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R-Functions and Nonlinear Galerkin Method for Solving the Nonlinear Stationary Problem of Flow around Body of Revolution

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Abstract: In the paper the steady flow of viscous incompressible fluid around a body of revolution is considered. The mathematical model of the process under consideration is the external boundary value problem for the stream function. For solving this problem a numerical method is proposed. The method is based on the joint use of the R-functions by V.L. Rvachev and the nonlinear Galerkin method. With the help of the R-functions, the problem solution structure is constructed. The structure exactly satisfies all the boundary conditions of the problem and has the necessary behavior at infinity. To approximate the uncertain components of the structure, the nonlinear Galerkin method is used. A computational experiment was carried out for the problem of the flow around a sphere, two touching, and two jointed spheres at different Reynolds numbers.

Keywords: *steady flow; viscous incompressible fluid; external boundary value problem; stream function; R-functions method; nonlinear Galerkin method.*

Mathematics Subject Classification (2010): 65N30, 76D05, 76D17.

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1 Introduction

Let us consider the nonlinear steady-state problem of the viscous incompressible fluid flow past a body of revolution in a spherical coordinate system [11, 16]:

$$\nu E^2\psi = \frac{1}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial E\psi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial E\psi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\left(2\text{ctg}\theta\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial\psi}{\partial\theta}\right)E\psi \text{ in } \Omega, \quad (1)$$

$$\psi|_{\partial\Omega} = 0, \left.\frac{\partial\psi}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0, \quad (2)$$

$$\lim_{r\to+\infty}\psi\cdot r^{-2} = \frac{1}{2}U_\infty\sin^2\theta, \quad (3)$$

where $\nu = \text{Re}^{-1}$ is the coefficient of viscosity, Re is the Reynolds number, $\psi = \psi(r,\theta)$ is the stream function, $E\psi = \dfrac{\partial^2\psi}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial\psi}{\partial\theta}\right)$, $E^2\psi = E\left(E\psi\right)$, $\mathbf{n}$ is the outer normal to $\partial\Omega$, U_∞ is the unperturbed fluid velocity at infinity.

The methods of solving problem (1) – (3) have not been sufficiently developed. This is due to the fourth order and nonlinearity of equation (1), as well as the unboundedness of the region in which equation (1) is considered.

Mathematical modeling is becoming an increasingly effective tool for researchers in the study of viscous fluid dynamics. The need to model such flows arises, for example, in hydrodynamics, thermal energy, chemical kinetics, biomedicine, radio electronics, etc. [2,11,14–16]. Due to using a computer, one can obtain an overall picture of the entire fluid flow and graphically visualize the velocity, pressure, or temperature fields throughout the flow region.

The purpose of the paper is to develop a new method of mathematical modeling for the nonlinear stationary problem of the flow of viscous incompressible fluid around a body of revolution on the basis of the R-functions method and nonlinear Galerkin method.

The use of the R-functions method [17, 18] to construct the boundary value problem solution structure will allow us to accurately take into account the geometric and analytical information included into the statement of the problem. Using further the nonlinear Galerkin method [6, 13] to approximate the uncertain components of the structure will allow us to obtain an approximate solution in an analytical form.

2 R-Functions Method

The R-functions method applied to hydrodynamics problems of viscous fluid (steady and unsteady flows) in bounded domains or in the presence of helical symmetry was used in [1, 3, 12]. The problems of the steady flow of viscous fluid past bodies of revolution were solved using the R-functions method in [4,5,7–10], but there the authors considered the slow flow of viscous incompressible fluid past bodies (the Stokes linearization) or the application of the R-functions method, successive approximations and Galerkin-Petrov method for calculating the axisymmetric steady flows of viscous incompressible fluid.

To apply the R-functions method to the problems of hydrodynamics it is necessary:

1) To construct such a function that is equal to zero at the boundary points, positive inside the region and whose normal derivative (in the direction of the outer normal)

on the boundary is equal to -1. It will allow to accurately describe analytically the geometry of the computational domain and to continue the functions and operators, defined on the boundary, at the interior points of the area.

2) To construct the general structure of the solution, i.e., such a formula that depends on some indeterminate functions and exactly satisfies all the boundary conditions of the problem for any choice of these functions.

3) To construct an approximate solution by approximating the undefined functions included in the structure by the chosen numerical method.

Let us consider the general principles of the R-functions method theory [9, 17, 18].

Definition 2.1 A function whose sign is completely determined by the signs of its arguments is called an R-function (V.L. Rvachev's function) corresponding to the partition of the numerical axis into intervals $(-\infty, 0)$ and $[0, +\infty)$, i.e., a function $z = f(x, y)$ is called the R-function if there exists a Boolean function F such that $S[z(x, y)] = F[S(x), S(y)]$, where $S(x) = \begin{cases} 0, x < 0, \\ 1, x \geq 0 \end{cases}$ is a two-valued predicate. In this case, the Boolean function F is called a companion function.

Each R-function is associated with a Boolean function. It allows us to use logic algebra methods to describe complex geometric objects.

The following system $\Re_\alpha$ is the most commonly used system of the R-functions:

$$\bar{x} \equiv -x,$$

$$x \wedge_\alpha y \equiv \frac{1}{1+\alpha}\left(x + y - \sqrt{x^2 + y^2 - 2\alpha xy}\right),$$

$$x \vee_\alpha y \equiv \frac{1}{1+\alpha}\left(x + y + \sqrt{x^2 + y^2 - 2\alpha xy}\right),$$

where $-1 < \alpha(x, y) \leq 1$, $\alpha(x, y) \equiv \alpha(y, x) \equiv \alpha(-x, y) \equiv \alpha(x, -y)$. Their companion Boolean functions are, respectively, negation, conjunction and disjunction.

Suppose that a geometric object Ω with a piecewise-smooth boundary $\partial\Omega$ is given in $\mathbb{R}^2$. Let us assume that Ω can be constructed from auxiliary (supporting) loci $\Sigma_1 = \{\omega_1(x, y) \geq 0\}$, ..., $\Sigma_m = \{\omega_m(x, y) \geq 0\}$ according to the logical rules defined by the Boolean function F, by means of the operations of union, intersection, and complement:

$$\Omega = F(\Sigma_1, \Sigma_2, \ldots, \Sigma_m),$$

and all functions $\omega_i(x, y)$ $(i = 1, 2, ..., m)$ are elementary. Replacing Ω by $\omega(x, y)$, Σ_i by $\omega_i(x, y)$ $(i = 1, 2, ..., m)$, and the symbols $\{\cap, \cup, \neg\}$ by the R-operations symbols $\{\wedge_\alpha, \vee_\alpha, ^-\}$, we obtain an analytic expression that defines in the elementary functions the equation of the boundary $\omega(x, y) = 0$. In this case, $\omega(x, y) > 0$ for the interior points of the region, and $\omega(x, y) < 0$ for the external points.

Thus, the equation $\omega(x, y) = 0$ in an implicit form determines the locus of points representing the boundary $\partial\Omega$ of the domain Ω, and the function $\omega(x, y) = 0$ has the form of a single analytic expression.

Definition 2.2 The equation $\omega(x, y) = 0$ is called normalized to the n-th order if

$$\omega|_{\partial\Omega} = 0, \ \left.\frac{\partial \omega}{\partial \mathbf{n}}\right|_{\partial\Omega} = -1, \ \left.\frac{\partial^l \omega}{\partial \mathbf{n}^l}\right|_{\partial\Omega} = 0 \ (l = 2, 3, ..., n),$$

where $\mathbf{n}$ is a vector of the outer normal to $\partial\Omega$.

The equation $\omega(x, y) = 0$, normalized to the first order, can be obtained from the equation $\omega_1(x, y) = 0$ as follows.

Theorem 2.1 *If* $\omega_1(x, y) \in C^m(\mathbb{R}^2)$ *satisfies the conditions* $\omega_1|_{\partial\Omega} = 0$ *and* $\left.\frac{\partial \omega_1}{\partial \mathbf{n}}\right|_{\partial\Omega} > 0$, *then the function* $\omega \equiv \frac{\omega_1}{\sqrt{\omega_1^2 + |\nabla\omega_1|^2}} \in C^{m-1}(\mathbb{R}^2)$ *satisfies the conditions* $\omega|_{\partial\Omega} = 0$, $\left.\frac{\partial \omega}{\partial \mathbf{n}}\right|_{\partial\Omega} = -1$ *at all regular points of the boundary* $\partial\Omega$.

To construct the equation normalized to the first order, one can also use the formula

$$\omega \equiv \frac{\omega_1}{|\nabla\omega_1|}$$

if $|\nabla\omega_1| \neq 0$ in $\overline{\Omega} = \Omega \cup \partial\Omega$.

Let us construct the normalized equation $\omega(x, y) = 0$ of the boundary of the closed area $\bar{\Omega} = \left\{0 \leq x \leq 3, \ 3 - \sqrt{9 - x^2} \leq y \leq 3\right\}$ with the help of the system $\Re_0$ (Figure 1).

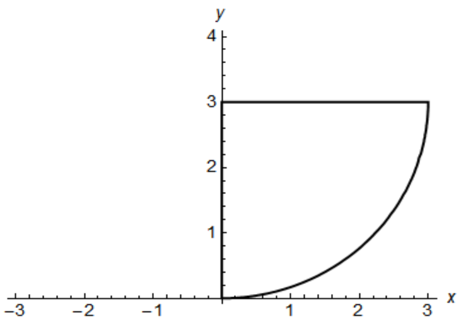

Figure 1: The area $\bar{\Omega}$.

The area $\bar{\Omega}$ can be constructed from the following primitive regions:

- the interior of a circle of radius 3 centered at the point $(0, 3)$: $\Sigma_1 = \left(\frac{1}{6}\left(9 - x^2 - (y-3)^2\right) \geq 0\right)$,
- the half-plane below the line $y = 3$: $\Sigma_2 = (3 - y \geq 0)$,
- the half-plane to the right of the line $x = 0$: $\Sigma_3 = (x \geq 0)$.

Then $\bar{\Omega} = \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3$ and the equation of the boundary of the area Ω is determined by the equation $\omega(x, y) = 0$, where

$$\begin{gathered}\omega(x,y) = \left[\frac{1}{6}\left(9 - x^2 - (y-3)^2\right)\right] \wedge_0 [3-y] \wedge_0 x = \\ = \left[\frac{1}{6}\left(9 - x^2 - (y-3)^2\right)\right] \wedge_0 \left[3 - y + x - \sqrt{(3-y)^2 + x^2}\right] = \\ = \frac{1}{6}\left(9 - x^2 - (y-3)^2\right) + 3 - y + x - \sqrt{(3-y)^2 + x^2} - \\ -\sqrt{\left[\frac{1}{6}\left(9 - x^2 - (y-3)^2\right)\right]^2 + \left[3 - y + x - \sqrt{(3-y)^2 + x^2}\right]^2}.\end{gathered} \tag{4}$$

The contour lines of the obtained normalized boundary equation (4) are shown in Figure 2.

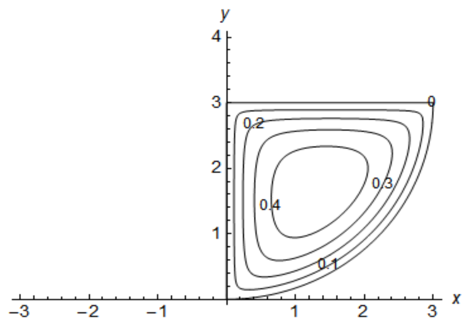

Figure 2: The area $\bar{\Omega}$.

The constructed function (4) is positive inside the area Ω and negative outside Ω. If it is necessary to obtain a function that is positive in the exterior of the finite area Ω, then it is required to use the function $-\omega(x, y)$.

Let us consider the problem

$$Au = f, \tag{5}$$

$$L_i u|_{\partial\Omega_i} = \varphi_i, \ i = \overline{1, m}, \tag{6}$$

where A and L_i are some differential operators; f and φ_i are functions defined inside the region Ω and on its boundary regions $\partial\Omega_i$.

Definition 2.3 The expression $u = B(\Phi, \omega, \{\omega_i\}_{i=1}^m, \{\varphi_j\}_{j=1}^m)$ is called the general structure of the solution of the boundary value problem (5) – (6) if it exactly satisfies the boundary conditions (6) for any choice of the indeterminate component Φ. Here, B is an operator that depends on the geometry of the area Ω and parts $\partial\Omega_i$ of its border, as well as the operators of the boundary conditions, but does not depend on the type of the operator A and the function f.

The solution structure extends the boundary conditions inside the region.

The undefined component Φ of the solution structure in the R-functions method is represented as a sum

$$\Phi(x,y) \approx \Phi_n(x,y) = \sum_{k=1}^{n} c_k \varphi_k(x,y),$$

where $\varphi_k(x,y)$ are known elements of the complete functional sequence, and c_k $(k = 1, 2, ..., n)$ are unknown coefficients of the expansion. To determine unknown coefficients one can use, for example, variational methods (Ritz, least squares, etc.), projection methods (Galerkin, collocations, etc.), grid methods and others.

3 The Method for Solving Problem (1) – (3)

For an exact analytical description of the geometry of computational domain, let us introduce a function $\omega(r,\theta)$ satisfying the conditions:

$$\text{a)}\ \ \omega(r,\theta) > 0 \text{ in } \Omega, \quad \text{b)}\ \ \omega(r,\theta)|_{\partial\Omega} = 0, \quad \text{c)}\ \ \left.\frac{\partial \omega}{\partial \mathbf{n}}\right|_{\partial\Omega} = -1,$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega$.

Let us introduce the function [3]

$$\omega_M = f_M(\omega) = \begin{cases} 1 - \exp\dfrac{M\omega}{\omega - M}, & 0 \le \omega < M; \\ 1, & \omega \ge M \ (M = \text{const} > 0), \end{cases} \tag{7}$$

that satisfies the conditions:

$$1)\ \omega_M > 0 \text{ in } \Omega,\ \ 2)\ \omega_M|_{\partial\Omega} = 0,\ \ 3)\ \left.\frac{\partial \omega_M}{\partial \mathbf{n}}\right|_{\partial\Omega} = -1,\ \ 4)\ \omega_M \equiv 1 \text{ if } \omega_M \ge M.$$

The introduction of the function (7) allows us to carry out calculations in the finite region since function (7) differs from unity only in some finite ring-shaped region $\{0 \le \omega(r,\theta) < M\}$ adjacent to the contour $\partial\Omega$.

Let us construct the general structure of the solution. In [7, 9, 10] it was proved that for any choice of sufficiently smooth functions Φ_1 and Φ_2 ($\Phi_1 \cdot r^{-2} \to 0$ as $r \to +\infty$) the boundary conditions (2) and the condition at infinity (3) are exactly satisfied by a function of the form

$$\psi = \omega_M^2(\psi_0 + \Phi_1) + \omega_M^2(1 - \omega_M)\Phi_2, \tag{8}$$

where $\psi_0 = \dfrac{1}{4} U_\infty (r - R)^2 \left(2 + \dfrac{R}{r}\right) \sin^2\theta$ is the Stokes solution for the problem of the flow past a sphere of radius R (the sphere of radius R lies entirely inside the streamlined body). Thus, the function (8) is the structure of the solution of the boundary value problem (1) – (3).

Let us construct an approximate solution by approximating the undefined components Φ_1 and Φ_2 of structure (8) by the nonlinear Galerkin method [6, 13]. The functions Φ_1 and Φ_2 will be presented in the form

$$\Phi_1 \approx \Phi_1^{m_1} = \sum_{k=1}^{m_1} \alpha_k \cdot \varphi_k,\ \ \Phi_2 \approx \Phi_2^{m_2} = \sum_{j=1}^{m_2} \beta_j \cdot \tau_j,$$

where

$$\{\varphi_k(r,\theta)\} = \left\{r^{1-k}\, J_k(\cos\theta),\ k=2,3,...;\ r^{3-k}\, J_k(\cos\theta),\ k=4,5,...\right\}$$

is a complete system of particular solutions of the equation $E^2\psi = 0$ with respect to the exterior of a sphere of finite radius;

$$\{\tau_j(r,\theta)\} = \left\{rJ_2(\cos\theta),\ J_3(\cos\theta),\ r^j\, J_j(\cos\theta,\ r^{j+2}\, J_j(\cos\theta),\ j=2,3,...\right\}$$

is a complete system of particular solutions of the equation $E^2\psi = 0$ relative to the domain $\{\omega(r,\theta) < M\}$, $J_k(\cos\theta)$ are the Gegenbauer functions of the first kind.

Thus, the approximate solution of the problem (1) – (3) is sought in the form

$$\psi_N = \omega_M^2 \left(\frac{1}{4}U_\infty (r-R)^2 \left(2+\frac{R}{r}\right)\sin^2\theta + \sum_{k=1}^{m_1}\alpha_k\cdot\varphi_k\right) + \omega_M^2(1-\omega_M)\cdot\sum_{j=1}^{m_2}\beta_j\cdot\tau_j.$$

The complete with respect to the whole plane sequence of functions has the form

$$\{\phi_i(r,\theta)\} = \left\{\omega_M^2(r,\theta)\varphi_k(r,\theta), \omega_M^2(r,\theta)(1-\omega_M(r,\theta))\tau_j(r,\theta)\right\}. \tag{9}$$

The values of the coefficients α_k $(k=1,2,...,m_1)$ and β_j $(j=1,2,...,m_2)$ in accordance with the nonlinear Galerkin method [6,13] will be found from the condition of the residual orthogonality to the first N $(N=m_1+m_2)$ elements of the sequence (9):

$$\left(\nu E^2\psi_N - \frac{1}{r^2\sin\theta}\left(\frac{\partial\psi_N}{\partial\theta}\frac{\partial E\psi_N}{\partial r} - \frac{\partial\psi_N}{\partial r}\frac{\partial E\psi_N}{\partial\theta}\right) - \right.$$
$$\left. -\frac{1}{r^2\sin\theta}\left(2\mathrm{ctg}\theta\frac{\partial\psi_N}{\partial r} - \frac{2}{r}\frac{\partial\psi_N}{\partial\theta}\right)E\psi_N,\ \phi_i\right) = 0,\ i=\overline{1,N}.$$

As a result, a system of nonlinear equations is obtained, where each equation is a quadratic function with respect to α_k and β_j. This system can be solved by the Newton method. As an initial approximation, a set of α_k and β_j is chosen corresponding to the solution of the Stokes problem, or, for large Reynolds numbers, to the solution obtained for smaller Reynolds numbers.

4 Computational Experiment

A computational experiment was carried out for the problems of the flow around a sphere, two touching, and two jointed spheres. The double integrals in the systems for determining α_k and β_j were taken approximately by the Gauss formula with 50 nodes for each variable.

4.1 First problem

The problem of the flow past a sphere $x^2+y^2+z^2=1$ at $U_\infty = 1$, $M=10$, $m_1=10$, $m_2=14$, Re $=10$; 20; 25, is solved.

The normalized equation of the boundary (in the plane $\varphi=0$) has the form

$$\omega(x,z) = \frac{1}{2}(1-x^2-z^2) = 0.$$

The streamlined contours of the obtained approximate solution are shown in Figure 3. Figure 4 shows detailed pictures of the streamlined contours and vector fields of velocities behind the sphere. For small Reynolds numbers, the flow around a sphere is symmetrical, without the formation of a detachment zone in the aft region of the body. With an increase in the Reynolds number to approximately $20-25$, the secondary vortices appear behind the body and then their size and intensity increase.

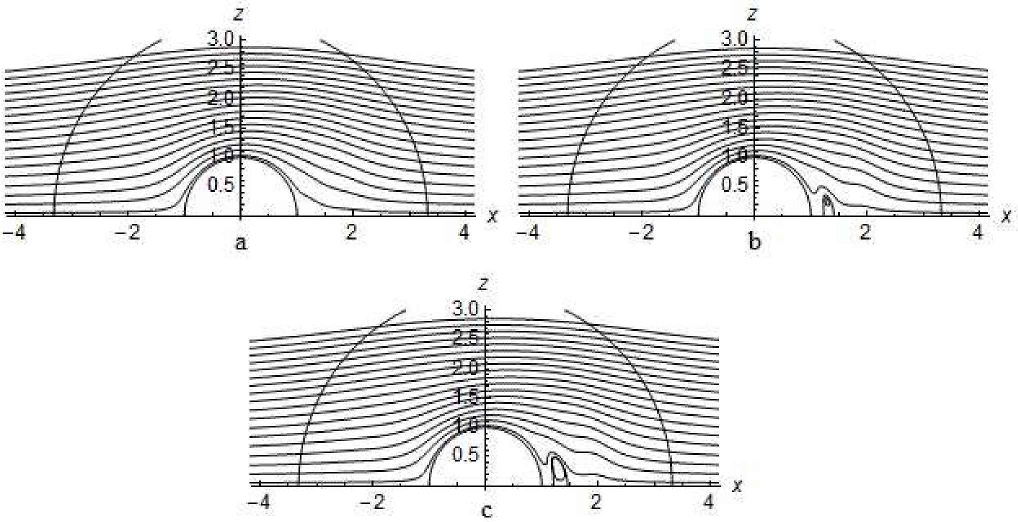

Figure 3: The streamlined contours for the problem of the flow past a sphere: (a) Re = 10, (b) Re = 20, (c) Re = 25.

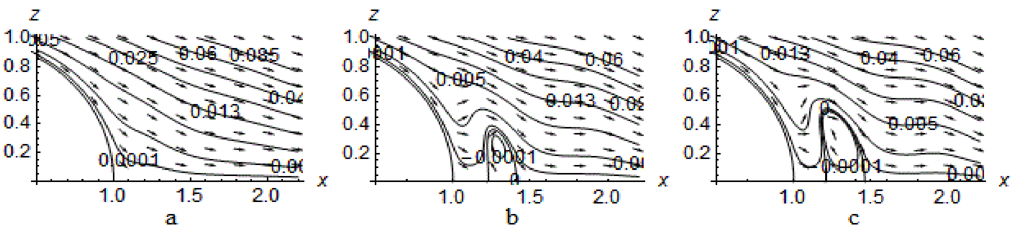

Figure 4: Detailed pictures of streamlined contours and vector fields of velocities behind the sphere: (a) Re = 10, (b) Re = 20, (c) Re = 25.

The results obtained are in good agreement with the results obtained by the method of successive approximations [8] (for Re ≤ 10), known results of physical experiments [20] and results obtained by other authors [2, 19], which indicates the effectiveness of the developed numerical method.

4.2 Second problem

The problem of the flow past two touching spheres, bounded by surfaces $(x-1)^2+y^2+z^2=1$, $(x+1)^2+y^2+z^2=1$, at $U_\infty=1$, $M=10$, $m_1=10$, $m_2=14$, $\mathrm{Re}=30;\ 60;\ 70$, is solved.

The normalized equation of the boundary (in the plane $\varphi=0$) has the form

$$\omega(x,z)=\left[\frac{1}{2}\left(1-(x-1)^2-z^2\right)\right]\wedge_0\left[\frac{1}{2}\left(1-(x+1)^2-z^2\right)\right]=0.$$

The streamlined contours of the obtained approximate solution are shown in Figure 5. Figure 6 shows detailed pictures of the streamlined contours and vector fields of velocities behind the spheres and in the hollow between them. The computational experiment showed that as the Reynolds number increases to approximately 60, vortices appear behind the body.

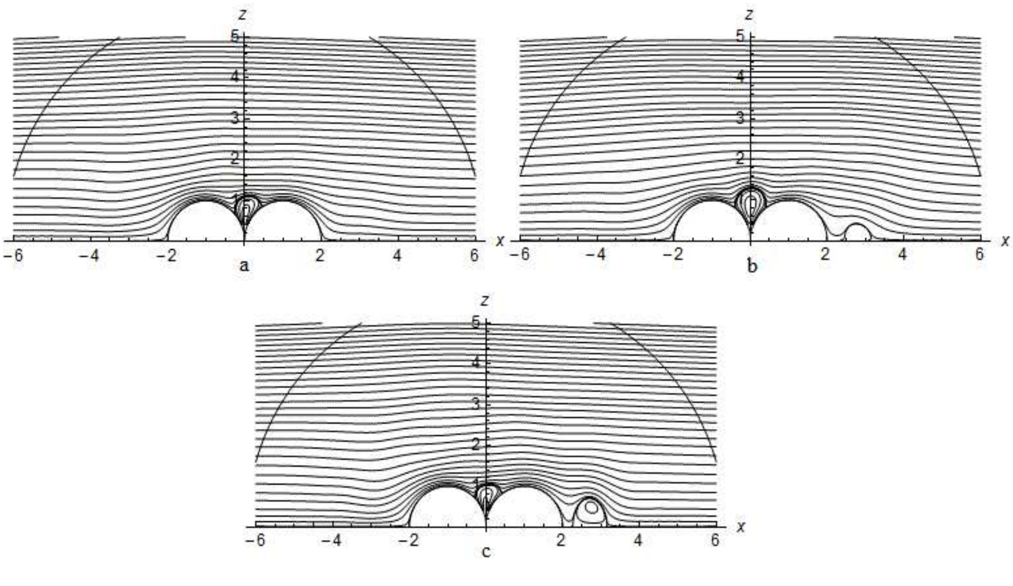

Figure 5: The streamlined contours for the problem of the flow past two touching spheres: (a) $\mathrm{Re}=30$, (b) $\mathrm{Re}=60$, (c) $\mathrm{Re}=70$.

4.3 Third problem

The problem of the flow past two jointed spheres, bounded by surfaces $\left(x-\frac{1}{2}\right)^2+y^2+z^2=1$, $\left(x+\frac{1}{2}\right)^2+y^2+z^2=1$, at $U_\infty=1$, $M=10$, $m_1=10$, $m_2=14$, $\mathrm{Re}=5;\ 10;\ 30$, is solved.

The normalized equation of the boundary (in the plane $\varphi=0$) has the form

$$\omega(x,z)=\left[\frac{1}{2}\left(1-\left(x-\frac{1}{2}\right)^2-z^2\right)\right]\wedge_0\left[\frac{1}{2}\left(1-\left(x+\frac{1}{2}\right)^2-z^2\right)\right]=0.$$

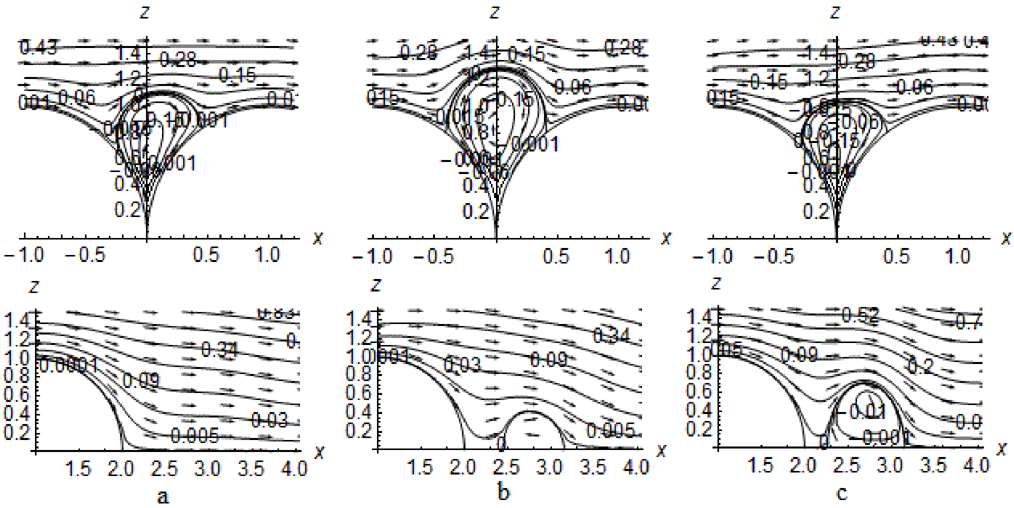

Figure 6: Detailed pictures of streamlined contours and vector fields of velocities behind the spheres in the hollow between them: (a) Re = 30, (b) Re = 60, (c) Re = 70.

The streamlined contours of the obtained approximate solution are shown in Figure 7. Figure 8 shows detailed pictures of the streamlined contours and vector fields of velocities behind the spheres. The computational experiment showed that as the Reynolds number increases to approximately 10, vortices appear behind the body.

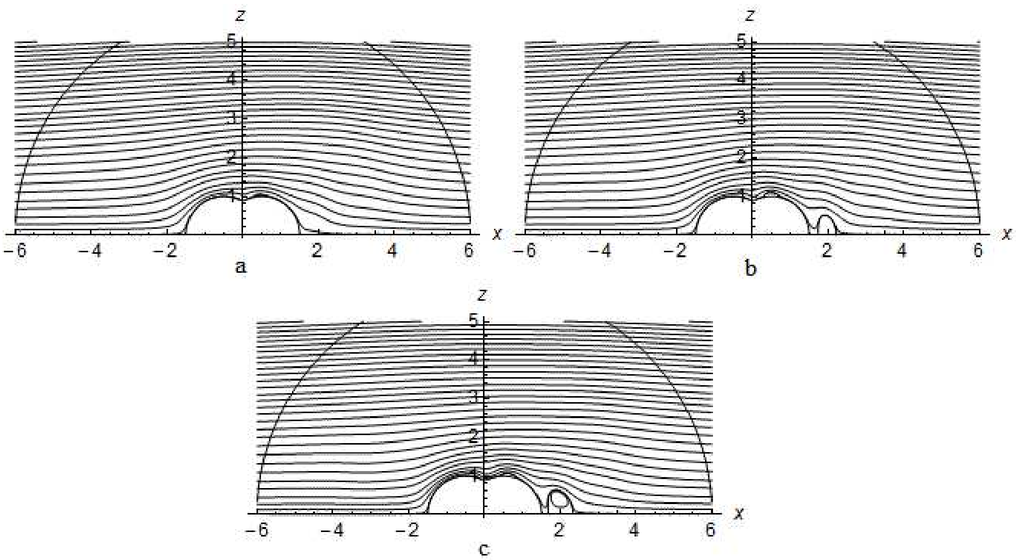

Figure 7: The streamlined contours for the problem of the flow past two jointed sphere: (a) Re = 5, (b) Re = 10, (c) Re = 30.

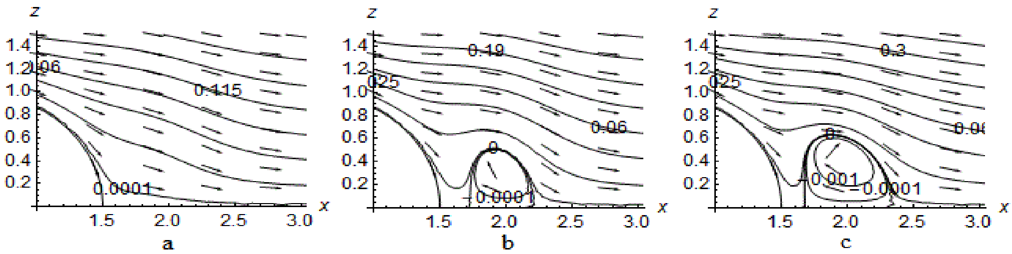

Figure 8: Detailed pictures of streamlined contours and vector fields of velocities behind the spheres: (a) Re = 5, (b) Re = 10, (c) Re = 30.

5 Conclusions

A new numerical method for solving the problem of the flow of viscous incompressible fluid past a body of revolution is proposed based on the joint application of the R-function method and the nonlinear Galerkin method. The advantage of the proposed method is that the method algorithm does not change when the domain geometry is changed, and the solution structure accurately takes into account both the boundary conditions on the boundary of the streamlined body and the condition at infinity. For various Reynolds numbers, the stationary problem of the flow past a body of revolution in a spherical coordinate system for a sphere, two touching, and two jointed spheres is solved numerically. The Reynolds numbers, at which secondary vortices appear behind the body, are experimentally determined for each body.

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