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Averaging Method and Boundary Value Problems for Systems of Fredholm Integro-Differential Equations

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Abstract: In this paper, an analogue of Bogolyubov's first theorem of the averaging method for systems of Fredholm integro-differential equations is established. The averaging method is also applied to boundary value problems for such systems. It is shown that if a boundary value problem for an averaged system, which is a system of ordinary differential equations, has a solution, then the original problem is solvable as well.

Keywords: *Fredholm integro-differential equation; boundary-value problem; averaging method.*

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1 Introduction

In this paper, we study systems of Fredholm integro-differential equations

$$\frac{dx}{dt} = \dot{x} = \varepsilon X(t, x, \int_0^{\frac{T}{\varepsilon}} \varphi(t, s, x(s))ds) \tag{1}$$

subject to the Cauchy conditions

$$x(0) = x_0, \tag{1'}$$

or to the boundary conditions

$$F(x(0), x(\tfrac{T}{\varepsilon})) = 0, \tag{1''}$$

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where $\varepsilon > 0$ is a small parameter, X and F are d-dimensional vector functions, φ is an m-dimensional vector function, $T > 0$ is a fixed number. We define the integral average $X_0(x)$ as

$$X_0(x) = \lim_{A\to\infty} \frac{1}{A} \int_0^A X\big(t, x, \varphi_1(t,x)\big)dt, \tag{2}$$

where $\varphi_1(t,x) = \int_0^t \varphi\big(t,s,x\big)ds$, and put the problems (1′) and (1″) in correspondence with the averaged problems

$$\dot{y} = \varepsilon X_0(y), \tag{3}$$

$$y(0) = x_0, \tag{3'}$$

$$F(y(0), y(\tfrac{T}{\varepsilon})) = 0, \tag{3''}$$

or, on the slow time scale $\tau = \varepsilon t$,

$$\frac{dy}{dt} = X_0(y), \quad F(y(0), y(T)) = 0. \tag{4}$$

The main results of the present paper are the justification of the averaging method for the Cauchy problem and the statement that if the problem (3) has a solution, then for small values of the parameter ε the problem (2) has a solution as well, in a small neighborhood of the solution of the boundary value problem (3). The exact statement of the problems and the results formulation are presented in the main part of the paper.

It should be noted that the averaging method has not lost its relevance and is widely used in the study of various problems, for example, optimal control [16,19], systems with a multi-valued right-hand side [13], and many others.

Integro-differential equations arise as mathematical models of various processes in natural sciences; for instance, in population dynamics [1], chemical kinetics, fluid dynamics [2, 12, 22], epidemiology [21]. Interaction of modeling objects with the environment leads to boundary value problems for integro-differential equations. These problems have been studied by many authors [3,4,6–8,17].

In [20], boundary value problems for systems of Volterra integro-differential equations are investigated by using the averaging method. It is shown that, for small ε, the existence of a solution of a boundary value problem for an averaged system (3) implies that of the original boundary value problem (1); the proximity between corresponding solutions is proven. The result of [20] is a generalization of the classical result [18] concerning boundary value problems for systems of ordinary differential equations.

Note that the averaging method has already been used for solving boundary value problems for systems of Volterra integro-differential equations (see [15] and the references therein). However, in these works only an estimate of proximity between the solutions of the exact and averaged problems was established. The very fact of the existence of a solution was only postulated.

The present work is devoted to the further development of the ideas [20] as applied to the studying boundary value problems for Fredholm equations. Analogues of the Bogolyubov first theorem for Fredholm equations, unlike those for Volterra equations, were obtained only in a very special case, when the right-hand part is a sum of an ordinary term and an integral part, and the integration is carried out over a finite interval (see [9, 11]). Again, the existence of a solution is only a postulate. However, this theorem plays an essential role in obtaining results analogous to those of [20].

The paper is organized as follows. In Section 2, the problem statement and the main results are formulated. Section 3 contains some auxiliary results which are also of independent interest. For a Cauchy problem for systems of Fredholm integro-differential equations, we prove the existence and uniqueness theorem and investigate the continuous dependence of solutions on initial data. Section 4 is devoted to the justification of the averaging method. In Section 5, the existence of a solution of the boundary value problem is proved.

2 Problem Statement and Main Results

Throughout the rest of this paper, we denote by $|\cdot|$ the norm of a vector in $\mathbb{R}^d$ and by $\|\cdot\|$ the matrix norm consistent with a vector norm.

The following theorem justifies the averaging method.

Theorem 2.1 *Let the following conditions hold:*

(1.1) $X(t,x,y)$ *is defined and continuous in a domain* $Q=\{t\geq 0, x\in\mathbb{R}^d, y\in\mathbb{R}^m\}$, *bounded by a constant* M *in this domain, and satisfies a Lipschitz condition with respect to the variables* x *and* y *in the following sense: there exists a function* $\alpha(t)\geq 0$ *such that*

$$|X(t,x,y)-X(t,x_1,y_1)|\leq \alpha(t)\big(|x-x_1|+|y-y_1|\big); \tag{5}$$

(1.2) $\varphi(t,s,z)$ *is defined and continuous in* $Q_1=\{t\geq 0, s\geq 0, z\in\mathbb{R}^d\}$, *bounded by a constant* $M>0$, *and satisfies a Lipschitz condition in the following sense: there exists a function* $\mu(t,s)\geq 0$ *such that*

$$|\varphi(t,s,z)-\phi(t,s,z_1)|\leq \mu(t,s)|z-z_1|. \tag{6}$$

Besides, there exists a constant $\mu_0>0$ *such that* $\mu(t,s)\leq\mu_0$, $\int_0^\infty \mu(t,s)ds\leq\mu_0$, *and*

$$\frac{1}{t}\int_0^t d\tau\int_0^\tau \mu(\tau,s)ds\to 0,\quad t\to\infty; \tag{7}$$

there also exists $\bar{\varepsilon}>0$ *such that for* $\varepsilon\in(0,\bar{\varepsilon}]$

$$\varepsilon\bigg(\int_0^{\frac{T}{\varepsilon}}\alpha(s)ds+\int_0^{\frac{T}{\varepsilon}}\alpha(s)\bigg(\int_0^{\frac{T}{\varepsilon}}\mu(\tau,s)d\tau\bigg)ds\bigg)<1; \tag{8}$$

(1.3) the limits (2) and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\bigg(\int_\tau^\infty|\varphi(\tau,s,x)|ds\bigg)d\tau=0 \tag{9}$$

exist uniformly with respect to $x\in D$ *(*D *is a domain in* $\mathbb{R}^d$*); and*

(1.4) the averaged system (3) has a solution $y(\tau)=y(\varepsilon t)$ *that belongs to* D *together with some* ρ*-neighborhood, for* $\tau\in[0,T]$.

Then, for every $\eta>0$, *there exists* $\varepsilon_0=\varepsilon_0(\eta)\leq\bar{\varepsilon}$ *such that for* $\varepsilon\in[0,\varepsilon_0]$ *the Cauchy problem* $x(0)=y(0)=x_0$ *for (1) has a unique solution* $x(t,\varepsilon)$ *defined on* $[0,\frac{T}{\varepsilon}]$, *and the following inequality holds:*

$$|y(\varepsilon t)-x(t,\varepsilon)|\leq\eta,\quad t\in[0,\tfrac{T}{\varepsilon}]. \tag{10}$$

For boundary value problem $(1)-(1'')$, the following statement holds true.

Theorem 2.2 *Let conditions (1.1)-(1.3) hold. Suppose, in addition, that the averaged boundary value problem* (3)*-*$(3'')$ *has a solution* $y = y(\tau) = y(\varepsilon t)$ *belonging to* D *together with some* ρ*-neighborhood, in which* $X_0(x)$, $F(x, y)$ *have continuous partial derivatives* $\frac{\partial X_0(x)}{\partial x}$, $\frac{\partial F}{\partial x}$, *and* $\frac{\partial F}{\partial y}$, *and*

$$det \frac{\partial F_0(x_0)}{\partial x_0} \neq 0, \tag{11}$$

where $x_0 = y(0), F_0(x_0) = F(x_0, y(T, x_0))$.

Then there exists $\varepsilon_0 > 0$ *such that, for* $\varepsilon \in (0, \varepsilon_0)$, *boundary value problem* $(1') - (1'')$ *has a solution* $x(t, \varepsilon)$, *and one can specify a function* $\xi = \xi(\varepsilon) \to 0$, $\varepsilon \to 0$, *such that*

$$|x(t, \varepsilon) - y(\varepsilon t)| \leq \xi(\varepsilon), \quad t \in [0, \tfrac{T}{\varepsilon}]. \tag{12}$$

3 Cauchy Problem for Fredholm Integro-Differential Equations

In this section, we consider the Cauchy problem

$$\dot{x} = X(t, x, \int_0^T \varphi\big(t, s, x(s)\big)ds), \quad x(0) = x_0, \tag{13}$$

where $[0, T]$ is a fixed interval.

Theorem 3.1 *Let the following conditions be satisfied:*

(2.1) *the function* $X(t, x, y)$ *is defined in a domain* $Q = \{t \in [0, T], x \in \mathbb{R}^d, y \in D\}$ *(*D *is a domain in* $\mathbb{R}^m$*) and satisfies a Lipschitz condition*

$$|X(t, x, y) - X(t, x_1, y_1)| \leq \alpha(t)(|x - x_1| + |y - y_1|), \tag{14}$$

as well as a linear growth condition with respect to x, y*; that is, there exists a constant* $M > 0$ *such that, for* $t \in [0, T], x \in \mathbb{R}^d, y \in D$

$$|X(t, x, y)| \leq M(1 + |x| + |y|); \tag{15}$$

(2.2) *the function* $\varphi(t, s, z)$ *is defined and continuous in a domain* $Q_1 = \{t \in [0, T], s \in [0, T], z \in \mathbb{R}^d\}$, *bounded by a constant* M_1 *in* Q_1, *and, with respect to* z, *satisfies a Lipschitz condition*

$$|\varphi(t, s, z) - \varphi(t, s, z_1)| \leq \mu(t, s)|z - z_1|; \tag{16}$$

(2.3) *the inequality*

$$\int_0^T \alpha(t)dt + \int_0^T \alpha(t)\bigg(\int_0^T \mu(t, s)ds\bigg)dt < 1 \tag{17}$$

holds;

(2.4) *the region* D *contains a closed ball* $\bar{B}_{TM_1}(0)$ *of radius* TM_1, *centered at the origin.*

Then, for all $x_0 \in \mathbb{R}^d$, *the Cauchy problem (13) has a unique solution* $x(t, x_0)$ $(x(0, x_0) = x_0)$ *on* $[0, T]$, *which depends continuously on the initial data* x_0.

Remark 3.1 The behavior of systems of kind (13) is substantially different from that of similar systems of Volterra type. In [5, p.71], an example is provided for the following equation

$$\dot{x} = Ax + \frac{1}{2\pi}\int_0^{2\pi} Bx(s)ds + f(t), x \in \mathbb{R}^2, \quad t \in (0; 2\pi), \tag{18}$$

with matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$.

It was shown that (18) is solvable only if the following condition is met:

$$\int_0^{2\pi} \big(- f_1(t)\sin t + (1-\cos t) f_2(t)\big) dt = 0.$$

Note that equation (18) does not satisfy conditions of Theorem 3.1.

Proof. The proof of Theorem 3.1 falls into three parts.

1. Uniqueness. Let the Cauchy problem have two solutions $x(t)$ and $y(t)$ on $[0, T]$, such that $\sup\limits_{t\in[0,T]} |x(t) - y(t)| = \gamma > 0$. Note that $x(t)$ and $y(t)$ satisfy the equations

$$x(t) = x_0 + \int_0^t X\bigg(\tau, x(\tau), \int_0^T \varphi(\tau, s, x(s))ds\bigg) d\tau \tag{19}$$

and

$$y(t) = x_0 + \int_0^t X\bigg(\tau, y(\tau), \int_0^T \varphi(\tau, s, y(s))ds\bigg) d\tau, \tag{20}$$

respectively. Thus, by (15) and (16), we get the estimate

$$|x(t) - y(t)| \le \int_0^t \alpha(\tau)\big|x(\tau) - y(\tau)\big| d\tau + \int_0^t \alpha(\tau)\bigg(\int_0^T \mu(\tau, s)|x(s) - y(s)|ds\bigg) d\tau \le$$

$$\le \bigg(\int_0^T \alpha(\tau)d\tau + \int_0^T \alpha(\tau)\bigg(\int_0^T \mu(\tau, s)ds\bigg)d\tau\bigg) \sup_{t\in[0,T]} |x(t) - y(t)|,$$

which contradicts (17).

2. Existence. We construct a system of functions $\{x_n(t)\}$ in the following way: $x_0(t) \equiv x_0$ and $x_n(t)$ is a solution of the following Cauchy problem for a system of differential equations

$$\dot{x}_n = X\bigg(t, x_n, \int_0^T \varphi(t, s, x_{n-1}(s))ds\bigg), \quad x_n(0) = x_0.$$

Let us show that this sequence is defined correctly. Indeed, we have

$$\dot{x}_1 = X\bigg(t, x_1, \int_0^T \varphi(t, s, x_0)ds\bigg), \quad x_1(0) = x_0. \tag{21}$$

Since $\varphi(t, s, x_0)$ is continuous jointly in its variables, then $g(t) = \int\limits_0^T \varphi(t, s, x_0)ds$ is continuous with respect to t as well. Moreover, $\left|\int\limits_0^T \varphi(t, s, x_0)ds\right| \le M_1 T$; hence,

$\int_0^T \varphi(t,s,x_0)ds \in D$. It follows that the function $Y(t,x) = X(t,x,g(t))$ is defined, continuous with respect to $t \in [0,T], x \in \mathbb{R}^d$, and satisfies a linear growth condition with respect to $x \in \mathbb{R}^d$.

So, the Cauchy problem (21) has the global solution $x_1(t)$ on $[0,T]$. In the same manner, we can see that the whole sequence $\{x_n(t)\}$ is defined on $[0,T]$ as well.

We proceed to show that $x_n(t)$ is uniformly convergent on $[0,T]$. We have

$$x_n(t) = x_0 + \int_0^t X\left(\tau, x_n(\tau), \int_0^T \varphi(\tau,s,x_{n-1}(s))ds\right)d\tau, \tag{22}$$

$$x_{n-1}(t) = x_0 + \int_0^t X\left(\tau, x_{n-1}(\tau), \int_0^T \varphi(\tau,s,x_{n-2}(s))ds\right)d\tau.$$

We thus get

$$\big|x_n(t) - x_{n-1}(t)\big| \le \int_0^T \alpha(\tau)d\tau \sup_{t\in[0,T]} |x_n(t) - x_{n-1}(t)| +$$

$$+ \int_0^T \alpha(\tau)\left(\int_0^T \mu(\tau,s)ds\right)d\tau \sup_{t\in[0,T]} |x_{n-1}(t) - x_{n-2}(t)|.$$

Then

$$\sup_{t\in[0,T]} |x_n(t) - x_{n-1}(t)| \le \frac{\int_0^T \alpha(\tau)\left(\int_0^T \mu(\tau,s)ds\right)d\tau}{1 - \int_0^T \alpha(\tau)d\tau} \sup_{t\in[0,T]} |x_{n-1}(t) - x_{n-2}(t)|. \tag{23}$$

But it follows from (17) that

$$\frac{\int_0^T \alpha(\tau)\left(\int_0^T \mu(\tau,s)ds\right)d\tau}{1 - \int_0^T \alpha(\tau)d\tau} = A < 1.$$

Thus, in view of (22), we can conclude that the sequence $x_n(t)$ uniformly converges to a limit function $x^*(t)$ on $[0,T]$. We can now easily obtain from (14) and (16) that

$$\int_0^t X\left(\tau, x_n(\tau), \int_0^T \varphi(\tau,s,x_{n-1}(s))ds\right)d\tau \to \int_0^t X\left(\tau, x^*(\tau), \int_0^T \varphi(\tau,s,x^*(s))ds\right)d\tau$$

as $n \to \infty$. From this it follows that $x^*(t)$ is a solution of the Cauchy problem (13).

3. Continuous dependence on initial data. Assume that continuous dependence does not hold. Then there exist $\varepsilon > 0$, a sequence of initial data x_n converging to x_0 as $n \to \infty$, and a sequence $\{t_n\}$, $t_n \in (0,T]$ such that

$$|x(t_n,x_n) - x(t_n,x_0)| = \varepsilon. \tag{24}$$

Here $x(t,x_n)$ is a solution of the system (13) subject to initial data $x(0,x_n) = x_n$. Let us show that the sequence $x(t,x_n)$ is compact in $C([0,T])$. Indeed, from (15) we obtain

$$|x(t,x_n)| \le |x_n| + \int_0^t M|1 + |x_n(\tau)|d\tau + \int_0^t |\varphi(\tau,s,x_n(s))ds|d\tau \le$$

$$\leq |x_n| + MT + M\int_0^t |x_n(\tau)|d\tau + T^2MM_1.$$

By Gronwall's lemma, we get

$$|x(t,x_n)| \leq \Big(|x_n| + MT + T^2MM_1\Big)e^{MT} \leq C \tag{25}$$

due to the boundedness of the sequence $\{x_n\}$.

Further, for $t_1 < t_2$, $t_1, t_2 \in [0,T]$,

$$|x(t_2,x_n) - x(t_1,x_0)| \leq \int_{t_1}^{t_2} M(1 + C + TM_1), \tag{26}$$

whence it follows that the sequence $\{x(t,x_n)\}$ is equicontinuous. Consequently, $\{x(t,x_n)\}$ contains a uniformly convergent on $[0,T]$ subsequence $\{x(t,x_{n_k})\}$. It is clear that this subsequence can be chosen so that the number sequence $\{t_{n_k}\}$ converges simultaneously to some $t^* \in [0,T]$. Thus, $x(t,n_k) \rightrightarrows x^*(t)$, $n_k \to \infty$, and

$$x(t,n_k) = x_{n_k} + \int_0^t X\Big(\tau, x_{n_k}(\tau), \int_0^T \varphi(\tau,s,x_{n_k}(s)ds\Big)d\tau. \tag{27}$$

Letting $n_k \to \infty$ in (27), we obtain

$$x^*(t) = x_0 + \int_0^t X\Big(\tau, x^*(\tau), \int_0^T \varphi(\tau,s,x^*(s)ds\Big)d\tau.$$

Hence $x^*(t)$ is a solution of the Cauchy problem (13) as well. Let us show that $x^*(t)$ does not coincide identically with $x(t,x_0)$. Taking into account that $x(t,x_{n_k})$ converges uniformly to $x^*(t)$, which is continuous, from the inequality

$$|x(t_{n_k},x_{n_k}) - x^*(t^*)| \leq |x(t_{n_k},x_{n_k}) - x^*(t_{n_k})| + |x^*(t_{n_k}) - x^*(t^*)|$$

we conclude that $x(t_{n_k},x_{n_k}) \to x^*(t^*)$, $n_k \to \infty$. Therefore, passing to the limit, as $n_k \to \infty$, in (24), we get

$$|x^*(t^*) - x(t^*,x_0)| = \varepsilon. \tag{28}$$

Note that $t^* \neq 0$, since otherwise (24) would not hold for large n. Thus (28) contradicts the uniqueness of a solution of the Cauchy problem. The proof is complete. □

4 Averaging Method for Systems (1)

In this section, we prove Theorem 2.1 on a justification of the averaging method.

4.1 Averaging Lemma

Assume the condition (1.4) of Theorem 2.1 to be fulfilled. Fix $K > 0$.

Definition 4.1 We say that a function $a(t,\varepsilon)$ belongs to a class A_K if:

(i) $a(t,\varepsilon)$ is defined for $\varepsilon > 0$, $t \geq 0$, and takes on values in a ρ-neighborhood of $y(\tau)$, which is a solution of the averaged Cauchy problem (3)–(3');

(ii) for $t \geq 0, s \geq 0$, and $\varepsilon > 0$, the following inequality holds:

$$|a(t,\varepsilon) - a(s,\varepsilon)| \leq K\varepsilon |t-s|. \tag{29}$$

Lemma 4.1 *Let the conditions of Theorem 2.1, except (8), be fulfilled. Then, for every $\eta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\eta, K)$ such that, for $\varepsilon \in (0, \varepsilon_0]$, the system*

$$\dot{x} = \varepsilon X\biggl(t, x, \int_0^{\frac{T}{\varepsilon}} \varphi\bigl(t, s, a(s,\varepsilon)\bigr) ds\biggr) \tag{30}$$

has a solution $x(t)$, $x(0) = y(0) = x_0$, defined on $[0, \frac{T}{\varepsilon}]$, and the inequality

$$|y(\varepsilon t) - x(t)| \leq \eta, \quad t \in [0, \tfrac{T}{\varepsilon}], \tag{31}$$

holds.

Remark 4.1 In the above lemma, ε_0 does not depend on x_0 and is uniform throughout the class A_K.

Proof. As in the proof of Theorem 3.1, we can show that, for all ε, the Cauchy problem

$$\dot{x} = \varepsilon X\biggl(t, x, \int_0^{\frac{T}{\varepsilon}} \varphi\bigl(t, s, a(s,\varepsilon)\bigr) ds\biggr), \quad x(0) = x_0, \tag{32}$$

has a solution $x(t,\varepsilon)$ defined on $[0, \frac{T}{\varepsilon}]$.

Fix $\eta > 0$. Let us estimate the difference between $x(t)$ and $y(t)$ (for the convenience of notation, we will omit the dependence on ε). We have

$$\begin{aligned}
\bigl|x(t) - y(t)\bigr| &= \varepsilon \int_0^t \biggl[X\biggl(\tau, x(\tau), \int_0^{\frac{T}{\varepsilon}} \varphi(\tau, s, a(s)) ds - X\biggl(\varepsilon, x(\tau), \int_0^{\tau} \varphi(\tau, s, a(s,\varepsilon)\biggr) ds\biggr] d\tau + \\
&+\varepsilon \int_0^t \biggl[X\biggl(\tau, x(\tau), \int_0^{\tau} \varphi(\tau, s, a(s)) ds\biggr) - X_0(y(\tau))\biggr] d\tau = \\
&= I_1(t) + \varepsilon \int_0^t \biggl[X\biggl(\tau, x(\tau), \int_0^{\tau} \varphi(\tau, s, a(s)) ds\biggr) - X\biggl(\tau, y(\tau), \int_0^{\tau} \varphi(\tau, s, y(s)) ds\biggr)\biggr] d\tau + \\
&+\varepsilon \int_0^{\tau} \biggl[X\biggl(\tau, y(\tau), \int_0^{\tau} \varphi(\tau, s, y(s)) ds\biggr) - X\biggl(\tau, y(\tau), \int_0^{\tau} \varphi(\tau, s, y(\tau)) ds\biggr)\biggr] d\tau + \\
&+\varepsilon \int_0^{\tau} \biggl[X(\tau, y(\tau), \int_0^{\tau} \varphi(\tau, s, y(\tau)) ds\biggr) - X_0(y(\tau))\biggr] d\tau = I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{33}$$

Let us now estimate each term of (33) separately. Due to the Lipschitz condition in (1.1), we have

$$\begin{aligned}
\bigl|I_1(t)\bigr| &\leq \varepsilon \int_0^t L \biggl| \int_0^{\frac{T}{\varepsilon}} \varphi\bigl(\tau, s, a(s)\bigr) ds - \int_0^{\tau} \varphi(\tau, s, a(s)) ds \biggr| d\tau \leq \\
&\leq \varepsilon L \int_0^t \biggl(\int_{\tau}^{\frac{T}{\varepsilon}} \bigl|\varphi\bigl(\tau, s, a(s)\bigr)\bigr| ds \biggr) d\tau.
\end{aligned} \tag{34}$$

Let us divide $[0, \frac{T}{\varepsilon}]$ into n subintervals of equal length by points t_i, $t_0 = 0 < t_1 < ... < t_n = \frac{T}{\varepsilon}$. Then

$$\varepsilon L \int_0^t \left(\int_\tau^{\frac{T}{\varepsilon}} \big|\varphi(\tau, s, a(s))\big| ds \right) d\tau = \varepsilon L \int_0^t \left(\sum_i^{n-1} \int_{t_i}^{t_{i+1}} \big|\varphi(\tau, s, a(s)) - \varphi(\tau, s, a(t_i))\big| ds + \right.$$

$$\left. + \sum_i^{n-1} \int_{t_i}^{t_{i+1}} \big|\varphi(\tau, s, a(s)) - \varphi(\tau, s, a(t_i))\big| ds \right) d\tau = I_{11}(t) + I_{12}(t). \tag{35}$$

Here, for every $\varepsilon \in [0, t]$, the summation is performed over such indices i that $[t_i, t_{i+1})$ cover the interval $[\tau, \frac{T}{\varepsilon}]$.

We now proceed to estimate each term of (35) separately. To estimate I_{11}, note that, by virtue of (1.2) and (29),

$$\big|\varphi(\tau, s, a(s)) - \varphi(\tau, s, a(t_i))\big| \leq \mu_0 \varepsilon K |s - t_i| \leq \varepsilon \mu_0 \varepsilon K \frac{T}{\varepsilon n} = \mu_0 \frac{KT}{n}. \tag{36}$$

Then

$$|I_{11}(t)| \leq \varepsilon L \int_0^t \left(\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mu_0 \frac{KT}{n} ds \right) d\tau \leq \varepsilon L \frac{T^2}{\varepsilon} \frac{\mu_0 K}{n} = \frac{\mu_0 K T^2 L}{n}. \tag{37}$$

We now turn to estimation of $I_{11}(t)$. Observe that, by virtue of (9), there exists such a function $\beta(t)$, continuous and decreasing monotonically to zero as $t \to \infty$, that

$$\int_0^t \left(\int_\tau^{\frac{\tau}{\varepsilon}} \bigg| \varphi\big(\tau, s, a(s_i)\big) \bigg| ds \right) d\tau \leq t\beta(t). \tag{38}$$

If t belongs to any subinterval $[t_i, t_{i+1}]$ except the first one, then it follows from (38) that

$$|I_{12}(t)| \leq \varepsilon L t \beta(t) \leq LT\beta(\tfrac{T}{\varepsilon n}). \tag{39}$$

For fixed n, the right-hand side of (39) approaches zero as $\varepsilon \to 0$.

If $t \in [0, t_1]$, we obtain by virtue of (38) and Dini's theorem that

$$|I_{12}(t)| \leq \varepsilon \beta(t) \leq \sup_{\tau \in [0,T]} \tau \beta(\tfrac{\tau}{\varepsilon}) \to 0, \quad \varepsilon \to 0. \tag{40}$$

Let us estimate the term $I_2(t)$ of (33). We have

$$|I_2(t)| \leq \varepsilon L \int_0^t |x(\tau) - y(\tau)| d\tau + \varepsilon L \int_0^t \left(\int_0^\tau |\varphi(\tau, s, a(s)) - \varphi(\tau, s, y(s))| ds \right) d\tau.$$

But, under the conditions of Lemma 4.1, $a(s, \varepsilon)$ belongs to a ρ-neighborhood of $y(\tau)$, which is a bounded on $[0, T]$ solution of the averaged problem. Therefore, for all $\varepsilon > 0$, $s \geq 0$, the function $a(s, \varepsilon)$ is bounded by a constant $R = R(\rho, y(\tau))$ independent of that function. Consequently,

$$|I_2(t)| \leq \varepsilon L \int_0^t \left(\int_0^\tau \mu(\tau, s)|a(s) - y(s)| ds \right) d\tau \leq 2R\varepsilon \int_0^t \left(\int_0^\tau \mu(\tau, s) ds \right) d\tau.$$

Similarly to the previous case, it follows from (7) that there exists a function $\beta_1(t)$, monotonically approaching zero as $t \to \infty$, such that

$$|I_2(t)| \leq 2R\varepsilon t \beta_1(t) \leq 2RT\beta_1(t). \tag{41}$$

Hence, for chosen $\eta > 0$, there exists T_0 such that

$$|I_2(t)| \le \tfrac{\eta}{4} \tag{42}$$

for $t \ge T_0$.

Obviously, we can assume $T_0 \in \big[0, \frac{T}{\varepsilon}\big]$. The estimate (42) for $t \in \big[0, T_0\big]$ is obtained by choosing a small ε_0, taking into account that $\int_0^\infty \mu(t,s)ds \le \mu_0$.

An estimate of $I_3(t)$ is obtained similarly to that of $I_2(t)$ due to the fact that the function $y(\tau)$ is bounded on $[0, T]$.

The term $I_4(t)$ is estimated in the same way as in the proof of Theorem 3.3 in [10], taking into account the first condition of (1.3). The method of estimation is similar to that for $I_1(t)$.

For given $\eta > 0$, we choose n and T_0 large enough to make the terms (39) sufficiently small to satisfy the estimate (42). Once such n and T_0 are fixed, we choose $\varepsilon_0 > 0$ such that, for $\varepsilon \le \varepsilon_0$, the terms (39),(40) and (42) for $t \in [0, T_0]$ are sufficiently small. The application of Gronwall's inequality completes the proof. □

4.2 Proof of Theorem 2.1

Proof. Choose $\eta > 0$ such that $\eta < \frac{\rho}{2}$ and keep it fixed. Let us construct a functional sequence $\{x_n(t,\varepsilon)\}$ in the following way: $x_0(t) = x_0$ and $x_n(t,\varepsilon)$, for every $\varepsilon > 0$, are defined recurrently as solutions of the Cauchy problems

$$\dot{x}_n = \varepsilon X\left(t, x_n, \int_0^{\frac{T}{\varepsilon}} \varphi(t,s,x_{n-1}(s,\varepsilon))ds\right). \tag{43}$$

As in the proof of Theorem 3.1, by virtue of (8), we can show that, for all $0 < \varepsilon < \bar{\varepsilon}$, the sequence $\{x_n(t,\varepsilon)\}$ converges uniformly with respect to $t \in [0, \frac{T}{\varepsilon}]$ as $n \to \infty$, and its limit function $x(t,\varepsilon)$ is a unique solution of the Cauchy problem for equation (1), $x(0) = x_0$, on $[0, \frac{T}{\varepsilon}]$. Clearly, the following estimate is valid for functions $\{x_n(t,\varepsilon)\}$:

$$|x_n(t_2,\varepsilon) - x_n(t_1,\varepsilon)| \le \varepsilon M|t_2 - t_1|. \tag{44}$$

Further, the system

$$\dot{x}_1(t,\varepsilon) = \varepsilon X\left(t, x_1(t,\varepsilon), \int_0^{\frac{T}{\varepsilon}} \varphi(t,s,x_0)ds\right), \tag{45}$$

$$x_1(0,\varepsilon) = x_0,$$

is a system of kind (30) in the Averaging Lemma with the function $a(t,\varepsilon) = x_0$, which obviously satisfies the conditions of Lemma 4.1.

Thus, for chosen $\eta > 0$, there exists $\varepsilon_0 \le \bar{\varepsilon}$ such that, for $\varepsilon < \varepsilon_0$, the estimate

$$|y(\varepsilon t) - x_1(t,\varepsilon)| \le \eta < \frac{\rho}{2}, \quad t \in [0, \tfrac{T}{\varepsilon}], \tag{46}$$

holds.

By (44), the function $x_1(t,\varepsilon)$ belongs to the class A_K introduced above, with $K = M$. Therefore, the system of equations for determining $x_2(t,\varepsilon)$ is a system of kind (30) with $a(t,\varepsilon) = x_1(t,\varepsilon)$. Hence, for $\varepsilon \le \varepsilon_0$, the function $x_2(t,\varepsilon)$ satisfies inequality (46) as well.

Now, setting $a(t,\varepsilon) = x_{n-1}(t,\varepsilon)$ for every n, we can conclude that all functions $x_n(t,\varepsilon)$ satisfy (26) with $K = M$, and hence

$$|x_n(t,\varepsilon) - y(\varepsilon t)| \le \eta < \frac{\rho}{2}, \quad t \in [0, \tfrac{T}{\varepsilon}], \tag{47}$$

for all $\varepsilon \le \varepsilon_0 = \varepsilon_0(\eta, M)$. We can therefore choose one and the same ε_0 for all n. In (47), passing to the limit as $n \to \infty$ for every $\varepsilon \le \varepsilon_0$ and taking into account the convergence of $x_n(t,\varepsilon)$ to $x(t,\varepsilon)$, we obtain the assertion of Theorem 2.1. $\square$

5 Proof of Theorem 2.2

Proof. Let $y(\tau) = y(\varepsilon t)$ be a solution of the boundary value problem (3) – (3′′). According to (1.4), for $t \in [0,T]$ this solution belongs to a domain D with some ρ-neighborhood.

Let $x_0 = y(0)$ be an initial value of this solution. We now seek a solution of (1) – (1′′) in the form

$$x(t,\varepsilon) = x(t, x_0 + \bar{x}, \varepsilon), \tag{48}$$

where $\bar{x}$ is chosen in some neighborhood of zero. We consider a solution $y(\tau, x_0 + \bar{x})$, $y(0, x_0 + \bar{x}) = x_0 + \bar{x}$ of the averaged problem. It follows from condition (8) and definition (2) of the averaged system that the function $X_0(x)$ satisfies the Lipschitz condition with a constant $L \le \frac{1}{T}$ (according to the problem statement, T is fixed).

By Gronwall's lemma, the following estimate

$$|y(\tau) - y(\tau, x_0 + \bar{x})| \le |\bar{x}| e^{LT} \tag{49}$$

holds until $y(\tau, x_0 + \bar{x})$ reaches the boundary of D. Therefore, if

$$|\bar{x}| < \frac{\rho}{2} e^{-LT}, \tag{50}$$

then a solution $y(\tau, x_0 + \bar{x})$ exists for $\tau \in [0,T]$ and belongs to a $\frac{\rho}{2}$-neighborhood of $y(\tau)$. Hence $y(\tau, x_0 + \bar{x})$, together with its $\frac{\rho}{2}$-neighborhood, belong to D.

We determine an unknown parameter $\bar{x}$ in (48) from the equation

$$F(x_0 + \bar{x}, x(\tfrac{T}{\varepsilon}, x_0 + \bar{x}, \varepsilon)) = 0. \tag{51}$$

Note that Theorem 2.1 applies to the solution $x(t, x_0 + \bar{x}, \varepsilon)$. Therefore, for $\varepsilon > 0$ sufficiently small, $x(t, x_0 + \bar{x}, \varepsilon)$ exists on $[0, \frac{T}{\varepsilon}]$. Moreover, for any $\eta > 0$, there exists $\varepsilon_0(\eta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the following estimate is valid:

$$|x(t, x_0 + \bar{x}, \varepsilon) - y(\varepsilon t, x_0 + \bar{x})| \le \eta(\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{52}$$

From this we see that, for $\varepsilon \in (0, \varepsilon_0)$, the mapping $F(x_0 + \bar{x}, x(\frac{T}{\varepsilon}, x_0 + \bar{x}, \varepsilon))$, with respect to $\bar{x}$, is well-defined in a ball $B_r(0)$, where $r \le \frac{\rho}{2} e^{-LT}$.

We note also that the points $x_0 + \bar{x}$ and $x(\frac{T}{\varepsilon}, x_0 + \bar{x}, \varepsilon)$ belong to the ρ-neighborhood of $y(\tau)$, for $\varepsilon \in (0, \varepsilon_0)$.

Then, by virtue of conditions (1.4) imposed upon the function $F(x,y)$, there exists a constant $N(r) > 0$ such that $\|\frac{\partial F}{\partial x}\| \le N(r)$ and $\|\frac{\partial F}{\partial y}\| \le N(r)$, for $\bar{x} \in B_r(0)$.

Let us represent $F(x_0 + \bar{x}, x(\frac{T}{\varepsilon}, x_0 + \bar{x}, \varepsilon))$ in the following way:

$$F(x_0 + \bar{x}, x(\tfrac{T}{\varepsilon}, x_0 + \bar{x}, \varepsilon)) = F(x_0 + \bar{x}, x(\tfrac{T}{\varepsilon}, x_0 + \bar{x}, \varepsilon)) -$$

$$-F(x_0+\bar{x}, y(T, x_0+\bar{x})) + F(x_0+\bar{x}, y(T, x_0+\bar{x}))-$$
$$-F(x_0, y(T, x_0)) = R_1(\bar{x}, \varepsilon) + M_1(\bar{x}, \varepsilon).$$

For $R_1(\bar{x}, \varepsilon)$, the estimate

$$|R_1(\bar{x}, \varepsilon)| \leq |N(r)(x(\tfrac{T}{\varepsilon}, x_0+\bar{x}, \varepsilon) - y(T, x_0+\bar{x}))| \leq N(r)\eta(\varepsilon) \to 0, \quad \varepsilon \to 0, \tag{53}$$

holds due to (52).

Under conditions of Theorem 2.2, solutions of the averaged problem depend smoothly on initial data, hence

$$M_1(\bar{x}, \varepsilon) = \Big(\frac{\partial F(x_0, y(T, x_0))}{\partial x} + \frac{\partial F(x_0, y(T, x_0))}{\partial y} \cdot \frac{\partial y(T, x_0)}{\partial x_0} \Big) \bar{x} +$$
$$+ \int_0^1 \Big(\frac{\partial F(x_0+s\bar{x}, y(T, x_0+s\bar{x}))}{\partial x} - \frac{\partial F(x_0, y(T, x_0))}{\partial x} \Big) \bar{x} ds +$$
$$+ \int_0^1 \Big(\frac{\partial F(x_0+s\bar{x}, y(T, x_0+s\bar{x}))}{\partial y} \cdot \frac{\partial y(T, x_0+s\bar{x})}{\partial z} \Big|_{z=x_0+s\bar{x}} -$$
$$- \frac{\partial F(x_0, y(T, x_0))}{\partial y} \cdot \frac{\partial y(T, x_0)}{\partial z} \Big|_{z=x_0} \Big) \bar{x} ds =$$
$$= \Big(\frac{\partial F(x_0, y(T, x_0))}{\partial x} + \frac{\partial F(x_0, y(T, x_0))}{\partial y} \cdot \frac{\partial y(T, x_0)}{\partial z} \Big|_{z=x_0} \Big) \bar{x} + R_2(\bar{x})\bar{x} + R_3(\bar{x})\bar{x}. \tag{54}$$

Let us consider each term of (54) separately. Using the notation of $F_0(x_0)$ in (11), the first term can be represented as

$$\Big(\frac{\partial F(x_0, y(T, x_0))}{\partial x} + \frac{\partial F(x_0, y(T, x_0))}{\partial y} \cdot \frac{\partial y(T, x_0)}{\partial z} \Big|_{z=x_0} \Big) \bar{x} = \frac{\partial F_0}{\partial x_0} \bar{x}.$$

Regarding $R_2(\bar{x})$, by the uniform continuity of partial derivatives and (49), for $|\bar{x}| \leq r$, we get the estimate

$$|R_2(\bar{x})| \leq \delta(r) \to 0, \quad r \to 0, \tag{55}$$

where $r \leq \frac{\rho}{2} e^{-LT}$.

To estimate $R_3(\bar{x})$, note that the derivative $\frac{\partial y(T,z)}{\partial z}$ with respect to initial data satisfies a linear variational equation and hence is a continuous function of a parameter z. So, similarly as above, for $|\bar{x}| \leq r$, we get the estimate

$$|R_3(\bar{x})| \leq \delta_1(r) \to 0, \quad r \to 0. \tag{56}$$

Now, equation (51) for determining $\bar{x}$ can be represented in the form

$$|\bar{x}| = -\Big(\frac{\partial F_0}{\partial x_0} \Big)^{-1} \big(R_1(\bar{x}, \varepsilon) + \big(R_2(\bar{x}) + R_3(\bar{x}) \big) \bar{x} \big),$$

or

$$\bar{x} = \Big(\frac{\partial F_0}{\partial x_0} \Big)^{-1} M(\bar{x}, \varepsilon), \tag{57}$$

where $M(\bar{x}, \varepsilon)$ satisfies the inequality

$$|M(\bar{x}, \varepsilon)| \leq N(r)\eta(\varepsilon) + \delta_2(r)\bar{x}, \tag{58}$$

where $\eta(\varepsilon) \to 0$, $\varepsilon \to 0$, $\delta_2(r)$, $r \to 0$.

Let $C = \|(\frac{\partial F_0}{\partial x_0})^{-1}\|$. Choose r so that

$$\delta_2(r) \leq \frac{1}{2}, \tag{59}$$

and then choose $\varepsilon_1 \leq \varepsilon_0$ such that

$$\eta(\varepsilon) \leq \frac{r}{2CN(r)}. \tag{60}$$

Then, for $|\bar{x}| \leq r$, from (40) we obtain

$$\left\| \left(\frac{\partial F_0}{\partial x_0} \right)^{-1} M(\bar{x}, \varepsilon) \right\| \leq C(N(r)\eta(\varepsilon) + \delta_2(r)|\bar{x}|) \leq \frac{r}{2} + \frac{r}{2} = r.$$

Thus, if (59) and (60) hold, $(\frac{\partial F_0}{\partial x_0})^{-1} M(\bar{x}, \varepsilon)$ maps the ball $B_0(r)$ into itself. Note also that, by Theorem 3.1, there exists a solution $x(t, x_0 + \bar{x}, \varepsilon)$ that is unique on $\left[0, \frac{T}{\varepsilon}\right]$ and continuously depends on $\bar{x}$. Therefore the mapping $(\frac{\partial F_0}{\partial x_0})^{-1} M(\bar{x}, \varepsilon)$ is well-defined and continuous, and, by Brouwer's theorem, it has a fixed point $\bar{x}^* = \bar{x}^*(\varepsilon, r)$, which is the initial value of the solution of the boundary value problem $(1) - (1'')$.

Let us now pick r, as a function of a parameter ε, so that $r(\varepsilon) \to 0, \varepsilon \to 0$. We then pick $\varepsilon_1 \leq \varepsilon_0$ so that the function $\eta(\varepsilon)$ in (53) satisfies the inequality

$$\frac{\eta(\varepsilon)}{r(\varepsilon)} \leq \frac{1}{2CN(r(\varepsilon))}.$$

Note that such a choice is possible, since a function $N(r(\varepsilon))$, by which the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are bounded in the ball $B_0(r)$, does not increase as $r(\varepsilon)$ decreases.

The estimate (12) now follows from (49) and (52), and the proof is complete. □

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