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Weak Solutions for Anisotropic Nonlinear Discrete Dirichlet Boundary Value Problems in a Two-Dimensional Hilbert Space

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Abstract: Using a minimization method we study the existence of weak solutions for a family of nonlinear discrete Dirichlet boundary value problems where the solution lies in a discrete $(T_1 \times T_2)$-Hilbert space. The originality of this work is the study done on a two-dimensional Hilbert space.

Keywords: *discrete boundary value problem; critical point; weak solution; two dimensional discrete Hilbert space; electrorheological fluids.*

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1 Introduction

In the last few years, great attention has been paid to the study of fourth-order nonlinear difference equations. These equations have been widely used to study discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. For background and recent results, we refer the reader to [3]– [12], [13] and the references therein.

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The main purpose of the present paper is to extend the study of difference equations in two dimensions. These models are of independent interest since their mathematical structure has a different nature. In the literature, to our knowledge, no scientific study has concerned these types of problems which are nevertheless discrete variants of the anisotropic or isotropic partial differential equations and are usually studied in connection with numerical analysis.

We study a p-Laplacian difference equation on the subset of integers. So, for $i, j \in \mathbb{N}$ with $i \leq j$, we define $\mathbb{N}[i, j]$ as the discrete interval $\{i, i+1, \ldots, j\}$ and we investigate the existence of solutions for the following nonlinear discrete Dirichlet boundary value problem:

$$\begin{cases} -\Delta\big(a(k-1, h-1, \Delta u(k-1, h-1))\big) = f(k, h), \ (k, h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2], \\ u(k, h) = 0, \quad \forall (h, k) \in \Gamma, \end{cases} \tag{1}$$

where

$$\Gamma = (\{0, T_1 + 1\} \times \mathbb{N}[0, T_2 + 1]) \cup (\mathbb{N}[0, T_1 + 1] \times \{0, T_2 + 1\})$$

is the boundary of the domain $\mathbb{N}[0, T_1+1]\times\mathbb{N}[0, T_2+1]$; $\Delta u(k, h) = u(k+1, h+1)-u(k, h)$ is the forward difference operator and

$$a : \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2] \times \mathbb{R} \longrightarrow \mathbb{R}, \quad f : \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2] \longrightarrow \mathbb{R}$$

are functions to be defined later.

Our goal is to use a minimization method in order to establish some existence results of solutions of (1). The idea of the proof is to transfer the problem of the existence of solutions for (1) into the problem of existence of a minimizer for some associated energy functional. This method was successfully used by Bonanno et al. [2] for the study of an eigenvalue nonhomogeneous Neumann problem, where, under an appropriate oscillating behavior of the nonlinear term, they proved the existence of a determined open interval of positive parameters for which the problem under consideration admits infinitely many weak solutions that strongly converge to zero, in an appropriate Orlicz-Sobolev space.

The remaining part of this paper is organized as follows. Section 2 is devoted to mathematical preliminaries. The main existence result is stated and proved in Section 3. In the last section of this paper we study an extension of the problem (1).

2 Mathematical Preliminaries

We define the $(T_1 \times T_2)$-dimensional Hilbert space

$$H = \{u : \mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1] \longrightarrow \mathbb{R} \quad \text{such that} \quad u(k, h) = 0, \quad \forall\ (h, k) \in \Gamma\}$$

with the inner product

$$\langle u, v\rangle = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} u(k, h)v(k, h)$$

and the associated norm defined by

$$\|u\| = \left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k, h)|^2\right)^{1/2}.$$

However, we introduced another norm on the space H, namely

$$|u|_m = \left(\sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k,h)|^m \right)^{1/m}, \quad \forall\, m \geq 2.$$

Due to equivalence of $\|.\|$ and $|.|_m$ there exist constants $C_2 \geq C_1 > 0$ such that

$$C_1 \|u\| \leq |u|_m \leq C_2 \|u\|, \quad \forall u \in H. \tag{2}$$

For the data f and a we impose the following conditions:

$$f \in H, \quad a(k,h,.) : \mathbb{R} \longrightarrow \mathbb{R} \text{ is continuous } \forall (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \tag{3}$$

and there exists a mapping $A : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$a(k,h,\xi) = \frac{\partial}{\partial \xi} A(k,h,\xi) \qquad \text{and} \qquad A(k,h,0) = 0 \quad \forall (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]. \tag{4}$$

We also assume that there exists a positive constant C_3 such that

$$|a(k,h,\xi)| \leq C_3 \left(1 + |\xi|^{p(k,h)-1} \right). \tag{5}$$

The following relations hold true for all $(k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]$:

$$\left(a(k,h,\xi) - a(k,h,\eta) \right) (\xi - \eta) > 0, \ \forall\, \xi, \eta \in \mathbb{R} \text{ with } \xi \neq \eta \tag{6}$$

and

$$|\xi|^{p(k,h)} \leq a(k,h,\xi)\xi \leq p(k,h) A(k,h,\xi), \ \forall\, \xi \in \mathbb{R}. \tag{7}$$

Example 2.1 We can give the following function:

$$A(k,h,\xi) = \frac{1}{p(k,h)} \left(\left(1 + |\xi|^2\right)^{p(k,h)/2} - 1 \right),$$

where

$$a(k,h,\xi) = \left(1 + |\xi|^2\right)^{(p(k,h)-2)/2} \xi, \ \ \forall\ (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2], \ \xi \in \mathbb{R}$$

and conditions on the function a are checked.

In this paper, we assume that the function

$$p : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \longrightarrow (1, +\infty). \tag{8}$$

We will use the following notations:

$$p^- = \min_{k \in \mathbb{N}[1,T_1]} \left(\min_{h \in \mathbb{N}[1,T_2]} p(k,h) \right) \quad \text{and} \quad p^+ = \max_{k \in \mathbb{N}[1,T_1]} \left(\max_{h \in \mathbb{N}[1,T_2]} p(k,h) \right). \tag{9}$$

The discrete Wirtinger type inequalities can be generalized in two dimensions as follows.

Lemma 2.1 *For any function $u \in H$, the following inequality holds:*

$$4\sin^2\left(\frac{\pi}{2(T_1+1)}\right)\sum_{h=1}^{T_2}\sum_{k=1}^{T_1}|u(k,h)|^2 \leq \sum_{h=1}^{T_2}\sum_{k=1}^{T_1}|\Delta u(k-1,h-1)|^2 \leq 4\cos^2\left(\frac{\pi}{2(T_1+1)}\right)\sum_{h=1}^{T_2}\sum_{k=1}^{T_1}|u(k,h)|^2.$$

Proof. Let $u \in H$. For h fixed in $\mathbb{N}[0, T_2+1]$, since $u(0,h) = 0 = u(T_1+1,h)$, the discrete Wirtinger type inequalities hold (see Theorem 12.6.1, page 860 in [1]). So, just apply the sum for $h = 0, \dots, T_1+1$ and make a variable change. $\square$

We need the following auxiliary result throughout our paper.

Lemma 2.2 *For any function $u \in H$ with $\|u\| > 1$, there exist constants $C_4, C_5 > 0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)} \geq C_4\|u\|^{p^-} - C_5. \tag{10}$$

Proof. Fix $u \in H$ with $\|u\| > 1$.
Let

$$v : \mathbb{N}[0, T_1+1] \longrightarrow \mathbb{R},\ k \mapsto v(k) = u(k,h)$$

and

$$q : \mathbb{N}[0, T_1] \longrightarrow (1, +\infty),\ k \mapsto q(k) = p(k,h) \text{ with } h \text{ fixed in } \mathbb{N}[0, T_2+1].$$

According to Lemma 1 in [9] we have

$$\sum_{k=1}^{T_1+1}|\Delta v(k-1)|^{q(k-1)} \geq T_1^{(2-q^-)/2}\|v\|^{q^-} - T_1.$$

Then, there exist two constants $C_4, C_5 > 0$ such that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)} \geq C_4\|u\|^{p^-} - C_5. \ \square$$

3 Main Results

In this section we study the existence of weak solution that we state in the following theorem.

Theorem 3.1 *Assume that (3)-(8) are satisfied. Then there is at least one weak solution for problem (1).*

By a weak solution for problem (1) we understand a function $u \in H$ such that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}f\big(k,h\big)v(k,h) \tag{11}$$

for any $v \in H$. The energy functional $J : H \longrightarrow \mathbb{R}$ corresponding to problem (1) is defined by the formula

$$J(u)=\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}f\big(k,h\big)u(k,h). \qquad (12)$$

This energy functional is vastly different from the energy functions defined before this work. Thus we indicate its properties. It is easy to see that the functional J is continuous, Gateaux differentiable and its Gateaux derivative J' at u reads

$$\langle J'(u),v\rangle=\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}f\big(k,h\big)v(k,h) \qquad (13)$$

for all $v \in H$. If $u \in H$ is a critical point to J, namely $\langle J'(u),v\rangle = 0$ for all $v \in H$, we observe that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}f\big(k,h\big)v(k,h)=0.$$

Since v is any in H, we see that the critical point u to J satisfies the problem (1).

The following results prove Theorem 3.1.

Lemma 3.1 *The functional J is coercive and bounded from below.*

Proof. We will only prove that the energy functional is coercive since the boundedness from below of J is a consequence of coerciveness.

$$\begin{aligned}
J(u) &= \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}f\big(k,h\big)u(k,h)\\
&\geq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{1}{p(k-1,h-1)}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|f(k,h)||u(k,h)|\\
&\geq \frac{1}{p^+}\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\\
&- \left[\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|f(k,h)|^2\right]^{\frac{1}{2}}\left[\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^2\right]^{\frac{1}{2}}\\
&\geq \frac{C_4}{p^+}\|u\|^{p^-}-C_5-C_6\|u\|. \qquad (14)
\end{aligned}$$

Hence, since $p^- > 1$, the functional J is coercive. □

Lemma 3.2 *The functional J is weakly lower semi-continuous.*

Proof. For any $u \in H$, let

$$I(u)=\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1)).$$

According to the convexity of A, the function I is convex. Thus it is enough to show that I is lower semi-continuous.

Let us fix $u \in H$ and $\varepsilon > 0$. Since I is convex, we have $I(v) - I(u) \geq \langle I'(u), v - u\rangle$ for any $v \in H$. Therefore

$$\begin{aligned}
I(v) &\geq I(u) + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a\Big(k-1, h-1, \Delta u(k-1,h-1)\Big) \times \\
&\qquad \Big(\Delta v(k-1,h-1) - \Delta u(k-1,h-1)\Big) \\
&\geq I(u) - \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \Big|a(k-1,h-1,\Delta u(k-1,h-1))\Big| \times \\
&\qquad \Big|\Delta v(k-1,h-1) - \Delta u(k-1,h-1)\Big| \\
&\geq I(u) - \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \Big|a(k-1,h-1,\Delta u(k-1,h-1))\Big| \times \\
&\qquad \Big|(v(k,h) - v(k-1,h-1)) - (u(k,h) - u(k-1,h-1))\Big| \\
&\geq I(u) - (\Lambda(u) + (\Phi(u)),
\end{aligned}$$

where

$$\Lambda(u) = \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))||v(k,h) - u(k,h)|$$

and

$$\Phi(u) = \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))||v(k-1,h-1) - u(k-1,h-1)|.$$

We use the Schwartz inequality to get

$$\begin{aligned}
\Lambda(u) &\leq \left[\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))|^2\right]^{\frac{1}{2}} \\
&\quad \times \left[\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |v(k,h) - u(k,h)|^2\right]^{\frac{1}{2}} \\
&\leq \left[\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))|^2\right]^{\frac{1}{2}} \|v-u\|
\end{aligned}$$

and

$$\Phi(u) \leq \left[\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))|^2\right]^{\frac{1}{2}} \|v-u\|.$$

Consequently, we have

$$\begin{array}{rcl} I(v) & \geq & I(u) - \left[1 + 2\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))|^2\right]^{\frac{1}{2}} \|v-u\| \\ & \geq & I(u) - \varepsilon \end{array}$$

for all $v \in H$ with $\|v-u\| < \sigma = \dfrac{\varepsilon}{K(T_1,T_2,u)}$, where

$$K(T_1,T_2,u) = \left[1 + 2\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))|^2\right]^{\frac{1}{2}}.$$

We conclude that the functional I is lower semi-continuous. This implies that the functional J is also semi-continuous. □

Proof of Theorem 3.1. Since J is proper, weakly lower semi-continuous and coercive on H, using the relation between critical points of J and problem (1), we deduce that J has a minimizer which is a weak solution of (1). □

4 Uniqueness of Solution

In this section we examine the uniqueness of the weak solution for the problem (1). To do this, let us consider u, $v \in H$ being two solutions to the problem. By choosing $u - v \in H$ as a test function, according to the notion of weak solution, we obtain

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta(u-v)(k-1,h-1) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f\big(k,h\big)(u-v)(k,h)$$

and

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta v(k-1,h-1))\Delta(u-v)(k-1,h-1) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f\big(k,h\big)(u-v)(k,h).$$

By subtracting the two equalities above, we have

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \left(a(k-1,h-1,\Delta u(k-1,h-1)) - a(k-1,h-1,\Delta v(k-1,h-1))\right) \times \Delta(u-v)(k-1,h-1) = 0.$$

Therefore, according to the assumption (6), necessarily

$$\Delta u(k-1,h-1) = \Delta v(k-1,h-1), \quad \text{for all} \quad (k,h) \in \mathbb{N}[1,T_1+1] \times \mathbb{N}[1,T_2+1],$$

so, using Lemma 2.1

$$\begin{array}{rcl} \|u-v\|^2 & = & \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)-v(k,h)|^2 \\ & \leq & \left(4\sin^2\left(\dfrac{\pi}{2(T_1+1)}\right)\right)^{-1} \sum_{h=1}^{T_2}\sum_{k=1}^{T_1} |\Delta u(k-1,h-1) - \Delta v(k-1,h-1)|^2 \\ & \leq & 0, \end{array}$$

which means that

$$u = v.$$

5 An Extension

In this section we are going to show that the existence result obtained for problem (1) can be extended to the problem

$$\begin{cases} -\Delta\big(a(k-1,h-1,\Delta u(k-1,h-1))\big) = f(k,h,u(k,h)), & (k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2], \\ u(k,h) = 0, \quad (h,k)\in\Gamma. \end{cases} \tag{15}$$

We shall replace the hypothesis on the source term f by the following. For each couple $(k,h)\in\mathbb{N}[0,T_1]\times\mathbb{N}[0,T_2]$, the function $f(k,h,.):\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and there exists a constant $C_7>0$ and $r:\mathbb{N}[0,T_1]\times\mathbb{N}[0,T_2]\longrightarrow[2,+\infty)$ such that

$$|f\,(k,h,u(k,h))\,| \le C_7\left(1+|u(k,h)|^{r(k,h)-1}\right), \tag{16}$$

where $2\le r(k,h)<p^-$ for all $(k,h)\in\mathbb{N}[0,T_1]\times\mathbb{N}[0,T_2]$.

In what follows, we denote by

$$r^- = \min_{\{(k,h)\in\mathbb{N}[0,T_1]\times\mathbb{N}[0,T_2]\}} r(k,h) \qquad \text{and} \qquad r^+ = \max_{\{(k,h)\in\mathbb{N}[0,T_1]\times\mathbb{N}[0,T_2]\}} r(k,h).$$

We denote

$$F(k,h,\xi) = \int_0^\xi f(k,h,s)ds \quad \text{for} \quad (k,h,\xi)\in\mathbb{N}[0,T_1]\times\mathbb{N}[0,T_2]\times\mathbb{R}$$

and we deduce that there exists a constant $C_8>0$ such that

$$|F(k,h,u)| \le C_8\left(1+|u(k,h)|^{r(k,h)}\right). \tag{17}$$

By a weak solution, we mean a function $u\in H$ such that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f\big(k,h,u(k,h)\big)v(k,h) \tag{18}$$

for any $v\in H$.

Let

$$L(u) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F\big(k,h,u(k,h)\big).$$

Then, for any $u,v\in H$,

$$\langle L'(u),v\rangle = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f\big(k,h,u(k,h)\big)v(k,h).$$

It is easy to see that L' is completely continuous and thus, the functional L is weakly lower semi-continuous. Therefore, the energy functional J associated with problem (15), defined by

$$J(u)=\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F\big(k,h,u(k,h)\big) \tag{19}$$

is such that $J\in C^1(H,\mathbb{R})$ and is weakly lower semi-continuous with

$$\begin{aligned}\langle J'(u),v\rangle &= \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)\\ &\quad -\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}f\big(k,h,u(k,h)\big)v(k,h)\end{aligned}$$

for all $v\in H$. This implies that the weak solution of problem (15) coincides with the critical points of the functional J. It suffices now to show that the energy functional J is coercive to conclude that the problem (15) admits at least one weak solution.
According to hypothesis (17) and using the relation (2), we have

$$\begin{aligned}L(u) &= \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F\big(k,h,u(k,h)\big)\\ &\le \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}C_8\left(1+|u(k,h)|^{r(k,h)}\right)\\ &\le C_8T_1T_2+C_8\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{r(k,h)}\\ &\le C_9+C_8\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{r^-}+C_8\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{r^+}\\ &\le C_9+C_{10}\left(\|u\|^{r^-}+\|u\|^{r^+}\right).\end{aligned}$$

Therefore the inequality (14) becomes

$$J(u)\ge\frac{C_4}{p^+}\|u\|^{p^-}-C_5-\left(C_9+C_{10}\left(\|u\|^{r^-}+\|u\|^{r^+}\right)\right), \tag{20}$$

namely

$$J(u)\ge\frac{C_4}{p^+}\|u\|^{p^-}-C_{10}\left(\|u\|^{r^+}+\|u\|^{r^-}\right)-C_{11}, \tag{21}$$

where C_{10} and C_{11} are positive constants. Hence, since $p^->r+\ge r^-\ge 2$, the functional J is coercive.

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