrm{Div}\,\sigma \in H\},$$

endowed with the norms $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$ induced by the inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx \ , \quad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx,$$
$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \ , \quad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\mathrm{Div}\,\sigma, \mathrm{Div}\,\tau)_H.$$

Let $\gamma : H_1 \to H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ be the trace operator. For every element $v \in H_1$, we also use the notation v to denote the trace γv of v on Γ. Recalling the boundary condition (5), we introduce the following closed subspace of H_1 given by

$$V = \{v \in H_1;\ v = 0 \ \text{ on } \ \Gamma_1\}.$$

Since Γ_1 is of non-zero measure, it follows from Korn's inequality that there exists $c_k > 0$ depending only on Ω and Γ_1 such that

$$\|v\|_{H_1} \le c_k \, \|\varepsilon(v)\|_{\mathcal{H}} \quad \text{for all} \quad v \in V. \tag{14}$$

We consider over the space V, the following inner product and associated norm:

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \|u\|_V = \|\varepsilon(v)\|_{\mathcal{H}} = (u, u)_V^{\frac{1}{2}}. \tag{15}$$

It follows from inequality (14) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Therefore, $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, by the Sobolev trace theorem, (14) and (15), there exists a constant $c_0 > 0$ depending only on Ω, Γ_3 and Γ_1 such that

$$\|v\|_{L^2(\Gamma_3)^d} \le c_0 \|v\|_V \text{ for all } v \in V. \tag{16}$$

For the real Hilbert space V previously defined, we recall the dense continuous and compact embeddings $V \subset H \subset V'$, where V' denotes the dual space of V, see [5,25]. For the electric unknowns, we introduce the following spaces:

$$W = \{\xi \in H^1(\Omega)\ ;\ \xi = 0 \ \text{ on } \ \Gamma_a\}, \quad \mathcal{W} = \{D \in H^1(\Omega)\ ; \quad \operatorname{div} D \in L^2(\Omega)\},$$

which are real Hilbert spaces for the norms $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$ induced by the inner products

$$(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H\ , \quad (D, E)_{\mathcal{W}} = (D, E)_{L^2(\Omega)^d} + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)}.$$

Since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\xi\|_W \le c_F\, \|\nabla\xi\|_{\mathcal{W}} \quad \text{for all} \quad \xi \in W, \tag{17}$$

for a constant $c_F > 0$ which depends only on Ω and Γ_a. Moreover, by the Sobolev trace theorem, there exists a constant $c_1 > 0$, depending only on Ω, Γ_a and Γ_3, such that

$$\|\xi\|_{L^2(\Gamma_3)} \le c_1 \|\xi\|_W \quad \text{for all} \quad \xi \in W. \tag{18}$$

Since $\Omega \subset \mathbb{R}^d$ is bounded, it follows from the Korn and Friedrichs-Poincaré inequalities that

$$\|v\|_{L^2(\Omega)^d} \le c_p\, \|v\|_V \quad \text{for all} \quad v \in V, \tag{19}$$

$$\|\xi\|_{L^2(\Omega)} \le c_p'\, \|\xi\|_W \quad \text{for all} \quad \xi \in W, \tag{20}$$

for some nonnegative constants c_p and c_p'. Finally, for any Hilbert space X, let X' denote the dual space of X, $\langle\cdot,\cdot\rangle_{X'\times X}$ denote the duality pairing between X' and X and the notations $C(0,T;X)$ and $C^1(0,T;X)$ stand for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, equipped with the following norms:

$$\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X\ , \quad \|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X.$$

In the study of Problem (P), we need the following assumptions on the data of the problem.

(h_1) The viscosity and elasticity tensors $\mathcal{A}$, $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ and the electric permittivity tensor $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy, for a.e. $x \in \Omega$, the following usual properties:

$$\begin{cases} (a): \exists\, M_{\mathcal{A}} > 0,\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d;\ \|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\| \le M_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\|, \\ (b): \exists\, m_{\mathcal{A}} > 0,\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d;\ \big(\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\big)\cdot\big(\varepsilon_1 - \varepsilon_2\big) \ge m_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\|^2, \\ (c): \text{the mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue-measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (e): \text{the mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ belongs to } \mathcal{H}, \text{ for all } \varepsilon \in \mathbb{S}^d, \end{cases} \tag{21}$$

$$\begin{cases} (a): \exists\, M_{\mathfrak{F}} > 0,\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d;\ \|\mathfrak{F}(x,\varepsilon_1) - \mathfrak{F}(x,\varepsilon_2)\| \le M_{\mathfrak{F}}\|\varepsilon_1 - \varepsilon_2\|, \\ (b): \exists\, m_{\mathfrak{F}} > 0,\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d;\ \big(\mathfrak{F}(x,\varepsilon_1) - \mathfrak{F}(x,\varepsilon_2)\big)\cdot\big(\varepsilon_1 - \varepsilon_2\big) \ge m_{\mathfrak{F}}\|\varepsilon_1 - \varepsilon_2\|^2, \\ (c): \text{the mapping } x \mapsto \mathfrak{F}(x,\varepsilon) \text{ is Lebesgue-measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (e): \text{the mapping } x \mapsto \mathfrak{F}(x,\varepsilon) \text{ belongs to } \mathcal{H}, \text{ for all } \varepsilon \in \mathbb{S}^d, \end{cases} \tag{22}$$

$$\begin{cases} (a): \exists\, M_{\beta} > 0,\ \forall \xi_1, \xi_2 \in \mathbb{R}^d;\ \|\beta(x,\xi_1) - \beta(x,\xi_2)\| \le M_{\beta}\|\xi_1 - \xi_2\|, \\ (b): \exists\, m_{\beta} > 0,\ \forall \xi_1, \xi_2 \in \mathbb{R}^d;\ \big(\beta(x,\xi_1) - \beta(x,\xi_2)\big)\cdot\big(\xi_1 - \xi_2\big) \ge m_{\beta}\|\xi_1 - \xi_2\|^2, \\ (c): \text{the mapping } x \mapsto \beta(x,\xi) \text{ is Lebesgue-measurable on } \Omega \text{ for all } \xi \in \mathbb{R}^d, \\ (e): \text{the mapping } x \mapsto \beta(x,\xi) \text{ belongs to } \mathcal{W}, \text{ for all } \xi \in \mathbb{R}^d. \end{cases} \tag{23}$$

(h_2) The piezoelectric tensor $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$. We note here that under hypotheses (h_2), $M_\mathcal{E} = \sup\limits_{i,j,k} \|e_{ijk}\|_{L^\infty(\Omega)}$ is well-defined.

(h_3) The function $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ with $r = e, \nu$ satisfies the following conditions:

(a) : $\exists\, M_{p_r} > 0,\ \forall s \in \mathbb{R};\quad 0 < p_r(x,s) \leq M_{p_r}$ a.e. $x \in \Gamma_3$,

(b) : $x \mapsto p_r(x,s)$ is measurable on Γ_3 for any $s \in \mathbb{R}$ and is zero for $s \leq 0$.

(h_4) The function $h_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ with $r = e, \nu$ satisfies the following conditions:

(a) : $\exists\, M_{h_e} > 0,\ \forall \varphi \in \mathbb{R};\quad |h_e(x,\varphi)| \leq M_{h_e}$ a.e. $x \in \Gamma_3$,

(b) : $\exists\, M_{h_\nu} > 0,\ \forall \varphi \in \mathbb{R};\quad 0 \leq h_\nu(x,\varphi) \leq M_{h_\nu}$ a.e. $x \in \Gamma_3$,

(c) : $\forall \varphi_1,\ \varphi_2 \in \mathbb{R};\quad \big(h_e(x,\varphi_1) - h_e(x,\varphi_2)\big)\big(\varphi_1 - \varphi_2\big) \geq 0,$ a.e. $x \in \Gamma_3$,

(d) : $x \mapsto h_r(x,\varphi)$ is measurable on Γ_3 for all $\varphi \in \mathbb{R}$.

(h_6) The mappings $s \mapsto p_r(x,s)$ and $\varphi \mapsto h_r(x,\varphi)$ are Lipschitz continuous, i.e.,

(a) : $\exists\, L_{p_r} > 0, \forall\, s_1, s_2 \in \mathbb{R};\quad |p_r(x,s_1) - p_r(x,s_2)| \leq L_{p_r}|s_1 - s_2|$ a.e. $x \in \Gamma_3$,

(b) : $\exists\, L_{h_r} > 0, \forall\, \varphi_1, \varphi_2 \in \mathbb{R};\quad |h_r(x,\varphi_1) - h_r(x,\varphi_2)| \leq L_{h_r}|\varphi_1 - \varphi_2|$ a.e. $x \in \Gamma_3$.

(h_7) The mapping $R : H^{-\frac{1}{2}}(\Gamma_3) \to L^\infty(\Gamma_3)$ is linear continuous. We denote $\|R\| = M_R$.

(h_8) The forces, the traction, the volume and surface charge densities satisfy

$$f_0 \in C(0,T; L^2(\Omega)^d)\,,\ f_2 \in C(0,T; L^2(\Gamma_2)^d),$$
$$q_0 \in C(0,T; L^2(\Omega))\,,\ q_2 \in C(0,T; L^2(\Gamma_b)).$$

(h_9) The friction coefficient, the contact surface potential and the gap function satisfy

$$\mu \in L^\infty(\Gamma_3),\ \mu \geq 0 \text{ a.e. on } \Gamma_3\,;\quad \varphi_F \in L^2(\Gamma_3)\,;\quad g \in L^2(\Gamma_3),\ g \geq 0 \text{ a.e. on } \Gamma_3.$$

Let $t \in (0,T)$, we use Riesz's representation to define $f(t) \in V$ and $q_e(t) \in W$ by

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v\, dx + \int_{\Gamma_2} f_2(t) \cdot v\, da \quad \text{for all } v \in V, \tag{24}$$
$$(q_e(t), \xi)_W = \int_\Omega q_0(t)\xi\, dx - \int_{\Gamma_b} q_2(t)\xi\, da \quad \text{for all } \xi \in W. \tag{25}$$

We consider the functionals j_1, j_2 and j_3 defined, respectively, as follows:

$$j_1(u,\varphi,v) = \int_{\Gamma_3} h_\nu(\varphi - \varphi_F) p_\nu(u_\nu - g)\, v_\nu\, da, \quad \text{for all } (u,\varphi,v) \in V \times W \times V, \tag{26}$$
$$j_2(\sigma, v) = \int_{\Gamma_3} \mu\, |R\sigma_\nu|\, \|v_\tau\|\, da, \quad \text{for all } (\sigma, v) \in \mathcal{H} \times V, \tag{27}$$
$$j_3(u,\varphi,\xi) = \int_{\Gamma_3} p_e(u_\nu) h_e(\varphi - \varphi_F)\, \xi\, da, \quad \text{for all } (u,\varphi,\xi) \in V \times W \times W. \tag{28}$$

Recalling (h_3)-(h_5) and (h_8)-(h_9), we find that the integrals in (24)-(28) are well-defined. Under these notations, the Green formula implies that if (u, σ, ϕ, D) are regular functions satisfying (3)-(11), we obtain the following weak formulation of Problem (P).

Problem (PV). *Find a displacement* $u:(0,T)\to V$*, an electric potential* $\varphi:(0,T)\to W$ *such that*

$$\begin{aligned}&\langle\ddot u(t),v-\dot u(t)\rangle_{V'\times V}+(\mathcal{A}\varepsilon(\dot u)(t),\varepsilon(v)-\varepsilon(\dot u)(t))_{\mathcal{H}}+(\mathfrak{F}\varepsilon(u)(t),\varepsilon(v)-\varepsilon(\dot u)(t))_{\mathcal{H}}\\&\quad+(\mathcal{E}^*\nabla\varphi(t),\varepsilon(v)-\varepsilon(\dot u)(t))_{L^2(\Omega)^d}+j_1(u(t),\varphi(t),v)-j_1(u(t),\varphi(t),\dot u(t))\\&\quad+j_2(\sigma(t),v)-j_2(\sigma(t),\dot u(t))\geq(f(t),v-\dot u(t))_V\ \text{ for all }\ v\in V\ \text{ a.e. }t\in(0,T),\end{aligned}\tag{29}$$

$$\begin{aligned}&(\beta\nabla\varphi(t),\nabla\xi)_{L^2(\Omega)^d}-(\mathcal{E}\varepsilon(u)(t),\nabla\xi)_{L^2(\Omega)^d}+j_3(\dot u(t),\varphi(t),\xi)\\&\quad=(q_e(t),\xi)_W\ \text{ for all }\ \xi\in W\ \text{ a.e. }t\in(0,T).\end{aligned}\tag{30}$$

We are now able to state our main result that we will prove in the next section.

Theorem 3.1 *Assume assumptions* (h_1)*–*(h_9) *hold. Then there exists a unique solution* (u,φ) *of Problem* (PV)*, which satisfies the following regularities:*

$$\ddot u\in L^2(0,T;V'),\quad u\in C^1(0,T;V),\quad \varphi\in C(0,T;W).$$

4 Proof of Theorem 3.1

We assume that (h_1)-(h_9) hold. The proof will be carried out in several steps. First, let $\eta=(\eta_1,\eta_2,\eta_3)\in L^2(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$ be given, we define the following functionals:

$$j_1^\eta(v)=\int_{\Gamma_3}\mu\,|R\eta_{3\nu}|\,\|v_\tau\|\,da\ \text{ for all }\ v\in V,\tag{31}$$

$$j_2^\eta(v)=\int_{\Gamma_3}\eta_2\,v_\nu\,da\ \text{ for all }\ v\in V.\tag{32}$$

For $\eta\in L^2(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$ known, we construct the following intermediate problem.

Problem (PV_1^η). *Find* $u_\eta:(0,T)\to V$ *such that for all* $v\in V$*, a.e.* $t\in(0,T)$*, we have*

$$\begin{aligned}&\langle\ddot u_\eta(t),v-\dot u_\eta(t)\rangle_{V'\times V}+(\mathcal{A}\varepsilon(\dot u_\eta)(t),\varepsilon(v)-\varepsilon(\dot u_\eta)(t))_{\mathcal{H}}+(\eta_1,\varepsilon(v)-\varepsilon(\dot u_\eta)(t))_{\mathcal{H}}\\&\quad+j_1^\eta(v)-j_1^\eta(\dot u_\eta(t))+j_2^\eta(v)-j_2^\eta(\dot u_\eta(t))\geq(f(t),v-\dot u_\eta(t))_V,\end{aligned}\tag{33}$$

$$\dot u(0)=0,\quad u(0)=0.\tag{34}$$

The unique solvability of Problem (PV_1^η) follows from the following lemma.

Lemma 4.1 *For a given* $\eta=(\eta_1,\eta_2,\eta_3)\in C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$*, Problem* (PV_1^η) *has a unique solution* u_η*, which satisfies* $\ddot u\in L^2(0,T;V')$ *and* $u\in C^1(0,T;V)$.

Proof. We consider the operator $A:V\to V'$ and the function $f_\eta:(0,T)\to V'$ defined, for all u, $v\in V$ and $t\in(0,T)$, by

$$\langle Au,v\rangle_{V'\times V}=(\mathcal{A}\varepsilon(u),\varepsilon(v))_{\mathcal{H}},\tag{35}$$

$$\langle f_\eta(t),v\rangle_{V'\times V}=(f(t),v)_V-(\eta_1(t),\varepsilon(v))_{\mathcal{H}}-j_2^\eta(v).\tag{36}$$

Hence, the inequality (33) can be rewritten, for all $v \in V$ and $t \in (0,T)$, as follows:

$$\begin{aligned} &\langle \ddot{u}_\eta(t), v - \dot{u}_\eta(t)\rangle_{V'\times V} + \langle A\dot{u}_\eta(t), v - \dot{u}_\eta(t)\rangle_{V'\times V} \\ &\qquad + j_1^\eta(v) - j_1^\eta(\dot{u}_\eta(t)) \geq \langle f_\eta(t), v - \dot{u}_\eta(t)\rangle_{V'\times V}. \end{aligned} \tag{37}$$

By assumption $(h_1)(21)$, the operator A is strongly monotone and Lipschitz continuous. Moreover, it follows from (31) that j_1^η is convex and Lipschitz continuous and then it is lower semi-continuous. From (36) it is easy to see that $f_\eta \in C(0,T;V')$. Then, by standard arguments on the first order nonlinear evolutionary inequalities (see [10]), there exists a unique solution u_η for Problem (PV_1^η), which satisfies

$$\ddot{u}_\eta \in L^2(0,T;V')\,,\ u_\eta \in C^1(0,T;V).$$

We use the solution u_η of Problem (PV_1^η) to consider the following auxiliary problem.

Problem (PV_2^η)**.** *Let $\eta \in C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$ be given, find $\varphi_\eta : (0,T) \to W$ such that*

$$\begin{aligned} &(\beta\nabla\varphi_\eta(t), \nabla\xi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_\eta)(t), \nabla\xi)_{L^2(\Omega)^d} \\ &\qquad + j_3(\dot{u}_\eta(t), \varphi_\eta(t), \xi) = (q_e(t), \xi)_W \quad \text{for all} \quad \xi \in W,\ a.e.\, t \in (0,T). \end{aligned} \tag{38}$$

The unique solvability of Problem (PV_2^η) follows from the following lemma.

Lemma 4.2 *Let $\eta = (\eta_1,\eta_2,\eta_3) \in L^2(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$ be known, then Problem (PV_2^η) has a unique solution φ_η which satisfies $\varphi_\eta \in C(0,T;W)$.*

Proof. Let $t \in (0,T)$, we use the Riesz representation theorem to introduce the element $q_\eta(t) \in W$ and the operator $A_\eta(t) : W \to W$, defined as follows:

$$(q_\eta(t), \xi)_W = (q_e(t), \xi)_W + (\mathcal{E}\varepsilon(u_\eta)(t), \nabla\xi)_{L^2(\Omega)^d} \quad \text{for all} \quad \xi \in W, \tag{39}$$

$$(A_\eta(t)\varphi, \xi)_W = (\beta\nabla\varphi_\eta(t), \nabla\xi)_{L^2(\Omega)^d} + j_3(\dot{u}_\eta(t), \varphi, \xi) \quad \text{for all} \quad \xi \in W. \tag{40}$$

From hypotheses $(h_1)(23)$, $(h_3)(a)$, $(h_4)(d)$ and $(h_6)(b)$, it follows that $A_\eta(t)$ is a strongly monotone, Lipschitz continuous operator on W, and therefore, there exists a unique element $\varphi_\eta(t) \in W$ such that

$$(A_\eta(t)\varphi_\eta(t), \xi)_W = (q_\eta(t), \xi)_W \quad \text{for all} \quad \xi \in W,\ t \in (0,T). \tag{41}$$

We combine (39) and (41) to find that $\varphi_\eta(t) \in W$ is the unique solution of the nonlinear variational Problem (PV_2^η), and by using Lemma 4.3 in [15], we deduce $\varphi_\eta \in C(0,T;W)$.

In the sequel, we will need the following result.

Lemma 4.3 *Let u_η and u'_η (resp. φ_η and φ'_η) be solutions of Problem (PV_1^η) (resp. Problem (PV_2^η)) for $\eta = (\eta_1,\eta_2,\eta_3)$ and $\eta' = (\eta'_1,\eta'_2,\eta'_3)$ of $C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$. Then there exist two constants $c > 0$ and $\tilde{c} > 0$ such that for all $t \in (0,T)$, we have*

$$\begin{aligned} &\|\dot{u}_\eta(t) - \dot{u}'_\eta(t)\|_V^2 + \int_0^t \|\dot{u}_\eta(s) - \dot{u}'_\eta(s)\|_V^2\, ds \\ &\quad \leq c\int_0^t \|\eta_1(s) - \eta'_1(s)\|_{\mathcal{H}}^2 + \|\eta_2(s) - \eta'_2(s)\|_{L^2(\Gamma_3)}^2 + \|\eta_3(s) - \eta'_3(s)\|_{\mathcal{H}}^2\, ds, \end{aligned} \tag{42}$$

$$\|\varphi_\eta(t) - \varphi'_\eta(t)\|_V \leq \tilde{c}\,\big(\|\dot{u}_\eta(t) - \dot{u}'_\eta(t)\|_V + \|u_\eta(t) - u'_\eta(t)\|_V\big). \tag{43}$$

Proof. It follows from (33) that for all $v \in V$ and $t \in (0,T)$, we have

$$\begin{aligned} &\langle \ddot{u}_\eta(t), v - \dot{u}_\eta(t)\rangle_{V'\times V} + (\mathcal{A}\varepsilon(\dot{u}_\eta)(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta)(t))_{\mathcal{H}} + (\eta_1(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta)(t))_{\mathcal{H}} \\ &\quad + j_1^\eta(v) - j_1^\eta(\dot{u}_\eta(t)) + j_2^\eta(v) - j_2^\eta(\dot{u}_\eta(t)) \geq (f, v - \dot{u}_\eta(t))_V, \end{aligned} \tag{44}$$

$$\begin{aligned} &\langle \ddot{u}'_\eta(t), \varepsilon(v) - \varepsilon(\dot{u}'_\eta)(t)\rangle_{V'\times V} + (\mathcal{A}\varepsilon(\dot{u}'_\eta)(t), \varepsilon(v) - \varepsilon(\dot{u}'_\eta)(t))_{\mathcal{H}} \\ &\quad + (\eta'_1(t), \varepsilon(v) - \varepsilon(\dot{u}'_\eta)(t))_{\mathcal{H}} + j_1^{\eta'}(v) - j_1^{\eta'}(\dot{u}'_\eta(t)) + j_2^{\eta'}(v) - j_2^{\eta'}(\dot{u}'_\eta(t)) \\ &\quad \geq (f, v - \dot{u}'_\eta(t))_V. \end{aligned} \tag{45}$$

Taking $v = \dot{u}'_\eta(t)$ in (44), $v = \dot{u}_\eta(t)$ in (45) and adding the obtained inequalities, we get

$$\begin{aligned} &\int_0^t \langle \ddot{u}_\eta(s) - \ddot{u}'_\eta(s), \dot{u}_\eta(s) - \dot{u}'_\eta(s)\rangle_{V'\times V}\, ds \\ &+ \int_0^t (\mathcal{A}\varepsilon(\dot{u}_\eta)(s) - \mathcal{A}\varepsilon(\dot{u}'_\eta)(s), \varepsilon(\dot{u}_\eta)(s) - \varepsilon(\dot{u}'_\eta)(s))_{\mathcal{H}}\, ds \\ &\leq - \int_0^t (\eta_1(s) - \eta'_1(s), \varepsilon(\dot{u}_\eta)(s) - \varepsilon(\dot{u}'_\eta)(s))_{L^2(\Omega)^d}\, ds \\ &+ \int_0^t j_1^\eta(\dot{u}'_\eta(s)) - j_1^\eta(\dot{u}_\eta(s)) + j_2^\eta(\dot{u}'_\eta(s)) - j_2^\eta(\dot{u}_\eta(s))\, ds \\ &+ \int_0^t j_1^{\eta'}(\dot{u}_\eta(s)) - j_1^{\eta'}(\dot{u}_\eta(s)') + j_2^{\eta'}(\dot{u}_\eta(s)) - j_2^{\eta'}(\dot{u}'_\eta(s))\, ds. \end{aligned} \tag{46}$$

Using the definition of the functional j_1^η, we deduce

$$\begin{aligned} &|j_1^\eta(\dot{u}'_\eta(s)) - j_1^\eta(\dot{u}_\eta(s)) + j_1^{\eta'}(\dot{u}_\eta(s)) - j_1^{\eta'}(\dot{u}'_\eta(s))| \\ &\leq \int_{\Gamma_3} \mu\, |R\eta_{3\nu}|\, \big(\|\dot{u}'_{\eta\tau}(s)\| - \|\dot{u}_{\eta\tau}(s)\|\big)\, da \\ &- \int_{\Gamma_3} \mu\, |R\eta'_{3\nu}|\, \big(\|\dot{u}'_{\eta\tau}(s)\| - \|\dot{u}_{\eta\tau}(s)\|\big)\, da, \\ &\leq \int_{\Gamma_3} \mu\, \big(|R\eta_{3\nu}| - |R\eta'_{3\nu}|\big)\, \big(\|\dot{u}_{\eta\tau}(s)\| - \|\dot{u}'_{\eta\tau}(s)\|\big)\, da, \\ &\leq c_0\, \|\mu\|_{L^\infty(\Gamma_3)} M_R\, \|\eta_3 - \eta'_3\|_{\mathcal{H}}\, \|\dot{u}_\eta(s) - \dot{u}'_\eta(s)\|_V. \end{aligned} \tag{47}$$

Moreover, we use the definition of the functional j_2^η to obtain

$$\begin{aligned} &|j_2^\eta(\dot{u}'_\eta(s)) - j_2^\eta(\dot{u}_\eta(s)) + j_2^{\eta'}(\dot{u}_\eta(s)) - j_2^{\eta'}(\dot{u}'_\eta(s))| \\ &\leq \int_{\Gamma_3} \eta_2\, \big(\dot{u}'_{\eta\nu}(s) - \dot{u}_{\eta\nu}(s)\big)\, da - \int_{\Gamma_3} \eta_2\, \big(\dot{u}'_{\eta\nu}(s) - \dot{u}_{\eta\nu}(s)\big)\, da, \\ &\leq \int_{\Gamma_3} \big(\eta_2 - \eta'_2\big)\, \big(\dot{u}_{\eta\nu}(s) - \dot{u}'_{\eta\nu}(s)\big)\, da, \\ &\leq c_0\, \|\eta_2 - \eta'_2\|_{L^2(\Gamma_3)}\, \|\dot{u}_{\eta\tau}(s) - \dot{u}'_\eta(s)\|_V. \end{aligned} \tag{48}$$

We combine the inequalities (46)-(48) and we use the assumption (h_1) to get

$$\begin{aligned}
&\frac{1}{2}\,\|\dot{u}_\eta(t)-\dot{u}'_\eta(t)\|_V^2+m_{\mathcal{A}}\int_0^t\|\dot{u}_\eta(s)-\dot{u}'_\eta(s)\|_V^2\,ds\\
&\le c_p\int_0^t\|\eta_1(s)-\eta'_1(s)\|_{L^2(\Omega)^d}\,\|\dot{u}_\eta(s)-\dot{u}'_\eta(s)\|_V\,ds\\
&+c_0\,\|\mu\|_{L^\infty(\Gamma_3)}M_R\int_0^t\|\eta_3-\eta'_3\|_{\mathcal{H}}\|\dot{u}_\eta(s)-\dot{u}_\eta(s)'\|_V\,ds\\
&+c_0\int_0^t\|\eta_2(s)-\eta'_2(s)\|_{L^2(\Gamma_3)}\|\dot{u}_\eta(s)-\dot{u}_\eta(s)'\|_V\,ds.
\end{aligned}\tag{49}$$

Finally, we apply Young's inequality $ab\le\epsilon a^2+\frac{b^2}{4\epsilon}$ to get, after some simplifications, that

$$\begin{aligned}
&\|\dot{u}_\eta(t)-\dot{u}'_\eta(t)\|_V^2+\int_0^t\|\dot{u}_\eta(s)-\dot{u}'_\eta(s)\|_V^2\,ds\\
&\le c\int_0^t\|\eta_1(s)-\eta'_1(s)\|^2_{L^2(\Omega)^d}+\|\eta_2(s)-\eta'_2(s)\|^2_{L^2(\Gamma_3)}+\|\eta_3-\eta'_3\|^2_{\mathcal{H}}\,ds.
\end{aligned}\tag{50}$$

Next, let φ_η and φ'_η be the corresponding solutions of (PV_2^η) for $\eta=(\eta_1,\eta_2,\eta_3)$ and $\eta'=(\eta'_1,\eta'_2,\eta'_3)$, respectively. From (39), we get, for all $t\in(0,T)$ and $\xi\in W$, that

$$(\beta\nabla\varphi_\eta(t),\nabla\xi)_{L^2(\Omega)^d}-(\mathcal{E}\varepsilon(u_\eta)(t),\nabla\xi)_{L^2(\Omega)^d}+j_3(\dot{u}_\eta,\varphi_\eta(t),\xi)=(q_e(t),\xi)_W,\tag{51}$$

$$(\beta\nabla\varphi'_\eta(t),\nabla\xi)_{L^2(\Omega)^d}-(\mathcal{E}\varepsilon(u'_\eta)(t),\nabla\xi)_{L^2(\Omega)^d}+j_3(\dot{u}'_\eta,\varphi'_\eta(t),\xi)=(q_e(t),\xi)_W.\tag{52}$$

Replacing ξ by $\varphi_\eta(t)-\varphi'_\eta(t)$ in (51) and (52), we subtract the obtained equations to find

$$\begin{aligned}
&(\beta\nabla\varphi_\eta(t)-\beta\nabla\varphi'_\eta(t),\nabla\varphi_\eta(t)-\nabla\varphi'_\eta(t))_{L^2(\Omega)^d}\\
&-(\mathcal{E}\varepsilon(u_\eta)(t)-\mathcal{E}\varepsilon(u'_\eta)(t),\nabla\varphi_\eta(t)-\nabla\varphi'_\eta(t))_{L^2(\Omega)^d}\\
&+j_3(\dot{u}_\eta,\varphi_\eta(t),\varphi_\eta(t)-\varphi'_\eta(t))-j_3(\dot{u}'_\eta,\varphi'_\eta(t),\varphi_\eta(t)-\varphi'_\eta(t))=0.
\end{aligned}\tag{53}$$

Using the assumptions (h_3)-(h_5) and the definition of the functional j_3, we obtain

$$\begin{aligned}
&|j_3(\dot{u}_\eta,\varphi_\eta(t),\varphi_\eta(t)-\varphi'_\eta(t))-j_3(\dot{u}'_\eta,\varphi'_\eta(t),\varphi_\eta(t)-\varphi'_\eta(t))|\\
&=\int_{\Gamma_3}\big(p_e(\dot{u}_\eta(t))\,h_e(\varphi_\eta(t)-\varphi_F)-p_e(\dot{u}'_\eta(t))\,h_e(\varphi'_\eta(t)-\varphi_F)\big)\big(\varphi_\eta(t)-\varphi'_\eta(t)\big)\,da,\\
&=\int_{\Gamma_3}p_e(\dot{u}_\eta(t))\big(h_e(\varphi_\eta(t)-\varphi_F)-h_e(\varphi'_\eta(t)-\varphi_F)\big)\big(\varphi_\eta(t)-\varphi'_\eta(t)\big)\,da\\
&+\int_{\Gamma_3}h_e(\varphi'_\eta(t)-\varphi_F)\,\big(p_e(\dot{u}_\eta(t))-p_e(\dot{u}'_\eta(t))\big)\big(\varphi_\eta(t)-\varphi'_\eta(t)\big)\,da,\\
&\le c_0\,c_1M_{h_e}\|\dot{u}_\eta(t)-\dot{u}'_\eta(t)\|_V\,\|\varphi_\eta(t)-\varphi'_\eta(t)\|_W.
\end{aligned}\tag{54}$$

By virtue of hypotheses $(h_1)(23)$ and (h_2), it follows from (53) and (54) that

$$\|\varphi_\eta(t)-\varphi'_\eta(t)\|_V\le\tilde{c}\,\big(\|\dot{u}_\eta(s)-\dot{u}'_\eta(s)\|_V+\|u_\eta(s)-u'_\eta(s)\|_V\big).\tag{55}$$

Hence, inequalities (42) and (43) of Lemma 4.3 are obtained.

In the next step, we consider the following operator:

$$\Lambda : C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H}) \to C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H}),$$

defined for $t\in(0,T)$ by $\Lambda\eta(t) = \big(\Lambda_1\eta(t),\Lambda_2\eta(t),\Lambda_3\eta(t)\big)$, where

$$\Lambda_1\eta(t) = \mathfrak{F}\varepsilon(u_\eta)(t) + \mathcal{E}^*\nabla\varphi_\eta(t), \tag{56}$$

$$\Lambda_2\eta(t) = h_\nu(\varphi_\eta(t)-\varphi_f(t))\,p_\nu(u_{\eta\nu}(t)-g), \tag{57}$$

$$\Lambda_3\eta(t) = \mathcal{A}\varepsilon(\dot{u}_\eta(t)) + \mathfrak{F}\varepsilon(u_\eta(t)) - \mathcal{E}^*E(\varphi_\eta(t)). \tag{58}$$

We have the following fixed point result.

Lemma 4.4 *There exists a unique $\eta^*\in C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$ such that*

$$\Lambda\eta^* = \eta^*.$$

Proof. Let $\eta=(\eta_1,\eta_2,\eta_3)$, $\eta'=(\eta_1',\eta_2',\eta_3')\in C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$. The definition of Λ_1 and Λ_3, and the assumptions (h_1) and (h_2) imply, after some algebras, that

$$\|\Lambda_1\eta(t)-\Lambda_1\eta'(t)\|_{\mathcal{H}} \le M_{\mathfrak{F}}\|u_\eta(t)-u_\eta'(t)\|_V + M_{\mathcal{E}}\|\varphi_\eta(t)-\varphi_\eta'(t)\|_W, \tag{59}$$

$$\begin{aligned}\|\Lambda_3\eta(t)-\Lambda_3\eta'(t)\|_{\mathcal{H}} \le{}& M_{\mathcal{A}}\|\dot{u}_\eta(t)+\dot{u}_\eta'(t)\|_V + M_{\mathfrak{F}}\|u_\eta(t)-u_\eta'(t)\|_V\\ &+ M_{\mathcal{E}}\|\varphi_\eta(t)-\varphi_\eta'(t)\|_W.\end{aligned} \tag{60}$$

Using the definition of Λ_2 and the properties of h_ν and p_ν, it is easy to verify that

$$\|\Lambda_2\eta(t)-\Lambda_2\eta'(t)\|_{\mathcal{H}} \le M_{h_\nu}L_{p_\nu}c_0\,\|u_\eta(t)+u_\eta(t)\|_V + M_{p_\nu}L_{h_\nu}c_1\,\|\varphi_\eta(t)-\varphi_\eta'(t)\|_W. \tag{61}$$

Then, from the inequalities (59)-(61), (42) and (43), there exists $c>0$ such that

$$\|\Lambda\eta(t)-\Lambda\eta'(t)\|^2_{\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H}} \le c\int_0^t \|\eta(s)-\eta'(s)\|^2_{\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H}}\,ds. \tag{62}$$

Reiterating the previous inequality n times, we get

$$\|\Lambda^n\eta-\Lambda^n\eta'\|_{C([0,T];\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})} \le \sqrt{\frac{c^nT^n}{n!}}\,\|\eta(s)-\eta'(s)\|_{C([0,T];\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})}. \tag{63}$$

Since $\lim\limits_{n+\infty}\frac{c^nT^n}{n!}=0$, the inequality (63) shows that for n sufficiently large, the operator Λ^n is a contraction on the Banach space $C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$. Thus, according to the Banach fixed point theorem, there exists a unique $\eta^*\in C(0,T;\mathcal{H}\times L^2(\Gamma_3)\times\mathcal{H})$ such that $\Lambda^n\eta^*=\eta^*$. Moreover, since $\Lambda^n(\Lambda\eta^*)=\Lambda(\Lambda^n\eta^*)=\Lambda\eta^*$, we deduce that $\Lambda\eta^*$ is also a fixed point of Λ^n, and by the uniqueness of the fixed point, we obtain $\Lambda\eta^*=\eta^*$. Therefore, η^* is a unique fixed point of Λ too.

Now, we have all the ingredients needed to prove Theorem 3.1. Indeed, let η^* be the unique fixed point of the operator Λ and let $u=u_{\eta^*}$ and $\varphi=\varphi_{\eta^*}$ be the unique solutions of the Problems $(PV_1^{\eta^*})$ and $(PV_2^{\eta^*})$, respectively. Therefore, (u,φ) is a solution of Problem (PV) and then the existence part is proved. The uniqueness part results from the uniqueness of the fixed point of the operator Λ. Then Theorem 3.1 is established.

5 Convergence Result

We are interested here in the dependence of the solution of Problem (PV) on the perturbations of the data. In the sequel, we assume that the assumptions (h_1)-(h_7) hold and let (u,φ) be the solution of Problem (PV) obtained in Theorem 3.1. For each $\epsilon>0$, let f_0^ϵ, q_0^ϵ, f_2^ϵ, q_2^ϵ and φ_F^ϵ denote the perturbations of f_0, q_0, f_2, q_2 and φ_F, respectively. We consider the operators $f^\epsilon:(0,T)\to V$ and $q_e^\epsilon:(0,T)\to W$ defined as follows:

$$(f^\epsilon(t),v)_V=\int_\Omega f_0^\epsilon(t)\cdot v\,dx+\int_{\Gamma_2} f_2^\epsilon(t)\cdot v\,da \quad \text{for all} \quad v\in V, \quad \text{a.e.} \quad t\in(0,T), \tag{64}$$

$$(q_e^\epsilon(t),\xi)_W=\int_\Omega q_0^\epsilon(t)\xi\,dx-\int_{\Gamma_b} q_2^\epsilon(t)\xi\,da \quad \text{for all} \quad \xi\in W, \quad \text{a.e.} \quad t\in(0,T). \tag{65}$$

We consider the functionals $j_1^\epsilon:V\times W\times V\to\mathbb{R}$ and $j_3^\epsilon:V\times W\times W\to\mathbb{R}$ given by

$$j_1^\epsilon(u,\varphi,v)=\int_{\Gamma_3} h_\nu(\varphi-\varphi_F^\epsilon)p_\nu(u_\nu-g)\,v_\nu\,da, \tag{66}$$

$$j_3^\epsilon(u,\varphi,\xi)=\int_{\Gamma_3} p_e(u_\nu)h_e(\varphi-\varphi_F^\epsilon)\,\xi\,da. \tag{67}$$

Next, we introduce the following perturbation of the variational Problem (PV).

Problem (PV^ϵ)**.** *Find a displacement $u^\epsilon:(0,T)\to V$ and an electric potential $\varphi^\epsilon:(0,T)\to W$ such that for all $\xi\in W$, $v\in V$ and a.e. $t\in(0,T)$, we have*

$$\begin{aligned}&\langle\ddot u^\epsilon(t),\varepsilon(v)-\varepsilon(\dot u^\epsilon)(t)\rangle_{V'\times V}+(\mathcal{A}\varepsilon(\dot u^\epsilon)(t),\varepsilon(v)-\varepsilon(\dot u^\epsilon)(t))_{\mathcal{H}}\\&+(\mathfrak{F}\varepsilon(u^\epsilon)(t),\varepsilon(v)-\varepsilon(\dot u^\epsilon)(t))_{\mathcal{H}}+(\mathcal{E}^*\nabla\varphi^\epsilon(t),\varepsilon(v)-\varepsilon(\dot u^\epsilon)(t))_{L^2(\Omega)^d}\\&+j_1^\epsilon(u^\epsilon(t),\varphi^\epsilon(t),v)-j_1^\epsilon(u^\epsilon(t),\varphi^\epsilon(t),\dot u^\epsilon(t))\\&+j_2(\sigma^\epsilon,v)-j_2(\sigma^\epsilon,\dot u^\epsilon(t))\ge(f^\epsilon(t),v-\dot u(t))_V,\end{aligned} \tag{68}$$

$$(\beta\nabla\varphi^\epsilon(t),\nabla\xi)_{L^2(\Omega)^d}-(\mathcal{E}\varepsilon(u^\epsilon)(t),\nabla\xi)_{L^2(\Omega)^d}+j_3(\dot u^\epsilon(t),\varphi^\epsilon(t),\xi)=(q_e^\epsilon(t),\xi)_W. \tag{69}$$

For each $\epsilon>0$, Theorem 3.1 implies that Problem (PV^ϵ) has a unique solution $(u^\epsilon,\varphi^\epsilon)$. On the other hand, we state the following convergence assumptions:

$$f_0^\epsilon\to f_0 \quad \text{in} \quad C(0,T;L^2(\Omega)^d) \quad as \quad \epsilon\to0, \tag{70}$$

$$q_0^\epsilon\to q_0 \quad \text{in} \quad C(0,T;L^2(\Omega)) \quad as \quad \epsilon\to0, \tag{71}$$

$$f_2^\epsilon\to f_2 \quad \text{in} \quad C(0,T;L^2(\Gamma_2)^d) \quad as \quad \epsilon\to0, \tag{72}$$

$$q_2^\epsilon\to q_2 \quad \text{in} \quad C(0,T;L^2(\Gamma_b)) \quad as \quad \epsilon\to0, \tag{73}$$

$$\varphi_F^\epsilon\to \varphi_F \quad \text{in} \quad C(0,T;L^2(\Gamma_3)) \quad as \quad \epsilon\to0. \tag{74}$$

Let $c>0$ be a generic constant which may depend on data, but does not depend on ϵ, and whose value may vary from place to place. We have the following convergence result.

Theorem 5.1 *Under assumptions* (70)-(74)*, the solution $(u^\epsilon,\varphi^\epsilon)$ of Problem (PV^ϵ) converges strongly to the solution (u,φ) of Problem (PV), i.e.,*

$$(u^\epsilon,\varphi^\epsilon)\to(u,\varphi) \quad as \quad \epsilon\to0. \tag{75}$$

Proof. Using inequalities (29) and (68), we obtain

$$\begin{aligned}
&\langle \ddot u(t)-\ddot u^\epsilon(t),\dot u(t)-\dot u^\epsilon(t)\rangle_{V'\times V}+(\mathcal{A}\varepsilon(\dot u)(t)-\mathcal{A}\varepsilon(\dot u^\epsilon)(t),\varepsilon(\dot u)(t)-\varepsilon(\dot u^\epsilon)(t))_{\mathcal{H}}\\
&\le -(\mathfrak{F}\varepsilon(u)(t)-\mathfrak{F}\varepsilon(u^\epsilon)(t),\varepsilon(\dot u)(t)-\varepsilon(\dot u^\epsilon)(t))_{\mathcal{H}}\\
&-(\mathcal{E}^*\nabla\varphi(t)-\mathcal{E}^*\nabla\varphi^\epsilon(t),\varepsilon(\dot u)(t)-\varepsilon(\dot u^\epsilon)(t))_{L^2(\Omega)^d}\\
&+\underbrace{j_1^\epsilon(u^\epsilon(t),\varphi^\epsilon(t),\dot u(t))-j_1^\epsilon(u^\epsilon(t),\varphi^\epsilon(t),\dot u^\epsilon(t))}_{=J_1^\epsilon}+\underbrace{j_2(\sigma^\epsilon,(\dot u)(t))-j_2(\sigma^\epsilon,\dot u^\epsilon(t))}_{=J_2^\epsilon}\\
&+\underbrace{j_1(u(t),\varphi(t),\dot u(t))-j_1(u(t),\varphi(t),\dot u^\epsilon(t))}_{=J_1}+\underbrace{j_2(\sigma,\dot u(t))-j_2(\sigma,\dot u^\epsilon(t))}_{=J_2}\\
&+(f(t)-f^\epsilon(t),\dot u(t)-\dot u^\epsilon(t))_V.
\end{aligned}\tag{76}$$

From the definition of the functionals j_1 and j_1^ϵ, we have

$$\begin{aligned}
&|J_1^\epsilon+J_1|\\
&\le\int_{\Gamma_3}\left|\left(p_\nu(u_\nu^\epsilon(t)-g)\,h_\nu(\varphi^\epsilon(t)-\varphi_F^\epsilon(t))-p_\nu(u_\nu(t)-g)\,h_\nu(\varphi(t)-\varphi_F(t))\right)(\dot u_\nu(t)-\dot u_\nu^\epsilon(t))\right|da.
\end{aligned}$$

Taking in mind the hypotheses (h_3) and (h_4), we find

$$\begin{aligned}
|J_1^\epsilon+J_1|\le{}& M_{p_\nu}L_{h_\nu}c_1c_0\,\|\varphi(t)-\varphi^\epsilon(t)\|_W\|\dot u(t)-\dot u^\epsilon(t)\|_V\\
&+M_{p_\nu}L_{h_\nu}c_0\,\|\varphi_F(t)-\varphi_F^\epsilon(t)\|_{L^2(\Gamma_3)}\,\|\dot u(t)-\dot u^\epsilon(t)\|_V\\
&+M_{h_\nu}L_{p_\nu}c_0^2\,\|u(t)-u^\epsilon(t)\|_V\|\dot u(t)-\dot u^\epsilon(t)\|_V.
\end{aligned}\tag{77}$$

Moreover, it follows from the definition of the functionals j_2 and j_2^ϵ that

$$\begin{aligned}
|J_2^\epsilon+J_2|={}&\int_{\Gamma_3}\mu\left(|R\sigma_\nu|-|R\sigma_\nu^\epsilon|\right)\left(\|\dot u\tau\|-\|\dot u^\epsilon\tau\|\right)da,\\
&\le c_0\,\|\mu\|_{L^\infty(\Gamma_3)}M_R\|\sigma-\sigma^\epsilon\|_{\mathcal{H}}\,\|\dot u(t)-\dot u^\epsilon(t)\|_V,\\
&\le c_0\,\|\mu\|_{L^\infty(\Gamma_3)}M_RM_{\mathcal{A}}\,\|\dot u(t)-\dot u^\epsilon(t)\|_V^2\\
&+c_0\,\|\mu\|_{L^\infty(\Gamma_3)}M_RM_{\mathfrak{F}}\,\|u(t)-u^\epsilon(t)\|_V\|\dot u(t)-\dot u^\epsilon(t)\|_V\\
&+c_0\,\|\mu\|_{L^\infty(\Gamma_3)}M_RM_{\mathcal{E}}\,\|\varphi(t)-\varphi^\epsilon(t)\|_W\|\dot u(t)-\dot u^\epsilon(t)\|_V.
\end{aligned}\tag{78}$$

We integrate (76) and use the assumptions (h_1)-(h_2) and the inequalities (77)-(78) to get

$$\begin{aligned}
&\frac12\,\|\dot u(t)-\dot u^\epsilon(t)\|_V^2+\frac12\,m_{\mathfrak{F}}\,\|u(t)-u^\epsilon(t)\|_V^2+m_{\mathcal{A}}\int_0^t\|\dot u(s)-\dot u^\epsilon(s)\|_V^2ds\\
&\le(M_{\mathcal{E}}+c_0\|\mu\|_{L^\infty(\Gamma_3)}M_RM_{\mathcal{E}}+M_{p_\nu}L_{h_\nu}c_1c_0)\int_0^t\|\varphi(s)-\varphi^\epsilon(s)\|_W\|\dot u(s)-\dot u^\epsilon(s)\|_Vds\\
&+M_{p_\nu}L_{h_\nu}c_0\int_0^t\|\varphi_F(s)-\varphi_F^\epsilon(s)\|_{L^2(\Gamma_3)}\,\|\dot u(s)-\dot u^\epsilon(s)\|_V\,ds\\
&+(M_{h_\nu}L_{p_\nu}c_0^2+c_0\,\|\mu\|_{L^\infty(\Gamma_3)}M_R\,M_{\mathfrak{F}})\int_0^t\|u(s)-u^\epsilon(s)\|_V\|\dot u(s)-\dot u^\epsilon(s)\|_Vds\\
&+c_0\|\mu\|_{L^\infty(\Gamma_3)}M_RM_{\mathcal{A}}\int_0^t\|\dot u(s)-\dot u^\epsilon(s)\|_V^2ds+\int_0^t\|f(s)-f^\epsilon(s)\|_V\|\dot u(s)-\dot u^\epsilon(s)\|_Vds.
\end{aligned}$$

Then we apply the α-inequality $ab < \alpha^2 a^2 + \frac{b^2}{\alpha^2}$ and the Gronwall inequality to find

$$\begin{aligned}&\|\dot{u}(t) - \dot{u}^\epsilon(t)\|_V^2 + \|u(t) - u^\epsilon(t)\|_V^2 + \int_0^t \|\dot{u}(s) - \dot{u}^\epsilon(s)\|_V^2\, ds \\ &\leq c \int_0^t \big(\|\varphi(s) - \varphi^\epsilon(s)\|_W^2 + \|\varphi_F(s) - \varphi_F^\epsilon(s)\|_{L^2(\Gamma_3)}^2 + \|f(s) - f^\epsilon(s)\|_V^2\big) ds.\end{aligned} \tag{79}$$

Furthermore, it follows from equations (30) and (69) that

$$\begin{aligned}&(\beta\nabla\varphi(t) - \beta\nabla\varphi^\epsilon(t), \nabla\varphi(t) - \nabla\varphi^\epsilon(t))_{L^2(\Omega)^d} \\ &- (\mathcal{E}\varepsilon(u)(t) - \mathcal{E}\varepsilon(u^\epsilon)(t), \nabla\varphi(t) - \nabla\varphi^\epsilon(t))_{L^2(\Omega)^d} + j_3(\dot{u}(t), \varphi(t), \varphi(t) - \varphi^\epsilon(t)) \\ &- j_3^\epsilon(\dot{u}^\epsilon(t), \varphi^\epsilon(t), \varphi(t) - \varphi^\epsilon(t)) = (q_e(t) - q_e^\epsilon(t), \varphi(t) - \varphi^\epsilon(t))_W.\end{aligned} \tag{80}$$

Using the definitions of j_3^ϵ and j_3 and the assumptions (h_3) and (h_4), we have

$$\begin{aligned}&|j_3(\dot{u}(t), \varphi(t), \varphi(t) - \varphi^\epsilon(t)) - j_3^\epsilon(\dot{u}^\epsilon(t), \varphi^\epsilon(t), \varphi(t) - \varphi^\epsilon(t))| \\ &\leq M_{h_e} L_{p_e} c_0\, c_1\, \|\dot{u}(t) - \dot{u}^\epsilon(t)\|_V \|\varphi(t) - \varphi^\epsilon(t)\|_W \\ &+ M_{h_e} L_{p_e} c_1\, \|\varphi_F(t) - \varphi_F^\epsilon(t)\|_{L^2(\Gamma_3)} \|\varphi(t) - \varphi^\epsilon(t)\|_W.\end{aligned} \tag{81}$$

Keeping in mind (80) and (81) and hypotheses (h_1) and (h_2), we deduce

$$\begin{aligned}\|\varphi(t) - \varphi^\epsilon(t)\|_W \leq & c\,\{\|u(t) - u^\epsilon(t)\|_V + \|\dot{u}(t) - \dot{u}^\epsilon(t)\|_V + \|q_e(t) - q_e^\epsilon(t)\|_W^2 \\ & + \|\varphi_F(t) - \varphi_F^\epsilon(t)\|_{L^2(\Gamma_3)}\} \ \text{ for all } \ t \in (0, T).\end{aligned} \tag{82}$$

Next, we combine (79) and (82) and we apply the Gronwall inequality to find

$$\begin{aligned}&\|\dot{u}(t) - \dot{u}^\epsilon(t)\|_V^2 + \|u(t) - u^\epsilon(t)\|_V^2 + \int_0^t \|\dot{u}(s) - \dot{u}^\epsilon(s)\|_V^2\, ds \\ &\leq c \int_0^t \big(\|q_e(t) - q_e^\epsilon(t)\|_W^2 + \|\varphi_F(s) - \varphi_F^\epsilon(s)\|_{L^2(\Gamma_3)}^2 + \|f(s) - f^\epsilon(s)\|_V^2\big)\, ds.\end{aligned} \tag{83}$$

Remembering the definitions (24), (25), (64) and (65) of f, q_e, f^ϵ and q_e^ϵ, we obtain

$$\|f(t) - f^\epsilon(t)\|_V \leq c_p\, \|f_0(t) - f_0^\epsilon(t)\|_{L^2(\Omega)} + c_0\, \|f_2(t) - f_2^\epsilon(t)\|_{L^2(\Gamma_2)}, \quad \forall\, t \in (0, T), \tag{84}$$

$$\|q_e(t) - q_e^\epsilon(t)\|_W \leq c_p'\, \|q_0(t) - q_0^\epsilon(t)\|_{L^2(\Omega)} + c_1\, \|q_2(t) - q_2^\epsilon(t)\|_{L^2(\Gamma_b)}, \quad \forall\, t \in (0, T). \tag{85}$$

Finally, we use the assumptions (70)-(74) together with (83)-(85) to establish (75).

6 Conclusion

Real applications in contact mechanics, where the dynamic behavior is linear, are rare. Usually, the contact phenomena involve largely nonlinearities due to the nature of the material (with a coupling constitutive law; here, an electro-elastic materials), and the friction and electrical conduction effects accompanying the mechanical contact process. Hence, the previous parameters can change the dynamic behavior of the whole mechanical system, and the modeling of this type of problem is therefore important to predict, for instance, the effects of friction and the electrical conduction on the material's body, and then to predict the evolution of the material state, particularly in the contact zone (wear and adhesion ..). Also, it is essential to correct prediction of the critical cases (for example, introduce lubrification effects to control friction wear and adhesion before the damage of the body).

In this paper, we presented a mathematical model for the dynamic contact problem of a nonlinear electro-elastic body and a conductive foundation. The unique weak solvability of this problem was established using arguments of evolutionary variational inequalities and a fixed point theorem. The obtained results represent an improvement of those existing in literature and will facilitate future research of other open problems arising from mathematical modeling in industrial engineering when it is necessary to take into account both the mechanical and the electrical properties. An interesting continuation of the current results would be their natural extensions to complicated piezoelectric contact problems with nontrivial electrical contact conditions. Moreover, such models lead to new evolutionary variational and hemi-variational inequalities.

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References

[1] A. Bachmar, B. Souraya and T. Serrar. Variational Analysis of a Dynamic Electroviscoelastic Problem with Friction. *Journal of Siberian Federal University. Mathematics & Physics* **12** (1) (2019) 68–78.

[2] M. Barboteu and M. Sofonea. Modelling and analysis of the unilateral contact of a piezoelectric body with a conductive support. *J. Math. Anal. Appl.* **358** (1) (2009) 110–124.

[3] M. Barboteu, J.R. Fernández and Y. Ouafik. Numerical analysis of a frictionless viscoelastic piezoelectric contact problem. *ESAIM Mathematical Modelling and Numerical Analysis* **42** (4) (2008) 667–682.

[4] M. Barboteu, J.R. Fernández, and R. Tarraf. Numerical analysis of a dynamic piezoelectric contact problem arising in viscoelasticity. *Comput. Methods Appl. Mech. Engrg.* **197** (45-48) (2008) 3724–3732.

[5] H. Brézis. *Functional analysis, Sobolev Spaces and Partial Differential Equations.* Universitext, Springer, New York, 2011.

[6] M. Cocu and R. Rocca. Existence results for unilateral quasistatic contact problems with friction and adhesion. *Mathematical Modelling and Numerical Analysis* **34**(5) (2000) 981–1001.

[7] Z. Denkowski, S. Migòrski and A. Ochal. A class of optimal control problems for piezoelectric frictional contact models. *N. Analysis: Real Wor. Appl.* **12** (3) (2011) 1883–1895.

[8] Z. Denkowski, S. Migòrski and A. Ochal. Existence and uniqueness to a dynamic bilateral frictional contact problem in viscoelasticity. *Acta Appl. Math.* **94** (2006) 251–276.

[9] G. Duvaut. Loi de frottement non locale. *J. Méc. Th. et Appl.* (1982) 73–78.

[10] G. Duvaut and J.L. Lions. *Inequalities in Mechanics and Physics.* Springer-Verlag, Berlin, 1976.

[11] P. Gamorski and S. Migòrski. Hemivariational inequalities modeling electro-elastic unilateral frictional contact problem. *Mathematics and Mechanics of Solids* **23** (2018) 329–347.

[12] S. Hüeber, A. Matei and B.I. Wohlmuth. A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity. *Bull. Math. Soc. Sci. Math. Roumanie* **48** (96) (2005) 209–232.

[13] M. Kabbaj and El-H. Essoufi. Slip-dependent friction in dynamic electroviscoelasticity. Glasnik Matematicki **45**(65) (2010) 125–137.

[14] M. Kabbaj and El-H. Essoufi. Existence of solutions of a dynamic Signorini's problem with nonlocal friction for viscoelastic piezoelectric materials. *Bull. Math. Soc. Sc. Math. de Roumanie* **48** (2) (2005) 181–195.

[15] Z. Lerguet, M. Shillor and M. Sofonea. A frictional contact problem for an electro-viscoelastic body. *Electronic Journal of Differential Equations* **170** (2007) 1–16.

[16] F. Maceri and P. Bisegna. The unilateral frictionless contact of a piezoelectric body with a rigid support. *Math. Comp. Modelling* **28** (4-8) (1998) 19–28.

[17] S. Migòrski. Hemivariational inequality for a frictional contact problem in elastopiezoelectricity. *Discrete Contin. Dyn. Syst. Ser. B* **6** (6) (2006) 1339–1356.

[18] S. Migòrski. A class of hemivariational inequality for electroelastic contact problems with slip dependent friction. *Discrete Contin. Dyn. Syst. Ser. S* **1** (1) (2008) 117–126.

[19] S. Migòrski, A. Ochal and M. Sofonea. Weak solvability of a piezoelectric contact problem. *European Journal of Applied Mathematics* **20** (2) (2009) 145–167.

[20] S. Migòrski, A. Ochal and M. Sofonea. Variational analysis of fully coupled electro-elastic frictional contact problems. *Mathematische Nachrichten* **283** (2010) 1314–1335.

[21] W. Nowacki. Foundations of linear piezoelectricity. In: *Electromagnetic Interactions in Elastic Solids*. Springer, Wein, Chapter 1, 1979.

[22] M. Shillor, M. Sofonea and J. J. Telega. *Models and Analysis of Quasistatic Contact*. Springer, Berlin, 2004.

[23] M. Sofonea, W. Han and M. Shillor. *Analysis and Approximation of Contact Problems with Adhesion or Damage*. Pure and Applied Mathematics, 276, Chapman & Hall/CRC press, Bocca Raton, Florida, 2005.

[24] M. Sofonea and El-H. Essoufi. Quasistatic frictional contact of a viscoelastic piezoelectric body. *Adv. Math. Sci. Appl.* **14** (1) (2004) 613–631.

[25] E. Zeidler. *Nonlinear Functional Analysis and Applications II A/B*. Springer, New York, 1990.