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About the Restricted Three-Body Problem with the Schwarszchild-de Sitter Potential

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Abstract: In this paper the restricted three body problem in the context of Schwarszchild-de Sitter's space-time is studied. The equations of motion that govern the bodies are derived using the Schwarszchild-de Sitter metric, by introducing a set known as the parameter domain, the existence of equilibrium points for any element of this set is shown. The stability conditions for the orbital motion of the system are established by the analysis of the eigenvalues of the linearized system.

Keywords: *restricted three body problem; Schwarszchild-de Sitter potential; relative equilibria; linear stability.*

Mathematics Subject Classification (2010): 70F15, 70F07.

1 Introduction

A de Sitter universe is an exact solution to the Einstein field equations of general relativity, named after Willem de Sitter. Setting the foundations of a particular cosmological universe, which is characterized as spatially flat and neglects ordinary matter, thus, the dynamics of the universe is dominated by a positive cosmological constant [7], or equivalent, de Sitter solution corresponds to a metric of a space-time of constant curvature. When the curvature is negative, the cosmological constant is too, and the corresponding universe is called anti-de Sitter space. In both cases, the metric corresponds to a general symmetry of Einsteins field equations, see Brinkmann's theorem [6]. The current observations indicate that the universe is expanding in an accelerated rate, and may approach de Sitter space asymptotically, that is, the concordance models of physical cosmology are converging on a consistent model that is best described as a de Sitter universe. See

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Carroll [3] and Zwicky [14] for a preliminary introduction, and [8] for a more detailed description and a consistent mathematical deduction.

Under the assumptions of this universe, we present a study of the Lagrangian triangular equilibria in the planar restricted three body problem, where the primaries are homogeneous spheroids rotating around their axis of symmetry and whose equatorial planes coincide throughout their motion. We follow closely the work of Arredondo *et al.* [1] for the Schwarszchild potential and the reference found there [9], but with the new ingredient of a potential associated to a more general metric, that is, in terms of relativistics effects, a new physical universe endowed with other qualities [4]. On the other hand, we introduce a new algebraic idea to give an analytical proof of the existence and uniqueness of a Lagrangian equilibrium, while as usual, linear stability of this equlibria is studied numerically.

2 Schwarszchild-de Sitter Potential

The Schwarzschild metric is the simplest solution of Einstein's equation with zero cosmological constant, while a de Sitter space is the simplest solution when a positive cosmological constant is considered [2], but both are obtained from considering a spherical symmetry [8]. As described in [10], a de Sitter-Schwarzschild space-time is just a combination of the two, and we can imagine it as the horizon of a black hole that is centered in a universe with de Sitter properties, which from the mathematical point of view, is properly described as a Riemannian space with one independent component of its curvature tensor. All the discussion behind this object and its beautiful developments can be found in Theorems 8.10 to 8.15 of [12]. For the purpose of this paper we just have to establish that the Schwarszchild-de Sitter metric is given by

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3}r^2\right)dt^2 - \left(1 - \frac{2GM}{c^2 r} - -\frac{\Lambda}{3}r^2\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where G is the universal gravitational constant, M is the mass of the filed source, c is the speed of light and Λ is the cosmological constant. It is known that the associated potential to this metric is given by the time-time component of the metric

$$U(r) = \frac{-(c^2 + g_{00})}{2} = \frac{k}{r} + \frac{B}{r^3} + Cr^2, \tag{2}$$

where $k = GM$, $C = \dfrac{\Lambda c^2}{6}$ and $B = \dfrac{GML^2}{c^2}$ (see [3] and [10] for details).

3 Approach to the Restricted Problem

Let us consider two bodies, m_1 and m_2, that interact mutually under the Schwarszchild-de Sitter potential, describing a circular orbit, and m_3 be the mass of a body with spherical symmetry such that $m_1, m_2 >> m_3$. Also, we assume that the center of mass of m_1, m_2 is fixed at the origin. As we consider m_1 and m_2 source of the potential of type (2), that we rewrite as

$$U(r) = G\frac{m_1 m_2}{r}\left(1 + \frac{B_1 + B_2}{r^2} + (C_1 + C_2)r^3\right), \tag{3}$$

the interaction among masses m_1 and m_2 is given by the equation

$$\left(\frac{m_1m_2}{m_1+m_2}\right)\ddot{R}=-\frac{dU(R)}{dR}=-\frac{d}{dR}\left(\frac{Gm_1m_2}{R}\left(1+\frac{B_1+B_2}{R^2}+(C_1+C_2)R^3\right)\right),$$

i.e.,

$$\left(\frac{m_1m_2}{m_1+m_2}\right)\ddot{R}=-\frac{Gm_1m_2}{R}\left(1+\frac{3(B_1+B_2)}{R^3}-2R^2(C_1+C_2)\right).$$

As it is supposed that m_1, m_2 are in an orbit with uniform circular movement, we have (R_0, ω). This is equivalent to finding the equilibrium points of the increased potential or effective potential [5]. Doing a rescaling, we consider $Gm_1m_2 = 1$; then the increased potential will be defined by

$$U_{aug}(R)=-\frac{1}{r}\left(1+\frac{B_1+B_2}{r^2}+(C_1+C_2)r^3\right)+\frac{r^2\omega^2}{2} \tag{4}$$

and the effective potential as

$$U_{eff}(r)=-\frac{1}{r}\left(1+\frac{B_1+B_2}{r}+(C_1+C_2)r^3\right)+\frac{L^2}{2r^2}. \tag{5}$$

Remember that equilibrium points are critical ones in the effective potential. So, operating and making $R = 1$, we have

$$\omega=\sqrt{1+3(B_1+B_2)-2(C_1+C_2)}. \tag{6}$$

Now, to guarantee orbit's stability, we use the fact that a critical point is further a minimal potential, namely, $U''_{eff}(R)|_{R=1} > 0$.

$$U''_{eff}(R)|_{R=1}=\Big[-\frac{2}{R^3}-12\frac{B_1+B_2}{R^4}-2(C_1+C_2)+\frac{3L^2}{R^4}\Big]_{R=1}>0, \tag{7}$$

and replacing (6) in (7) we get

$$-2-12(B_1+B_2)-2(C_1+C_2)+3(1+3(B_1+B_2)-2(C_1+C_2))>0.$$

$$1>3(B_1+B_2)+8(C_1+C_2). \tag{8}$$

In the other way, the expression inside the root of (6) must be positive. So, another constraint for the coefficients is

$$1+3(B_1+B_2)\geq 2(C_1+C_2). \tag{9}$$

With (8) and (9), it is possible to uncouple one pair of the coefficients:

$$\frac{1}{5}>C_1+C_2. \tag{10}$$

Also, in (8), since C_1 and C_2 are always non-negative, the other pair of coefficients is uncoupled:

$$\frac{1}{3}>B_1+B_2. \tag{11}$$

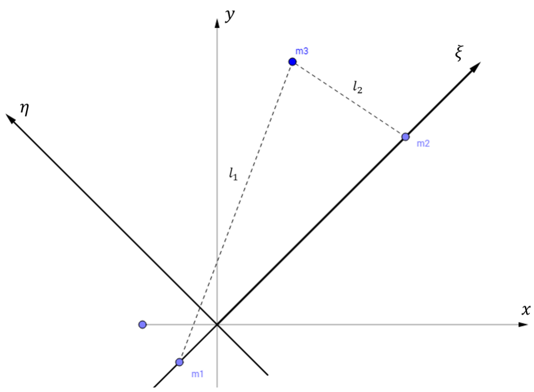

Figure 1: Representation of the restricted three body problem in the non-inertial system.

A particle's Hamiltonian in a central field is given by $H(r, \dot{r}) = \dfrac{1}{2}m\dot{r}^2 - U(r)$, then the Hamiltonian of m_3 in the inertial reference system is

$$H(r, \dot{r}) = \frac{1}{2}m\dot{r}^2 - \frac{(1-\mu)}{l_1}\left(1 + \frac{B_1}{l_1^2} + C_1 l_1^3\right) - \frac{\mu}{l_2}\left(1 + \frac{B_2}{l_2^2} + C_2 l_2^3\right), \tag{12}$$

where

$$l_1 = \sqrt{(\xi + \mu)^2 + \eta^2} \tag{13}$$

and

$$l_2 = \sqrt{(\xi + \mu - 1)^2 + \eta^2} \tag{14}$$

are the distances from the masses m_1, m_2 to the mass m_3, respectively.

Now, we name $m_1 = \mu$, located on ξ_1; and $m_2 = 1 - \mu$, located on ξ_2. In this order, $\mu \leq \frac{1}{2}$, $\xi_1 - \xi_2 = 1$ and $\mu\xi_2 + (1-\mu)\xi_1 = 0$. So, $\xi_1 = -\mu$ and $\xi_2 = 1 - \mu$. Also,

$$m_1 = \begin{cases} x = -\mu\cos(\omega t), \\ y = -\mu\sin(\omega t), \end{cases} \tag{15}$$

and

$$m_2 = \begin{cases} x = (1-\mu)\cos(\omega t), \\ y = (1-\mu)\sin(\omega t), \end{cases} \tag{16}$$

as in Figure 1.

Consider (ξ, η) as the coordinates of m_3 in the non-inertial system; therefore, the interaction between the mases m_1 and m_2 with m_3 is given by the following potential:

$$U_{m_3}(\xi, \eta) = \frac{(1-\mu)}{l_1}\left(1 + \frac{B_1}{l_1^2} + C_1 l_1^3\right) + \frac{\mu}{l_2}\left(1 + \frac{B_2}{l_2^2} + C_2 l_2^3\right), \tag{17}$$

and the Hamiltonian for m_3 in the non-inertial system is

$$H(\xi, \eta, P_\xi, P_\eta) = \frac{1}{2}(P_\xi^2 + P_\eta^2) + \omega(P_\xi\eta - P_\eta\xi) - U_{m_3}(\xi, \eta). \tag{18}$$

Apply Hamilton's motion equations

$$\frac{\partial H}{\partial P_\xi} = P_\xi + \omega\eta = \omega\dot{\xi}, \tag{19}$$

$$\frac{\partial H}{\partial P_\eta} = P_\eta - \omega\xi = \omega\dot{\eta}. \tag{20}$$

Multiply the equation (19) by ω and derive it with respect to time, knowing that $\dot{\omega} = 0$, since the circular movement is uniform:

$$\omega \dot{P}_\xi = \omega^2(\ddot{\xi} - \dot{\eta}),$$

$$\boxed{\dot{P}_\xi = \omega(\ddot{\xi} - \dot{\eta}).} \tag{21}$$

In an analogous way, multiply the equation (20) by ω and derive it with respect to time:

$$\boxed{\dot{P}_\eta = \omega(\ddot{\eta} + \dot{\xi}).} \tag{22}$$

Before continuing, the partial derivatives of U_{m_3} are going to be calculated, in order to facilitate the calculus of the other two Hamilton's motion equations:

$$\begin{aligned} \frac{\partial U_{m_3}(\xi,\eta)}{\partial \xi} &= (1-\mu)\frac{\partial l_1}{\partial \xi}\Big(-\frac{1}{l_1^2} - \frac{3B_1}{l_1^4} + 2C_1 l_1\Big) + \mu\frac{\partial l_2}{\partial \xi}\Big(-\frac{1}{l_2^2} - \frac{3B_2}{l_2^4} + 2C_2 l_2\Big), \\ &= -\frac{(1-\mu)(\xi+\mu)}{l_1^3}\Big(1 + \frac{3B_1}{l_1^2} - 2C_1 l_1^3\Big) - \frac{\mu(\xi+\mu-1)}{l_2^3}\Big(1 + \frac{3B_2}{l_2^2} - 2C_2 l_2^3\Big), \end{aligned} \tag{23}$$

on the other hand,

$$\frac{\partial U_{m_3}(\xi,\eta)}{\partial \eta} = -\eta\Big[\frac{(1-\mu)}{l_1^3}\Big(1 + \frac{3B_1}{l_1^2} - 2C_1 l_1^3\Big) + \frac{\mu}{l_2^3}\Big(1 + \frac{3B_2}{l_2^2} - 2C_2 l_2^3\Big)\Big]. \tag{24}$$

By the last two Hamilton's motion equations we have

$$\frac{\partial H}{\partial \xi} = -\omega \dot{P}_\xi, \tag{25}$$

$$\frac{\partial H}{\partial \eta} = -\omega \dot{P}_\eta. \tag{26}$$

Replacing (18) in these equations we get

$$-\omega P_\eta - \frac{\partial U_{m_3}}{\partial \xi} = -\omega \dot{P}_\xi . \omega P_\xi - \frac{\partial U_{m_3}}{\partial \eta} = -\omega \dot{P}_\eta. \tag{27}$$

Therefore, using (21) and (22) in the last couple of equations, it is obtained that

$$\omega^2(\ddot{\xi} - \dot{\eta}) = \omega P_\eta + \frac{\partial U_{m_3}}{\partial \xi}, \tag{28}$$

$$\omega^2(\ddot{\eta} + \dot{\xi}) = -\omega P_\xi + \frac{\partial U_{m_3}}{\partial \eta}. \tag{29}$$

Now, with the centrifuge potential

$$\boxed{\Omega(\xi,\eta) = \frac{\omega^2}{2}(\xi^2 + \eta^2) + U_{m_3}(\xi,\eta),} \tag{30}$$

one can find the critical points of m_3 by deriving it with respect to ξ and μ and making it equal to zero. Before doing that, one should consider the following equations:

$$\frac{\partial \Omega}{\partial \xi} = \omega^2(\ddot{\xi} - 2\dot{\eta}), \tag{31}$$

$$\frac{\partial \Omega}{\partial \eta} = \omega^2(\ddot{\eta} + 2\dot{\xi}). \tag{32}$$

Obtain summing (28)$-\omega\cdot$(20) and $\omega\cdot$(19)+(29), respectively. With this pair of equations, it is possible to deduce that the components (ξ, η) are orthogonal between them, but this is already known because of the nature of the problem and the coordinate axis. Consequently, the relation that is going to be used to find the critical points is

$$\frac{\partial \Omega}{\partial \xi} = \frac{\partial \Omega}{\partial \eta} = 0.$$

3.1 Collinear stability points

In order to obtain the collinear stability points, the partial derivative of Ω with respect to ξ is done, and all the η are replaced by zero. This gives the stability points that are in the ξ axis. After some algebra, one obtains that

$$\begin{aligned} -\mu x^4[3B_2 + (x-1)^2] - (x-1)^4[3B_1(\mu - 1) + 2C_1x^5(\mu - 1) - 2C_2\mu x^4(x-1) \\ + \omega^2 x^4(\mu - x) - x^2(\mu - 1)] = 0, \end{aligned} \tag{33}$$

where $x = \xi + \mu$. Since (33) is a ninth grade polynom, it has at least a real solution.

3.2 Non-collinear stability points ($\eta \neq 0$)

In this case, both partial derivatives of Ω are zero, but $\eta \neq 0$, so one has two equations, the derivative with respect to ξ and η of (30). These two equations can be written as

$$0 = \frac{(1-\mu)(\xi+\mu)}{l_1^3}\Big(1+\frac{3B_1}{l_1^2} - 2C_1l_1^3 - \omega^2l_1^3\Big) + \frac{\mu(\xi+\mu-1)}{l_2^3}\Big(1+\frac{3B_2}{l_2^2} - 2C_2l_2^3 - \omega^2l_2^3\Big) \tag{34}$$

and

$$0 = \eta\Big[\frac{(1-\mu)}{l_1^3}\Big(1+\frac{3B_1}{l_1^2} - 2C_1l_1^3 - \omega^2l_1^3\Big) + \frac{\mu}{l_2^3}\Big(1+\frac{3B_2}{l_2^2} - 2C_2l_2^3 - \omega^2l_2^3\Big)\Big], \tag{35}$$

respectively, due to the fact that $(1-\mu)(\xi+\mu) + \mu(\xi+\mu-1) = \xi$. Consider l_1, l_2 as an independent system of variables, last two equations hold if and only if

$$(\omega^2 + 2C_i)l_i^5 - l_i^2 - 3B_i = 0, \tag{36}$$

for $i = 1, 2$. Since (36) has a single change of sign, by Descartes's rule of signs, each equation has exactly one positive root. The next proposition shows that these roots satisfy the triangle inequalities.

Definition 3.1 [Parameter domain] The set of all possible combinations of the non-negative parameters (B_1, B_2, C_1, C_2) that satisfy the constraints (8) – (11) will be called D.

Theorem 3.1 *For every combination in D, there exists a unique non-collinear rotating equilibrium.*

Proof. It will be shown that every possible combination of D gives positive solutions in (36) that satisfy the triangle inequalities. It can be seen that l_1 and l_2 depend on the values of the constants in D, and moreover,

$$l_1 = l_1(B_1, B_2, C_1, C_2) = l_2(B_2, B_1, C_2, C_1) = l_2 \tag{37}$$

taking advantage of the symmetry in (36). Define $\bar{D}$ as the set D with its frontier, *i.e.,*

$$\bar{D} = D \cup \delta D.$$

It is known that a differentiable real-valued function whose domain is closed and bounded attains its extreme values either at a critical point or on the boundary. In this context, the functions

$$\begin{array}{rccc} l_i : & \bar{D} & \to & \mathbb{R}, \\ & (B_1, B_2, C_1, C_2) & \to & l_i = l_i(B_1, B_2, C_1, C_2), \end{array}$$

despite of being implicitly defined, are differentiable. A direct calculation proves that l_i does not accept critical points inside $\bar{D}$, so the extreme values of it must be in the frontier. All cases are shown below [11].

1. For $B_1 = 0$,

$$l_1 = \frac{1}{\sqrt[3]{1 + 3B_2 - 2C_2}}.$$

Given the constraints for the sum of two constants, it follows that $l_1^{\min} = \sqrt[3]{\frac{1}{2}} \approx 0.79$.

2. For $B_2 = 0$, the equation (36) becomes

$$l_1^5 - \frac{1}{(1 + 3B_1 - 2C_2)} l_1^2 - \frac{3B_1}{(1 + 3B_1 - 2C_2)} = 0.$$

To find a minimum bound, notice that the last polynomial can be rearranged as

$$l_1^2 \Big(l_1^3 - \frac{1}{1 + 3B_1 - 2C_2} \Big) = \frac{3B_1}{1 + 3B_1 - 2C_2},$$

from where it is deduced that

$$l_1 \geq \frac{1}{\sqrt[3]{1 + 3B_1 - 2C_2}} \geq \sqrt[3]{\frac{1}{2}} = l_1^{\min}.$$

3. For $C_1 = 0$, the minimum value for l_1 is given by the same arguments shown in the last case, so

$$l_1^{\min} = \sqrt[3]{\frac{1}{2}}.$$

4. For $C_2 = 0$, by similar reasons as in the previous cases, it follows that

$$l_1^{\min} = \sqrt[3]{\frac{1}{2}}.$$

5. For $C_1 + C_2 = \frac{1}{5}$, equation (36) can be written as

$$0 = (1 + 3(B_1 + B_2) - 2(C_1 + C_2) + 2C_1)l_1^5 - l_1^2 - 3B_1$$
$$= (\frac{3}{5} + 3(B_1 + B_2) + 2C_1)l_1^5 - l_1^2 - 3B_1.$$

Calculating the derivative of the last polynomial expression with respect to C_1 and clearing dl_1/dC_1 yield to

$$\frac{dl_1}{dC_1} = \frac{-6l_1^5}{5(3/5 + 3(B_1 + B_2) + 2C_1)l_1^4 - 2l_1} = \frac{-6l_1^6}{3l_1^2 + 15B_1} < 0,$$

since $5(3/5+3(B_1+B_2)+2C_1)l_1^5 = 5l_2^2+15B_1$. This implies that the function $l_1(C_1)$ with its other variables fixed is decreasing on $C_1 + C_2 = 1/5$. Then its minimum is reached when C_1 is maximum. Therefore, if $C_1 = 1/5$, notice that the polynomial equation can be rearranged as

$$l_1^2\Big(l_1^3 - \frac{1}{1 + 3(B_1 + B_2)}\Big) = \frac{3B_1}{1 + 3(B_1 + B_2)},$$

from where it is deduced that

$$l_1 \geq \frac{1}{\sqrt[3]{1 + 3(B_1 + B_2)}} \geq \frac{1}{\sqrt[3]{2}} = l_1^{\min}.$$

6. For $B_1 + B_2 = \frac{1}{3}$, equation (36) becomes

$$(2 - 2C_2)l_1^5 - l_1^2 - 3B_1 = 0.$$

Differentiating it with respect to B_1 and clearing dl_1/dB_1 lead to

$$\frac{dl_1}{dB_1} = \frac{3}{5(2 - 2C_2)l_1^4 - 2l_1} = \frac{3l_1}{5(2 - 2C_2)l_1^5 - 2l_1^2} = \frac{3l_1}{3l_1^2 + 15B_1} > 0$$

since $5(2-2C_2)l_1^5 = 5l_1^2+15B_1$. This implies that the function $l_1(B_1)$ with its other variables fixed is increasing on $B_1 + B_2 = 1/3$. Then its minimum value is reached when B_1 is minimum. Therefore, when $B_1 = 0$,

$$l_1^{\min} = \sqrt[3]{\frac{1}{2}}.$$

7. For $1 = 3(B_1 + B_2) + 8(C_1 + C_2)$, one writes equation (36) as

$$l_1^3\Big(l_1^3 - \frac{1}{2 - 8C_1 - 10C_2}\Big) = \frac{3B_1}{2 - 8C_1 - 10C_2},$$

replacing $3(B_1 + B_2)$ with $1 - 8(C_1 + C_2)$. Using the same argument as in the previous cases,

$$l_1 > \sqrt[3]{\frac{1}{2 - 8C_1 - 10C_2}} \geq \sqrt[3]{\frac{1}{2}} = l_1^{\min}.$$

Testing the triangular inequalities with $l_1^{\min} = \sqrt[3]{\frac{1}{2}}$, one gets that if $l_1^{\max}$ is in the vicinity

$$1 - \sqrt[3]{\frac{1}{2}} \leq l_1^{\max} \leq 1 + \sqrt[3]{\frac{1}{2}},$$

$l_1^{\max}$ and $l_1^{\min}$ satisfy the triangular inequalities. Therefore, a candidate to be an upper bound is $l_1^{\max} = 1 + \sqrt[3]{\frac{1}{2}}$. To show that it is, in fact, a valid bound, notice that replacing $l_1 = l_1^{\max}$ in (36) yields

$$(\omega^2 + 2C_1)(l_1^{\max})^5 - (l_1^{\max})^2 - 3B_1 \geq \frac{3}{5}\Big(1 + \sqrt[3]{\frac{1}{2}}\Big)^5 - \Big(1 + \sqrt[3]{\frac{1}{2}}\Big)^2 - 1 > 0.$$

Since the result is positive, independently of the constant value, $l_1^{\max}$ is effectively an upper bound for the real root of (36), because the polynomial is positive only after the root.

By (37), l_1 and l_2 share the same minimum and maximum values, so every combination of constants

$$(B_1, B_2, C_1, C_2) \in D$$

raises solutions of (36) for l_1 and l_2 that satisfy the triangular inequalities since their bounds satisfy them.

3.2.1 Isosceles cases

The distances between the primaries were normalized to be one. Thus, a possible isosceles solution is when $l_i = 1$, and for that (36) raises the following condition:

$$3B_i = 2C_i, \tag{38}$$

and with this, equation (36) for $j \neq i$ becomes

$$(\omega^2 + 2C_j)l_j^5 - l_j^2 - 2C_j = 0.$$

Therefore, if (38) holds, $l_i = 1$ and l_j is given by the last polynomial equation that can be numerically solved in terms of ω^2 and C_j (see Figure 2). Another possible case is when $l_1 = l_2$, and a sufficient condition for this to happen is the trivial case when the bodies m_1 and m_2 have the same constants and the same mass.

4 Stability

To study the movement near the equilibrium points in this problem, the Hamiltonian (18) is expanded through the Taylor series around these points, the linear terms in this are omitted because the equilibrium points are zeroes in the potential and the constant term does not affect the form of the motion equation, so it is not taken into account. The Hamiltonian function rises the Hamiltonian matrix

$$\begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ \dfrac{\partial^2 U_{m3}}{\partial \xi^2} & \dfrac{\partial^2 U_{m3}}{\partial \xi \partial \eta} & 0 & \omega \\ \dfrac{\partial^2 U_{m3}}{\partial \xi \partial \eta} & \dfrac{\partial^2 U_{m3}}{\partial \eta^2} & -\omega & 0 \end{pmatrix}, \tag{39}$$

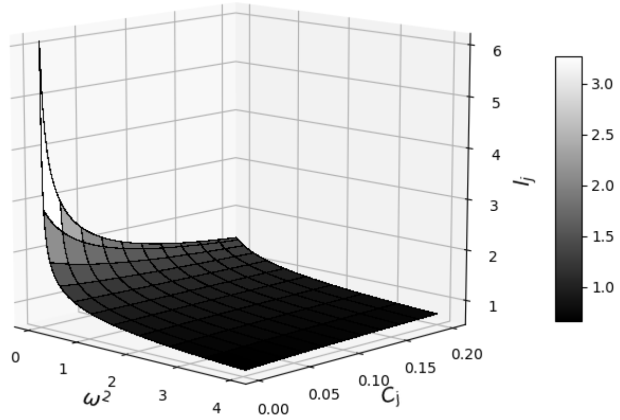

Figure 2: l_j in function as C_j and ω^2 when $l_i = 1$.

whose eigenvalues determine the behavior of the linearized system. The characteristic equation reads

$$\lambda^4 + \Big(2\omega^2 - \frac{\partial^2 U_{m3}}{\partial \xi^2} - \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)\lambda^2 + \Big(\frac{\partial^2 U_{m3}}{\partial \xi^2} + \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)\omega^2 + \omega^4 - \Big(\frac{\partial^2 U_{m3}}{\partial \xi \eta}\Big)^2 + \frac{\partial^2 U_{m3}}{\partial \xi^2}\frac{\partial^2 U_{m3}}{\partial \eta^2} = 0. \tag{40}$$

The conditions that insure linear stability are given by the root of the quadratic formula

$$G_1(B_1, B_2, C_1, C_2, \mu) \equiv \Big(2\omega^2 - \frac{\partial^2 U_{m3}}{\partial \xi^2} - \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)^2 - 4\Big(\Big(\frac{\partial^2 U_{m3}}{\partial \xi^2} + \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)\omega^2 + \omega^4 - \Big(\frac{\partial^2 U_{m3}}{\partial \xi \eta}\Big)^2 + \frac{\partial^2 U_{m3}}{\partial \xi^2}\frac{\partial^2 U_{m3}}{\partial \eta^2}\Big) > 0 \tag{41}$$

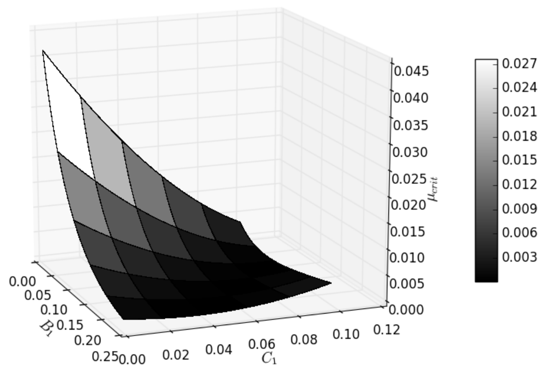

Figure 3: μ_{crit} as a function of B_1 and C_1 when $B_2 = C_2 = 0.1$.

and by the sign of the part outside the root

$$G_1(B_1, B_2, C_1, C_2, \mu) \equiv 2\omega^2 - \frac{\partial^2 U_{m3}}{\partial \xi^2} - \frac{\partial^2 U_{m3}}{\partial \eta^2} > 0. \tag{42}$$

Both conditions must be fulfilled in order to have spectral stability. Five dimentions are needed to visualize the regions of the parameter domain and the values of μ for which the spectral stability exists. One way to display the data in three dimentions is to make projections: fix B_1 and B_2 and graph μ_{crit} (the maximum value of μ that satisfies both conditions) as a function of B_2 and C_2 (see Figures 3 and 4).

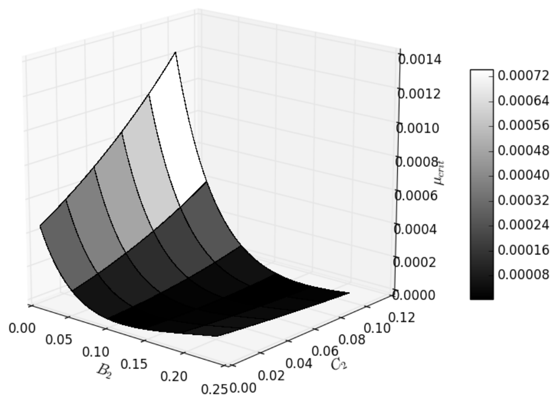

Figure 4: μ_{crit} as a function of B_2 and C_2 when $B_1 = C_1 = 0.1$.

5 Conclusion

We have shown that always the primaries are in a rotational equilibrium (a.k.a, when the coefficients belong to the parameter domain), there is a collinear and a non-collinear relative equilibrium in the restricted three body problem induced by this configuration. Knowing the exact numerical value of these coefficients allows a direct calculation of the position of these equilibrium points.

Also, we discussed the particular case when the non-collinear relative equilibrium is in an isosceles configuration with the primaries, plotting its value given ω^2, C_j and $l_i = 1$. Finally, we provided two conditions necessary to have spectral stability for a given non-collinear equilibrium point. With these conditions, we plotted μ_{crit} for some values of the parameter domain.

6 References

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