Nonlinear Dynamics and Systems Theory, **21** (2) (2021) 115–125

Capacity in Anisotropic Sobolev Spaces

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Received: November 28, 2019; Revised: March 16, 2021

Abstract: This paper is devoted to the study of the theory of capacity in an anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$, where Ω is a bounded set of $\mathbb{R}^N (N \geq 2)$, $\vec{p} = (p_0, p_1, ..., p_N)$ with $1 < p_0, p_1, ..., p_N < \infty$. We will define the $C_{k,\vec{p}}$ capacity and prove its main properties, especially, it will be shown that $C_{k,\vec{p}}$ defines a Choquet capacity. To illustrate our results, we will present an application of this capacity.

Keywords: *anisotropic Sobolev spaces; capacity; potential.*

Mathematics Subject Classification (2010): 31C15.

1 Introduction

The theory of capacity and non-linear potential in the classical Lebesgue space $L^p(\Omega)$ $(1 < p < \infty)$ was studied by Maz'ya and Khavin in [16] and Meyers in [18]. These authors introduced the concept of capacity and non-linear potential in these spaces and provided very rich applications in functional analysis, harmonic analysis, theory of partial differential equations and theory of probabilities.

It has been developed specially by Adams [1], by Hedberg in [13], by Hedberg and Wolff in [14] and others. The Sobolev capacity for constant exponent spaces has found a great number of applications (see [12, 15]) and, for example, Boccardo et al. [8] studied the existence and non existence of solutions of the following problem:

$$(\mathcal{P}) \begin{cases} -\triangle u + u \mid \nabla u \mid^2 = \mu & in \ \Omega, \\ u = 0 & on \ \partial\Omega, \end{cases}$$

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where Ω is a bounded open set in $\mathbb{R}^N, N \geq 2$ and μ is a radon measure on Ω.

More precisely, the authors proved the existence of a solution u in $H_0^1(\Omega)$ for the problem $(\mathcal{P})$ if and only if the measure μ does not charge the sets of capacity zero in Ω. Also, Kilpeläinen [17] introduced the weighted Sobolev capacity and discussed the role of capacity in the pointwise definition of functions in Sobolev spaces involving weights of Muckenhoupt's A_p-class. The previous concept was generalized by N. Aissaoui and A. Benkirane in [2], by replacing L^p with an Orlicz space. Later, this theory was studied by M. C Hassib, Y. Akdim, A. Benkirane and N. Aissaoui in Musielak-Orlicz spaces (see [3] and [4]).

The notion of capacity offers a standard way to characterize exceptional sets in various function spaces. Depending on the starting point of the study, the capacity of a set can be defined in many appropriate ways. A common property of capacities is that they measure small sets more precisely than the usual Lebesgue measure. The Choquet theory [10] provides a standard approach to capacities. Capacity is a necessary tool in both classical and non-linear potential theory.

The main purpose of this paper is to study the theory of capacity in an anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$. Our results generalize those in [18] obtained in Lebesgue spaces, in order to apply them to some problems of partial differential equations and harmonic analysis.

The present paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on the anisotropic Sobolev space and we recall main properties of capacities. In Section 3, we define the $C_{k,\vec{p}}$ -capacity in the anisotropic Sobolev space and we show some of its properties. As an application of our results, we consider a variational problem, where X is a subset of $\mathbb{R}^N$. We give a sufficient condition on the $C_{k,\vec{p}}$ capacity of X to ensure the existence and uniqueness of a $C_{k,\vec{p}}$ -capacitary distribution of X such that the $C_{k,\vec{p}}$ -capacitary potential of X is greater than or equal to one.

2 Preliminaries

2.1 Anisotropic Sobolev spaces

Let Ω be an open bounded domain in $\mathbb{R}^N (N \geq 2)$ with boundary $\partial\Omega$.
Let $1 < p_0, p_1, ..., p_N < \infty$, we denote

$$\vec{p} = (p_0, p_1, ..., p_N),\ D^0 u = u \text{ and } D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, ..., N.$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as follows:

$$W^{1,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \text{ } and \text{ } D^i u \in L^{p_i}(\Omega), i = 1, ..., N\}.$$

We recall that the $W^{1,\vec{p}}(\Omega)$ is a separable and reflexive Banach space (see [19])with respect to the norm

$$\|u\|_{W^{1,\vec{p}}(\Omega)} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i}(\Omega)}.$$

We denoted

$$W_+^{1,\vec{p}}(\Omega) = \{u \in W^{1,\vec{p}}(\Omega) \setminus u \geq 0\}.$$

The space $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. The theory of such anisotropic spaces was developed in [20–23]. It was proved that $C_0^\infty(\Omega)$ is dense in $W_0^{1,\vec{p}}(\Omega)$, and $W_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space.

For any $\vec{p} = (p_0, p_1, ..., p_N)$, with $1 < p_i < \infty$, $i = 0, 1, ..., N$, the dual space of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is equivalent to $W^{-1,\vec{p'}}(\Omega)$, where $\vec{p'} = (p'_0, p'_1, ..., p'_N)$ and $p'_i = \dfrac{p_i}{p_i - 1}$ for all $i = 0, 1, ..., , N$.

Proposition 2.1 *Let $p \in [1, +\infty[$ and $(f_n)_n$ be a sequence in $(L^p(\mu), \|.\|_p)$ whose series of norms $\sum_n \|f_n\|_p$ converges. Then the series of functions $\sum_n f_n$ converges for the norm $\|.\|_p$ and we have $\|\sum_n f_n\|_p \leq \sum_n \|f_n\|_p$.*

Proof. For $n \in \mathbb{N}^*$ fixed, according to the inequality of Minkowski, we have

$$\left\|\sum_{k=0}^{n} |f_k|\right\|_p \leq \sum_{k=0}^{n} \|f_k\|_p \leq \sum_{k=0}^{+\infty} \|f_k\|_p.$$

It follows from the monotone convergence theorem that

$$\left(\int_\Omega \left(\sum_{k=0}^{+\infty} |f_k|\right)^p d\mu\right)^{\frac{1}{p}} \leq \sum_{k=0}^{+\infty} \|f_k\|_p.$$

Thus,

$$\left\|\sum_{k=0}^{+\infty} f_k\right\|_p \leq \sum_{k=0}^{+\infty} \|f_k\|_p.$$

Lemma 2.1 *[see [9]] Let E be a Banach space. If $(f_n)_n$ converges weakly to f in E, then the sequence $\|f_n\|$ is bounded and $\|f\| \leq \liminf \|f_n\|$.*

2.2 Capacity

Definition 2.1 Let E be a topological space and T be the class of Borel sets in E, and a function $C : T \to [\ 0, +\infty\ ]$.
1) The function C is called a capacity if the following axioms are satisfied:
i) $C(\emptyset) = 0$.
ii) $X \subset Y \Rightarrow C(X) \leq C(Y)$ for all X and Y in T.
iii) For all sequences $(X_n) \subset T$,

$$C(\bigcup_n X_n) \leq \sum_n C(X_n).$$

2) The function C is called an outer capacity if, for all $X \in T$,

$$C(X) = \inf\{C(O) : O \supset X, O \text{ is open }\}.$$

3) The function C is called an interior capacity if, for all $X \in T$,

$$C(X) = \sup\{C(K) : K \subset X, K \text{ is compact}\}.$$

4) A property, that holds true except perhaps on a set of capacity zero, is said to be true C-quasi everywhere (abbreviated C - q.e.).

5) Let f and (f_n) be real-valued finite functions C-q.e. We say that (f_n) converges to f in C -capacity if

$$\forall \varepsilon > 0, \lim_{n \to +\infty} C(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

6) Let f and (f_n) be real-valued finite functions C-q.e. We say that (f_n) converges to f C -quasi-uniformly (abbreviated C -q.u) if $(\forall \varepsilon > 0), (\exists X \in T) : C(X) < \varepsilon$ and (f_n) converges to f uniformly on X^c.

3 Capacity in Anisotropic Sobolev Spaces

3.1 $C_{k,\vec{p}}$- capacity

Let k be a positive integrable function on $\mathbb{R}^N$ and $X \subset \mathbb{R}^N$ $(N \geq 2)$. We denote

$$S_{\vec{p}}(X) = \{f \in W^{1,\vec{p}}(\Omega) : k * f \geq 1 \text{ on } X\},$$

where $k * f$ is the convolution of k and f.

The anisotropic Sobolev $\vec{p}$-capacity of X is defined by

$$C_{k,\vec{p}}(X) = \inf_{f \in S_{\vec{p}}(X)} \{\|f\|_{W^{1,\vec{p}}}(\Omega)\}.$$

In the case where $S_{\vec{p}}(X) = \emptyset$, we set $C_{k,\vec{p}}(X) = \infty$.

Functions $f \in S_{\vec{p}}(X)$ are said to be $\vec{p}$- admissible for the X.

The anisotropic $\vec{p}$-capacity enjoys all relevant properties of general capacities, specifically, it will be shown that $C_{k,\vec{p}}(X)$ defines a Choquet capacity.

Theorem 3.1 *The anisotropic Sobolev $\vec{p}$-capacity $C_{k,\vec{p}}$ is an outer capacity.*

Remark 3.1 Let $B_{k,\vec{p}}(X) = \inf\{\|f\|_{W^{1,\vec{p}}(\Omega)}: f \in W^{1,\vec{p}}_+(\Omega)$ and $k * f \geq 1$ on $X\}$, then $C_{k,\vec{p}}(X) = B_{k,\vec{p}}(X)$.

Indeed, it is obvious that $C_{k,\vec{p}}(X) \leq B_{k,\vec{p}}(X)$.

On the other hand, let $f \in W^{1,\vec{p}}(\Omega)$, then $|f| \in W^{1,\vec{p}}_+(\Omega)$, and if $k * f \geq 1$ on X, then $k * |f| \geq 1$ on X, thus

$$B_{k,\vec{p}}(X) \leq \|f\|_{W^{1,\vec{p}}(\Omega)}.$$

Therefore,

$$B_{k,\vec{p}}(X) \leq C_{k,\vec{p}}(X).$$

A direct application of Proposition 2.1 is the following result.

Lemma 3.1 *Let $(f_n)_n$ be a sequence in $W^{1,\vec{p}}(\Omega)$ whose series of norms $\sum_n \|f_n\|_{W^{1,\vec{p}}(\Omega)}$ converges. Then we have*

$$\|\sum_n f_n\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_n \|f_n\|_{W^{1,\vec{p}}(\Omega)}.$$

Proof. **(Theorem 3.1)** It is obvious that $C_{k,\vec{p}}(\emptyset) = 0$ and $C_{k,\vec{p}}(X) \leq C_{k,\vec{p}}(Y)$ if $X \subset Y$. Let (X_i) be a subset of $\mathbb{R}^N$. If $\sum_{i=0}^{\infty} C_{k,\vec{p}}(X_i) = +\infty$, there is nothing to show. We may assume that

$$\sum_{i=0}^{\infty} C_{k,\vec{p}}(X_i) < +\infty, \text{ then } (\forall i \in \mathbb{N})\ C_{k,\vec{p}}(X_i) < +\infty,$$

thus,

$$(\forall i \in \mathbb{N})\ (\forall \varepsilon > 0)\ (\exists f_i \in W_+^{1,\vec{p}}(\Omega)) \text{ so that } k * f_i \geq 1 \text{ on } X_i,$$

and we have

$$\|f_i\|_{W^{1,\vec{p}}(\Omega)} \leq C_{k,\vec{p}}(X_i) + \frac{\varepsilon}{2^{i+1}}.$$

Let $f = supf_i$, we show that $f \in W_+^{1,\vec{p}}(\Omega)$. For all $i \geq 0$, we have by Lemma 3.1

$$\|\sup f_i\|_{W^{1,\vec{p}}(\Omega)} \leq \|\sum_{i=0}^{\infty} f_i\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_{i=0}^{\infty} \|f_i\|_{W^{1,\vec{p}}(\Omega)}.$$

Thus,

$$\|f\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_{i=0}^{\infty} \|f_i\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_{i=0}^{\infty} C_{k,\vec{p}}(X_i) + \varepsilon.$$

This implies that $f \in W_+^{1,\vec{p}}(\Omega)$. Since $k * f \geq 1$ *on* $\bigcup_{i \geq 0} X_i$, we deduce that

$$C_{k,\vec{p}}(\bigcup_{i=0}^{\infty} X_i) \leq \|f\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_{i=0}^{\infty} C_{k,\vec{p}}(X_i) + \varepsilon, \quad forall \ \ \varepsilon > 0.$$

The claim follows by letting $\varepsilon \to 0$.

Now, it remains only to verify that $C_{k,\vec{p}}(X)$ is an outer capacity. Let $X \subset \mathbb{R}^N$, we have

$$C_{k,\vec{p}}(X) \leq \inf\{C_{k,\vec{p}}(O), O \supset X, \text{ O is open}\}.$$

For the reverse inequality, if $C_{k,\vec{p}}(X) = +\infty$, there is nothing to show.
Assume that $C_{k,\vec{p}}(X) < +\infty$ and $0 < \varepsilon < 1$, then there exists $g \in W_+^{1,\vec{p}}(\Omega)$ so that $k * g \geq 1$ on X and

$$\|g\|_{W^{1,\vec{p}}(\Omega)} \leq C_{k,\vec{p}}(X) + \varepsilon.$$

We put $g_\varepsilon = \frac{g}{1-\varepsilon}$ and let the set $O_\varepsilon = \{x : (k * g_\varepsilon)(x) > 1\}$.

Thus O_ε is open, and

$$\forall x \in X; \quad (k * g_\varepsilon)(x) \geq \frac{1}{1-\varepsilon} > 1.$$

Hence $X \subset O_\varepsilon$. On the other hand, we have $C_{k,\vec{p}}(O_\varepsilon) \leq \|g_\varepsilon\|_{W^{1,\vec{p}}(\Omega)}$, and we deduce that

$$C_{k,\vec{p}}(O_\varepsilon) \leq \frac{1}{1-\varepsilon}\|g\|_{W^{1,\vec{p}}(\Omega)} \leq \frac{1}{1-\varepsilon}(C_{k,\vec{p}}(X) + \varepsilon), \forall \varepsilon > 0.$$

Thus,

$$\inf\{C_{k,\vec{p}}(O), O \supset X, \text{O open}\} \leq C_{k,\vec{p}}(X).$$

Proposition 3.1 *The anisotropic Sobolev $\vec{p}$-capacity $C_{k,\vec{p}}$ verifies the following properties:*
1) If there exists $f \in W^{1,\vec{p}}(\Omega)$ such that $|k * f| = +\infty$ on X, then $C_{k,\vec{p}}(X) = 0$.
2) If $C_{k,\vec{p}}(X) = 0$, then there exists $f \in W^{1,\vec{p}}_{+}(\Omega)$ such that $k * f = +\infty$.

Proof.

1) Let $f \in W^{1,\vec{p}}(\Omega)$ be such that $|k * f| = +\infty$ on X, then for all $\alpha > 0$, $|k * f| > \alpha$ on X, thus,

$$C_{k,\vec{p}}(X) \leq \frac{\|f\|_{W^{1,\vec{p}}(\Omega)}}{\alpha}, \qquad \forall \alpha > 0.$$

This means that

$$C_{k,\vec{p}}(X) = 0.$$

2) If $C_{k,\vec{p}}(X) = 0$, then $(\forall i \in \mathbb{N})$ $(\exists f_i \in W^{1,\vec{p}}_{+}(\Omega))$ with $k * f_i \geq 1$ on X and

$$\|f_i\|_{W^{1,\vec{p}}(\Omega)} \leq 2^{-i}.$$

Let $f = \sum_i f_i$. By Lemma 3.1 we have

$$\|f\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_i \|f_i\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_i 2^{-i}.$$

Then

$$\|f\|_{W^{1,\vec{p}}(\Omega)} < +\infty.$$

We conclude that $f \in W^{1,\vec{p}}_{+}(\Omega)$ such that $k * f = +\infty$ on X.

Theorem 3.2 *Let f and $(f_n)_n$ be in $W^{1,\vec{p}}(\Omega)$ and consider the following propositions:*
i) $f_n \to f$ strongly in $W^{1,\vec{p}}(\Omega)$.
ii) $k * f_n \to k * f$ $C_{k,\vec{p}}$-capacity.
iii) There is a subsequence (f_{n_j}) such that $k * f_{n_j} \to k * f$ $C_{k,\vec{p}}$ - q.u.
iv) $k * (f_{n_j}) \to k * f$ in $C_{k,\vec{p}}$.- q.e.
Then we have

$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv).$$

Proof.

- We show $i) \Rightarrow ii)$.

By Proposition 3.1, we have $k * f$ and $k * f_n$ are finite $C_{k,\vec{p}}$ -q.e, for all n. Let $\varepsilon > 0$, then

$$C_{k,\vec{p}}(\{x : |k * f_n - k * f|(x) > \varepsilon\}) \leq \frac{\|f_n - f\|_{W^{1,\vec{p}}(\Omega)}}{\varepsilon}.$$

- We show $ii) \Rightarrow iii)$.

Let $\varepsilon > 0$, there exists f_{n_j} such that

$$C_{k,\vec{p}}(\{x : |k * f_{n_j} - k * f|(x) > 2^{-j}\}) \leq \varepsilon \cdot 2^{-j}.$$

We put

$$E_j = \{x : |k * f_{n_j} - k * f|(x)\} > 2^{-j}\} \ \ and \ \ G_m = \bigcup_{j \geq m} E_j.$$

Then we have

$$C_{k,\vec{p}}(G_m) \leq \sum_{j \geq m} \varepsilon \cdot 2^{-j} < \varepsilon.$$

On the other hand,
$\forall x \in (G_m)^c, \forall j \geq m \ \ |k * f_{n_j} - k * f|(x) \leq 2^{-j}$, thus $k * f_{n_j} \to k * f$ $C_{k,\vec{p}}$ - q.u.

- We show $iii) \Rightarrow iv)$.

We have $\forall j \in \mathbb{N}, \exists X_j : \ \ C_{k,\vec{p}}(X_j) \leq \frac{1}{j}$ and $k * f_{n_j} \to k * f$ converges uniformly on $(X_j)^c$. We put $X = \bigcap_j X_j$, then $C_{k,\vec{p}}(X) = 0$ and $k * f_{n_j} \to k * f$ on X^C.

Theorem 3.3 *Let $(K_n)_n$ be a decreasing sequence of compacts and $K = \bigcap_n K_n$. Then*

$$\lim_{n \to +\infty} C_{k,\vec{p}}(K_n) = C_{k,\vec{p}}(K).$$

Proof. First, we observe that $C_{k,\vec{p}}(K) \leq \lim_{n \to +\infty} C_{k,\vec{p}}(K_n)$.

On the other hand, let O be an open set that satisfies $K \subset O$; then

$$K \cap O^c = \emptyset.$$

The sequence defined, for all n, $K_n^{'} = K_n \cap O^c$ is a decreasing sequence of compacts and satisfies $\bigcap_n K_n^{'} = \emptyset$. Then there exists n_0 such that $K_{n_0}^{'} = \emptyset$.
Hence $\forall n \geq n_0$, $K_n^{'} = \emptyset$, then $\forall n \geq n_0$, $K_n \subset O$. Therefore,

$$\lim_{n \to +\infty} C_{k,\vec{p}}(K_n) \leq C_{k,\vec{p}}(O).$$

Since $C_{k,\vec{p}}$ is an outer capacity, we have

$$\lim_{n \to +\infty} C_{k,\vec{p}}(K_n) \leq C_{k,\vec{p}}(K).$$

Proposition 3.2 *Let $(f_n)_n$, $f \in W^{1,\vec{p}}(\Omega)$ be such that $f_n \to f$ weakly in $W^{1,\vec{p}}(\Omega)$, then* $\liminf(k * f_n) \leq k * f \leq \limsup(k * f_n)$ $C_{k,\vec{p}}$*-q.e.*

Proof. Since $W^{1,\vec{p}}(\Omega)$ is a reflexive space, $f_n \to f$ weakly in $W^{1,\vec{p}}(\Omega)$, then by the Banach-Saks theorem, there is a subsequence denoted again by (f_n) such that the sequence $g_n = \frac{1}{n} \sum_{i=1}^{n} f_i$ converges to f strongly in $W^{1,\vec{p}}(\Omega)$.

By Theorem 3.2, there is a subsequence of (g_n), denoted again (g_n), such that

$$\lim_{n \to +\infty} (k * g_n) = (k * f) \quad\quad C_{k,\vec{p}} - q.e.$$

On the other hand,

$$\liminf(k * f_n) \leq \lim_{n\to+\infty}(k * g_n).$$

Therefore,

$$\liminf(k * f_n) \leq (k * f) \qquad C_{k,\vec{p}} - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality.

Theorem 3.4 *If $(X_n)_n$ is an increasing sequence of sets and $X = \bigcup_n X_n$, then*

$$\lim_{n\to+\infty} C_{k,\vec{p}}(X_n) = C_{k,\vec{p}}(X).$$

Proof. First, we have $\lim_{n\to+\infty} C_{k,\vec{p}}(X_n) \leq C_{k,\vec{p}}(X)$.
For the reverse inequality, if $\lim_{n\to+\infty} C_{k,\vec{p}}(X_n) = +\infty$, there is nothing to show.

We assume that the sequence $C_{k,\vec{p}}(X_n)$ converges to the finite ℓ. Let f_n be $\vec{p}$- admissible for (X_n) such that

$$\|f_n\|_{W^{1,\vec{p}}(\Omega)} \leq C_{k,\vec{p}}(X_n) + \frac{1}{n}. \tag{1}$$

Since (f_n) forms a bounded sequence in $W^{1,\vec{p}}_+(\Omega)$, there exists a subsequence denoted again (f_n) which converges weakly to a function $f \in W^{1,\vec{p}}_+(\Omega)$.

We have by Proposition 3.2

$$\forall i \in \mathbb{N}, \quad k * f \geq 1 \text{ on } X_n,\, C_{k,\vec{p}} - q.e.$$

Therefore,

$$k * f \geq 1 \; on \; X, C_{k,\vec{p}} - q.e. \tag{2}$$

Let B be a subset of X where $k * f \geq 1$, then from (1) and by Lemma 2.1 we have

$$C_{k,\vec{p}}(X) = C_{k,\vec{p}}(B) \leq \|f\|_{1,\vec{p}} \leq \ell, \tag{3}$$

the desired result is now a simple consequence of (3).

Corollary 3.1 *Let $(E_n)_n$ be a sequence of subsets of $\mathbb{R}^N$, then*

$$C_{k,\vec{p}}(\liminf E_n) \leq \liminf C_{k,\vec{p}}(E_n).$$

Proof. Let $E = \liminf E_n$, we have $E = \bigcup_n \Big(\bigcap_{i\geq n} E_i\Big)$.
We put $G_n = \bigcap_{i\geq n} E_i$, thus a sequence is increasing and by Theorem 3.4, we have

$$C_{k,\vec{p}}(E) = \lim_{n\to+\infty} C_{k,\vec{p}}(G_n).$$

Hence,

$$C_{k,\vec{p}}(G_n) \leq C_{k,\vec{p}}(E_n).$$

Therefore,

$$C_{k,\vec{p}}(E) \leq \liminf C_{k,\vec{p}}(E_n).$$

Definition 3.1 In the terminology of Choquet, C is called a capacity if it satisfies the following four properties:

i) $C(\emptyset) = 0$,

ii) C is increasing,

iii) If (E_n) is an increasing sequence of sets, then $\sup_n C(X_n) = C(\bigcup_n X_n)$,

iv) If (K_n) is a decreasing sequence of compacts, then $\inf_n C(K_n) = C(\bigcap_n K_n)$.

Remark 3.2 By Theorems 3.1, 3.3 and 3.4, $C_{k,\vec{p}}$ is a capacity in the sense of Choquet.

Definition 3.2 Let C be a capacity in the sense of Choquet. A subset $X \subset \mathbb{R}^N$ is called capacitable if

$$C(X) = \sup\{C(K) : K \subset X, K - compact\}.$$

Theorem 3.5 *All analytic sets are $C_{k,\vec{p}}$- capacitable.*

Proof. It is an immediate consequence of the Choquet theorem in [11].

3.2 Application of a $C_{k,\vec{p}}$ - capacity

In this subsection, we propose to study an application of $C_{k,\vec{p}}$ capacities, more precisely, we treat the following variational problem.

Let X be a subset of $\mathbb{R}^N$ such that $C_{k,\vec{p}}(X) < \infty$. There exists $f_0 \in W_+^{1,\vec{p}}(\Omega)$ such that $k * f_0 \geq 1$ $C_{k,\vec{p}}$- q.e on X, and

$$\|f_0\|_{W^{1,\vec{p}}(\Omega)} = \inf\{\|f\|_{W^{1,\vec{p}}(\Omega)} : f \in W_+^{1,\vec{p}}(\Omega) \ \ and \ \ k * f \geq 1 \ \ on \ \ X\}. \tag{4}$$

Definition 3.3 We call a solution, f_0, of problem (4) a $C_{k,\vec{p}}$ -capacitary distribution of X and we call $k * f_0$ a $C_{k,\vec{p}}$ -capacitary potential of X.

Theorem 3.6 *Let X be a subset of $\mathbb{R}^N$ such that $C_{k,\vec{p}}(X) < \infty$ and denote by Ω_X the set $\Omega_X = \{f \in W_+^{1,\vec{p}}(\Omega) : k * f \geq 1 \ \ C_{k,\vec{p}}(X) \ - q.e \ \ on \ \ X\}$.*
Then there exists a unique $f_0 \in W_+^{1,\vec{p}}(\Omega)$ such that:

i) $\|f_0\|_{W^{1,\vec{p}}(\Omega)} = \inf\{\|f\|_{W^{1,\vec{p}}(\Omega)} : f \in \Omega_X\}$.

ii) $k * f_0 \geq 1$ *on* X *and* $\|f_0\|_{W^{1,\vec{p}}(\Omega)} = C_{k,\vec{p}}(X)$.

Proof. i) Let the function $\theta : W^{1,\vec{p}}(\Omega) \longrightarrow \mathbb{R}^+$ be defined by $\theta(f) = \|f\|_{W^{1,\vec{p}}(\Omega)}$; $\forall f \in W^{1,\vec{p}}(\Omega)$. θ is lower semi-continuous on $W^{1,\vec{p}}(\Omega)$ and coercive. By Theorem 3.2, Ω_X is strongly closed in $W^{1,\vec{p}}(\Omega)$. On the other hand, Ω_X is convex. Since $W^{1,\vec{p}}(\Omega)$ is reflexive, there exists a unique $f_0 \in W_+^{1,\vec{p}}(\Omega)$ such that

$$\|f_0\|_{W^{1,\vec{p}}(\Omega)} = \inf\{\|f\|_{W^{1,\vec{p}}(\Omega)} : f \in \Omega_X\}.$$

ii) Let Y be a subset of X where $k * f_0 < 1$, then $C_{k,\vec{p}}(X) = C_{k,\vec{p}}(X - Y)$.

Since $k * f_0 \geq 1$ on $X - Y$, $C_{k,\vec{p}}(X - Y) \leq \|f_0\|_{W^{1,\vec{p}}(\Omega)}$, on the other hand, we have

$$\{f \in W_+^{1,\vec{p}}(\Omega) : k * f \geq 1 \ \ on \ \ X\} \subset \Omega_X.$$

Then

$$\|f_0\|_{W^{1,\vec{p}}(\Omega)} \leq C_{k,\vec{p}}(X).$$

4 Concluding Remarks

In this paper we defined the notion of $C_{k,\vec{p}}$ -capacity in the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ for $\vec{p} = (p_0, p_1, ..., p_N)$, with $1 < p_0, p_1, ..., p_N < \infty$. We showed that this capacity is an outer capacity and proved some convergence properties related to it. Moreover, we proved that $C_{k,\vec{p}}$ is a Choquet capacity. Finally, we gave an application of this capacity in anisotropic Sobolev spaces.

Note that the results obtained previously, especially, the properties of the anisotropic Sobolev $\vec{p}$ -capacity $C_{k,\vec{p}}$ will be useful in the study of some differential equations problems. Namely, for problems, studied previously in [5–7], we can treat solutions in anisotropic Sobolev spaces and we can assume that the right hand side is a measure data.

A perspective of this work will focus on the application of our results to a unilateral problem that was addressed in a previous study [4] in Musielak–Orlicz–Sobolev spaces.

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