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Tube-MPC Based on Zonotopic Sets for Uncertain System Stabilisation

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Abstract: This paper is dedicated to the model predictive control (MPC) for constrained discrete time systems with additive uncertainties to track a reference system. The tracking MPC problem distributes a control law appropriate for regulating constrained uncertain system to a given target system. Though, when the target operating point changes, the feasibility of the controller may be lost and the controller misses to track the reference. In this paper, a novel MPC for tracking variation system references is introduced. The main issue consists in minimizing a cost that penalizes the error between the state of the original system and the reference system state. The polyhedral invariant set for tracking is considired an extended terminal constraint. The properties of the proposed controller have been tested in two examples. Simulation results show that the proposed tracking MPC successfully achieves robust tracking in terms of control performance.

Keywords: *LTV systems; robust control; tracking MPC; polytopic invariant sets.*

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1 Introduction

An effective strategy for the practical implementation of the robust MPC is the tube-based MPC. The design of a robust control law guarantees the satisfaction of hard constraints and is addressed by means of calculating a sequence of state space regions, called an accessibility tube. The term tube is based on control techniques whose purpose is to maintain all the possible trajectories of an uncertain system inside a sequence of admissible regions using set-theory related tools. Such approach has been widely employed to robustify MPC [1–5].

The tube-based robust model predictive control (TMPC) is an advanced control algorithm that can deal with model uncertainty. The basic idea of the tube-based robust MPC is to maintain a state trajectory of an uncertain system inside a sequence of tubes [6]. The TMPC is motivated by the fact that a real state trajectory differs from a state trajectory of a nominal system due to uncertainty [17]. In 2001, [8] developed a tube-based robust model predictive controller for a linear time-invariant (LTI) system subject to bounded disturbance. The control law is obtained by solving an unconstrained LQR problem. The objective is to drive the state of an uncertain system to a terminal set while using the input as little as possible. Constraint fulfillment is guaranteed by replacing the original constraints with more stringent ones. A larger control horizon implies better control performance at the price of a higher computational load, so a suitable trade off is required. In 2004, [11] proposed a tube-based robust MPC using the time-varying control inputs instead of the LTI control law. A sequence of time-varying control inputs is obtained by solving an optimal control problem subject to additional constraints sets in order to guarantee robust stability. Since the control inputs are time-varying, the proposed MPC algorithm can achieve better control performances than the conventional tube-based MPC algorithm using the LTI control law.

Tube-based MPC approaches are motivated by the fact that the predicted evolution of a system obtained using a nominal model differs from the real evolution due to uncertainty. An MPC formulation that permits to consider this mismatch in the controller synthesis is the tube-based one, whose basis consists in computing the region around the nominal prediction that contains the state of the system under any possible uncertainties [8–10].

Gonzalez et al. [12] proposed a tube-based robust MPC for tracking of a linear time-varying (LTV) system subject to bounded disturbance. The proposed MPC algorithm requires an additional assumption that the time-varying parameter at each step within the prediction horizon is known a priori. Then a reachable set at each time step is calculated instead of a disturbance invariant set in order to reduce the conservativeness. Although the conservativeness is reduced, the computational problem is more severe because both the optimal control problem and the reachable set are computed on-line.

Bumroongsri and Kheawhom [13] proposed a strategy for the design of a tube-based output feedback MPC which is independent of the estimation method employed. They formulate a control policy by choosing a candidate estimate that is consistent with the reachable sets of the system under control. The proposed method can be combined with any estimation scheme as long as the assumed error bounds are satisfied. They show that the proposed method is recursively feasible, robustly exponentially stable, and performs better than other available strategies. The idea of the Tube MPC is motivated by robustness considerations for system dynamics affected by bounded disturbances.

In this work, a new strategy for formulation of an optimal problem of the robust tube MPC based on zonotopic invariant sets was established. This method contains two steps. The first one, off-line, calculates a sequence of state feedback control laws for global systems corresponding to a sequence of zonotopic invariant sets using the LMI technique proposed by [14]. For the nominal system, a feedback control law using the LQR problem is computed. The second step, on-line, at each sampling time, determines the smallest invariant tube containing the measured state and implements the computed state feedback control. Such a control is obtained from the last two control laws. Finally, the global control law is applied to the original process allowing to improve system control performances.

The paper is organized as follows. General problem setup is presented in Section 2. Then, in Section 3, the robust model predictive control is described. Polyhedral and zonotopic sets are introduced in Section 4. Main result is presented in Section 5. The implementation of the proposed algorithm is illustrated in two numerical examples in Section 6. Finally, the paper is concluded.

2 General Problem Setup

Consider the following discrete-time LTV system with disturbance:

$$x_{k+1} = A(k)x_k + B(k)u_k + w, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^n$ is the control input, $w \in \mathbb{R}^n$ is the bounded disturbance. The system is subject to the state constraint $x \in X$, the control constraint $u \in U$, and the disturbance constraint $w \in W$, where $X \subset R^n$, $U \subset R^m$ and $W \subset R^n$ are convex polytopes and each set contains the origin as an interior point.

Remark 2.1 $[A(k),\ B(k)] \in conv\,\{[A_j, B_j], \forall j \in 1,\ 2,\ ...,\ L\}$, where $[A_j, B_j]$ are vertices of the convex hull and L is the number of vertices of the convex hull. The pair $[A_j, B_j]$ is controllable.

Let the nominal system be defined by

$$x'_{k+1} = Ax'_k + Bu'_k, \tag{2}$$

where $x' \in R^n$ and $u' \in R^m$ are the state and control input of the nominal system, respectively. Now, we calculate the difference between the global and the nominal system:

$$\begin{aligned} x_{k+1} - x'_{k+1} &= Ax_k + Bu_k + w - (Ax'_k + Bu'_k) \\ &= A(x_k - x'_k) + B(u_k - u'_k) + w. \end{aligned} \tag{3}$$

The objective is to robustly stabilize the system (1). The presence of a persistent disturbance w means that it is not possible to regulate the state x to the origin. The best that can be hoped for, is to regulate the state to a neighborhood of the origin. Then the proposed idea is to compensate the mismatch between the real and the nominal state, and to steer the nominal system as close as possible to the reference without constraints violation. For that purpose, we consider the following control law:

$$u_k = K(x_k - x'_k) + u'_k, \tag{4}$$

where K is the disturbance rejection gain whose goal is to compensate the system realisation at each sampling instant.

The system (3) is rewritten as

$$x_{k+1} - x'_{k+1} = (A + BK)(x_k - x'_k) + w. \tag{5}$$

Then, using the nominal dynamics, an effective invariant tube is defined in the state space of the nominal system and a deterministic finite horizon optimization problem is formulated and solved on-line resulting in an optimal sequence of nominal controls $u'_k = \left\{u'_{k/k}, u'_{k+1/k}, ...\right\}$. Finally, the control law (4) is implemented to the process (1).

3 Robust Model Predictive Control

In this section, we present an off-line step for the MPC problem developed by [15], which consists in determining a sequence of feedback control law. In order to build an invariant tubes based on zonotopes, from the gains K_i to be found by this algorithm we establish the zonotopic invariant sets.

By solving the optimization problem presented in (6)-(10), we obtain a state feedback control law $u_k = K_i x_k$ with a state feedback gain $K_i = Y_i Q_i^{-1}$ that can stabilize the system while satisfying the input and output constraints.

The optimization problem is shown in the following LMI:

$$\min_{\gamma_i, Y_i, Q_i} \quad \gamma_i \tag{6}$$

subject to

$$\begin{bmatrix} 1 & x_i^T \\ x_i & Q_i \end{bmatrix} \geqslant 0, \tag{7}$$

$$\begin{bmatrix} Q_i & QA_j^T + Y_i^T B_j^T & Q_i \Theta^{1/2} & Y_i^T R^{1/2} \\ A_j Q_i + B_j Y_i & Q_i & 0 & 0 \\ \Theta^{1/2} Q_i & 0 & \gamma_i I & 0 \\ R^{1/2} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \geqslant 0 \ \forall j = 1, 2, \ldots, L, \tag{8}$$

$$\begin{bmatrix} X & Y_i \\ Y_i^T & Q_i \end{bmatrix} \geqslant 0, \ X_{hh} \leqslant u_{h,\max}^2, \ h = 1, 2, \ldots, n_u, \tag{9}$$

$$\begin{gathered} \begin{bmatrix} S & C((A_j Q_i + B_j Y_i) \\ (A_j Q_i + B_j Y_i)^T C^T & Q_i \end{bmatrix} \geqslant 0, \ S_{rr} \leqslant y_{r,\max}^2, \\ r = 1, 2, \ldots, n_y, \quad \forall j = 1, 2, \ldots, L, \end{gathered} \tag{10}$$

where Q is a symmetric matrix. For each K_i, calculate the sequence of zonotopic invariant sets as shown in [14].

4 Polyhedral and Zonotopic Sets

Definition 4.1 (G-representation of a zonotope) Given a vector $c \in R^n$ and a set $G = \{g_1, \ldots, g_m\}$ of vectors of R^n, $m \geqslant n$, a zonotope Z of order m is defined as follows:

$$Z = \left\{ x \in R^n, \, x = c + \sum_{i=1}^{p} \gamma_i g_i; \, -1 \leqslant \gamma_i \leqslant 1 \right\}. \tag{11}$$

The vector c is called the center of the zonotope Z. The vectors $g_1, ..., g_m$ are called generators of Z.

Definition 4.2 (V-representation of a polytope). Given r vertices $v_i \in R^n, P = conv\{v_1, ..., v_r\}$ is a convex polytope, where *conv* is the convex hull operator. To obtain zonotopic sets from polyhedral ones, we have to perform the following three steps:
Step 1: Compute the vertices $v_i \in R^n$ (V-representation) of all N polytopes $S_i, i = 1, ..., N$.
Step 2: Obtain the minimum and maximum values of each polytope i:

$$\begin{aligned} m_{\min} &= \min(V_i^1, ..., V_i^r)\ , \\ m_{\max} &= \max(V_i^1, ..., V_i^r), \end{aligned} \tag{12}$$

where V_i^r is the i-th component of the vector V^j and r is the number of the vertices of each polytope.
Step 3: Compute a G-representation of the n-dimensional interval $[m_{\min}, m_{\max}]$:

$$[m_{\min}, m_{\max}] = \left\{ x = c + \sum_{i=1}^{P} \gamma_i . g_i\ , -1 \leqslant \gamma_i \leqslant 1 \right\}, \tag{13}$$

where

$$c = 0.5(m_{\min} + m_{\max}), \tag{14}$$

$$g_i^{(i)} = \begin{cases} 0.5(m_{\max} - m_{\min}), & if\ i = j, \\ 0, & otherwise. \end{cases} \tag{15}$$

5 Main Result

In this part, we propose a robust tube-based MPC controller via invariant zonotopic sets. The idea of the Tube MPC is motivated by robustness considerations for system dynamics affected by bounded disturbances instead of considering each possible disturbance sequence separately in the prediction. The effect of the bounded disturbances is over-approximated by a sequence of sets which contains all possible state trajectories. In order to prevent these sets from growing too quickly within the prediction horizon, feedback is assumed in the predictions. Here we do not consider system dynamics affected by disturbances at each time instance, but instead consider (uncertain) initial offsets, which lead to a similar uncertainty in the predictions. In this way, we use robust MPC methods in order to approximate the input generated by a (nominal) MPC controller for an infinite number of initial conditions by a single robust MPC controller.

Determination of the invariant tube: Consider the zonotopic sets

$$Z = \left\{ x \in R^n,\ x = c + \sum_{i=1}^{p} \gamma_i g_i;\ -1 \leqslant \gamma_i \leqslant 1 \right\}. \tag{16}$$

The predicted state trajectory when the initial state $x^{'} = \left(x^{'}_{0/k}, x^{'}_{1/k}, \ldots, x^{'}_{N-1/k} \right)$, where $x^{'}_{k/k} \subseteq R^n$ is the k steps ahead prediction calculated from the set of initial conditions x_k.

An invariant tube is written in the form

$$T = \left(\left\{x'_{0/k}\right\} \oplus Z, \left\{x'_{1/k}\right\} \oplus Z, \ldots, \left\{x'_{N-1/k}\right\} \oplus Z\right), \tag{17}$$

where $\oplus$ is the Minkowski sum. Since $x_{k+1} - x'_{k+1}$ is bounded by T, we can control the nominal system $x'_{k+1} = Ax'_k + Bu'_k$ in such a way that the LTI system with disturbance $x_{k+1} = A(k)x_k + B(k)u_k + w$ satisfies the original state and control constraints $x \in X$ and $u \in U$, respectively. To achieve this, the tighter constraint sets for the nominal system are employed, $x'_i \in X \oplus T$, $u'_i \in U \oplus KT$ for $i \in \{0, \ldots, N-1\}$.

It has been demonstrated, that the control law $u_k = K(x_k - x'_k) + K_{lqr}x'_k$ keeps the states x of the original system $x_{k+1} = A(k)x_k + B(k)u_k + w$ close to the state x' of the nominal system $x'_{k+1} = Ax'_k + Bu'_k$. It is clear that if we can regulate x' to the origin, then x must be regulated to a robust positively invariant set T whose center is at the origin.

Terminal cost and terminal invariant set: A common technique to ensure the asymptotic stability of MPC is to incorporate both a terminal cost and a set of terminal constraints [16, 17]. In this part, we are interested in the two problems related to the nominal system (2). Note that the stability properties for analogous control approaches have been analyzed in the literature. The goal of the final cost is to provide closed-loop stability. For this reason, it requires the use of a Lyapunov function with a stabilization control law. In our case, a procedure similar to [1] was followed for the nominal system (2).

In order to ensure stability, an additional terminal constraint is implemented, $x'_N \in X'_f \subset X \oplus T$, where X'_f is the terminal constraint set.
Hence, $(A+BK)X'_f \subset X'_f$, $X'_f \subset X \oplus T$, $KX'_f \subset U \oplus KT$, and $V_f(A+BK)x' + l(x', u') \leqslant V_f(x')$, $\forall x' \in X'_f$,

where $u_k = K(x_k - x'_k) + K_{lqr}x'_k$ is the stage cost, and $V_f(x') = \frac{1}{2}(x')^T P x'$ is the terminal cost, Θ, R and P are the positive definite weighting matrices.

Proposition 5.1 *If $x_k \in x'_k \oplus T$, $x'_k \in X \oplus T$ and $K_{lqr}x'_k \in U \oplus T$, with K_{lqr} provided by solving an LQR problem optimisation for the nominal system (2). Then the control law $u_k = K(x_k - x'_k) + K_{lqr}x'_k$ of the global system (1) ensures satisfaction of the original constraints $x \in X$,$u \in U$ for $\forall w \in W$ and $[A(k),\ B(k)] \in conv\{[A_j, B_j], \forall j \in 1, 2, \ldots, L\}$.*

Proposition 5.1 states that the control law $u_k = K(x_k - x'_k) + u'_k$, where $u'_k = K_{lqr}x'_k$, ensures satisfaction of the original state and control constraints.

Theorem 5.1 *For the LTV system as shown in (1), given the control law $u_k = K(x_k - x'_k) + K_{lqr}x'_k$ with a state feedback gain $K = YQ^{-1}$ provided by solving the optimization problem presented in (6)-(10) and a state feedback gain K_{lqr} provided by solving an LQR optimisation problem for the nominal system (2), the invariant tubes as shown in (17) provide a set of states whereby the system will evolve to the origin without input and output constraints violation.*

Proof. The feedback gain $K_i = Y_i Q_i^{-1}$ used in the construction of the zonotopic invariant set Z, $Z = \left\{x \in R^n,\ x = c + \sum_{i=1}^{p} \gamma_i g_i;\ -1 \leqslant \gamma_i \leqslant 1\right\}$, is obtained by solving convex optimization problem with LMI constraints as shown in (6)-(10). The satisfaction of (8) for a state feedback gain K ensures that $\left(\left[A_j + BK\right]x_k\right)^T \gamma Q^{-1}\left(\left[A_j + BK\right]x_k\right) - x_k^T \gamma Q^{-1} x_k \leq \left[x_k^T \Theta x_k + u_k^T R u_k\right]$, $j = 1, ..., l$. Also, $V_k = x_k^T \gamma Q^{-1} x_k$ is a strictly decreasing Lyapunov function (negative derivative) and the closed-loop system is robustly stabilized by the state feedback gain K.

Hence, a set of initial states $T = \left(\left\{x'_{0/k}\right\} \oplus Z, \left\{x'_{1/k}\right\} \oplus Z, \ldots, \left\{x'_{N-1/k}\right\} \oplus Z\right)$ is constructed such that all predicted states remain inside $T(x_k \subset T)$, and approach to the origin without constraint violation. Moreover, the invariant tube T constructed is never an empty set ($T \notin \{\}$) because the given feedback gain K_{lqr} is a stabilizable gain.

Corollary 5.1 *The state of the LTV system with disturbance $x_{k+1} = Ax_k + Bu_k + w$ at each time step is restricted to lie within a tube whose center is the state of the nominal LTV system $x'_{k+1} = Ax'_k + Bu'_k$.*

So, in summary, with Theorem 1 and Proposition 1, the off-line tube robust MPC algorithm based on zonotopes for the LTV system with disturbance $x_{k+1} = Ax_k + Bu_k + w$ can be formulated as follows.

Off-line:

– Solve (6)-(10) using Yalmip toolbox MATLAB.

– Calculate the feedback gain $K = YQ^{-1}$.

– Construction of the corresponding zonotopic invariant sets:

$Z = \left\{x \in R^n,\ x = c + \sum\limits_{i=1}^{p} \gamma_i g_i;\ -1 \leqslant \gamma_i \leqslant 1\right\}$.

– Calculate the feedback gain K_{lqr}, for the nominal system (2) using the LQR problem.

On-line: At each sampling time, calculate x'_{k+1} from $x'_{k+1} = \left(A_j + BK_{lqr}\right)x'_k$, $j = 1, 2, ..., L$. Then obtain the invariant tubes $T = \left(\left\{x'_{0/k}\right\} \oplus Z, \left\{x'_{1/k}\right\} \oplus Z, \ldots, \left\{x'_{N-1/k}\right\} \oplus Z\right)$, and we determine the smallest tube invariant set containing the measured state and implement the corresponding state feedback control law $u_k = K_i(x_k - x'_k) + K_{lqr}x'_k$, $i = 1, 2, ..., N$, to the process.

6 Numerical Examples

6.1 Example 1

Let us consider an uncertain non-isothermal CSTR [12], where the exothermic reaction $A \to B$ takes place. The reaction is irreversible and the rate of reaction is first order with respect to component A. A cooling coil is used to remove heat that is released in the exothermic reaction. The uncertain parameters are: the reaction rate constant k_0 and the heat of reaction δH_{rxn}. The linearized model based on the component balance and the energy balance is given by the following state equations:

$$\left\{\begin{array}{c} \dot{x}(t) = Ax(t) + Bu(t) + w, \\ y(t) = Cx(t), \end{array}\right. \tag{18}$$

where $\begin{bmatrix} C_A \\ T \end{bmatrix}$ is the state vector $x(t)$ and $\begin{pmatrix} C_{A,F} \\ F_C \end{pmatrix}$ is the input control vector $u(t)$. Matrices are defined by

$$A = \begin{bmatrix} 0.85 - 0.0986\alpha(k) & -0.0014\alpha(k) \\ 0.9864\alpha(k)\beta(k) & 0.0487 + 0.01403\alpha(k)\beta(k) \end{bmatrix}, w = [0 \quad 0]^T, \qquad (19)$$
$$B = \begin{bmatrix} 0.15 & 0 \\ 0 & -0.912 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where C_A is the concentration of A in the reactor, $C_{A,F}$ is the feed concentration of A, T is the reactor temperature, and F_C is the coolant flow. The operating parameters are: $F = 1m^3/min, V = 1m^3$, $k_0 = 10^9 - 10^{10}min^{-1}$, $E/R = 8330.1K$, $-?H_{rxn} = 10^7 - 10^8 cal/kmol$, $\rho = 10^6 g/m^3$, $UA = 5.3410^6 cal/(Kmin?)$ and $C_p = 1cal/(gK)$.

Let $\overline{C}_A = C_A - C_{A,eq}$, $\overline{C}_{A,F} = C_{A,F} - C_{A,F,eq}$ and $\overline{F}_C = F_C - F_{C,eq}$, where the subscript eq is used to denote the corresponding variable at the equilibrium condition. By discretization, using a sampling time of $15min$, and the discrete-time model with $\begin{bmatrix} \overline{C}_A(k) \\ \overline{T}(k) \end{bmatrix}$ as a state vector $\begin{bmatrix} \overline{C}_{A,F} \\ \overline{F}_C(k) \end{bmatrix}$ as a control vector, is given as follows:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + w, \\ y(k) = Cx(k) \end{cases} \qquad (20)$$

$$\begin{cases} x(k+1) = \begin{bmatrix} \overline{C}_A(k+1) \\ \overline{T}(k+1) \end{bmatrix} \\ \qquad = \begin{bmatrix} 0.85 - 0.0986\alpha(k) & -0.0014\alpha(k) \\ 0.9864\alpha(k)\beta(k) & 0.0487 + 0.01403\alpha(k)\beta(k) \end{bmatrix} \begin{bmatrix} \overline{C}_A(k) \\ \overline{T}(k) \end{bmatrix} \\ \qquad\qquad + \begin{bmatrix} 0.15 & 0 \\ 0 & -0.912 \end{bmatrix} \begin{bmatrix} \overline{C}_{A,F} \\ \overline{F}_C(k) \end{bmatrix}, \\ y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{C}_A(k) \\ \overline{T}(k) \end{bmatrix}, \end{cases} \qquad (21)$$

where $1 \le \alpha(k) = k_0/10^9 \le 10$ and $1 \le \beta(k) = -\Delta H_{rxn}/10^7 \le 10$.

The two parameters $\alpha(k)$ and $\beta(k)$ are independent of each other. Then we consider the following polytopic uncertain model with four vertices:

$$\Omega = conv \left\{ \begin{matrix} \begin{bmatrix} 0.751 & -0.0014 \\ 0.986 & 0.063 \end{bmatrix}, \begin{bmatrix} 0.751 & -0.0014 \\ 9.864 & 0.189 \end{bmatrix} \\ \begin{bmatrix} 0.751 & -0.0014 \\ 0.986 & 0.063 \end{bmatrix}, \begin{bmatrix} 0.751 & -0.0014 \\ 9.864 & 0.189 \end{bmatrix} \end{matrix} \right\}. \qquad (22)$$

These matrices are used to calculate four off-line feedback gains K_{lqr} for the nominal system $x'_{k+1} = Ax'_k + Bu'_k$.

The objective is to regulate the concentration $\overline{C}_A$ and the reactor temperature $\overline{T}$ to the origin by manipulating $\overline{C}_{A,F}$ and $\overline{F}_C$, respectively. These variables are constrained by $|\overline{C}_{A,F}| \le 0.5\, kmol/m^3$ and $|\overline{F}_C| \le 1.5\, m^3/min$.

The weighting matrices in the cost function are given as $\Theta = I$ and $R = 0.1I$.

Let us choose a sequence of states

$$x_i = \left\{ \begin{matrix} (0.0525, 0.0525), (0.0475, 0.0475), \\ (0.0425, 0.0425), (0.0375, 0.0375), \\ (0.0325, 0.0325), (0.0275, 0.0275). \end{matrix} \right\}. \qquad (23)$$

The nominal feedback control laws $u' = K_{lqr}x'$ calculated with the LQR problem, are given by

$K_1 = \begin{bmatrix} -0.34 & 0 \\ 0.50 & 0.03 \end{bmatrix}$, $K_2 = \begin{bmatrix} -3.41 & 0 \\ 5.08 & 0.09 \end{bmatrix}$, $K_3 = \begin{bmatrix} 0.12 & 0.04 \\ 4.91 & 0.09 \end{bmatrix}$, $K_4 = \begin{bmatrix} -4.28 & 0.01 \\ 49.49 & 0.72 \end{bmatrix}$.

Six feedback gains calculated off-line using the LMI methods, are given by $K_1 = \begin{bmatrix} -1.51 & -0.01 \\ 24.66 & 0.13 \end{bmatrix}$, $K_2 = \begin{bmatrix} -1.54 & -0.01 \\ 27.39 & 0.15 \end{bmatrix}$, $K_3 = \begin{bmatrix} -1.56 & -0.01 \\ 30.64 & 0.18 \end{bmatrix}$, $K_4 = \begin{bmatrix} -1.60 & -0.01 \\ 38.83 & 0.21 \end{bmatrix}$, $K_5 = \begin{bmatrix} -1.66 & -0.01 \\ 38.83 & 0.22 \end{bmatrix}$, $K_6 = \begin{bmatrix} -0.93 & -0.00 \\ 47.32 & 0.34 \end{bmatrix}$.

Figure 1 shows six invariant polytopes computed off-line corresponding to six feedback gains previously defined.

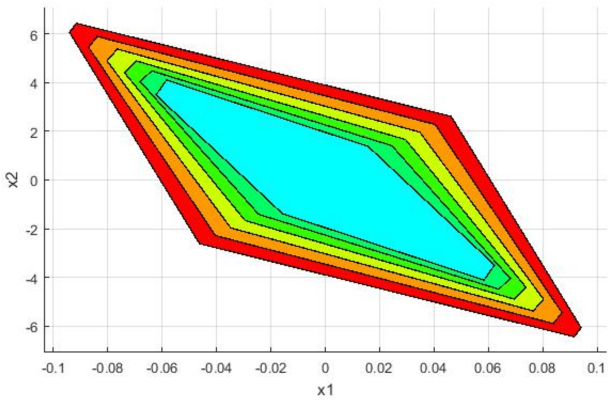

Figure 1: Polyhedral invariant sets constructed off-line.

Six zonotopes Z are defined by their centers:

$$c = \{2.9842,\ 3.1745,\ -1.3132,\ 1.3132,\ -3.1745,\ -2.9842\}\,. \tag{24}$$

The generators matrices are defined by $G_1 = \begin{bmatrix} 3.1047 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 \\ 3.2739 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $G_3 = \begin{bmatrix} 0 \\ 0 \\ 1.2955 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $G_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.2955 \\ 0 \\ 0 \end{bmatrix}$, $G_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3.2739 \\ 0 \end{bmatrix}$, $G_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3.1047 \end{bmatrix}$.

In Figure 2, the terminal cost $V_f(x)$ and the invariant tubes are obtained with the optimal finite horizon value $N = 10$ function for the unconstrained problem and X_f

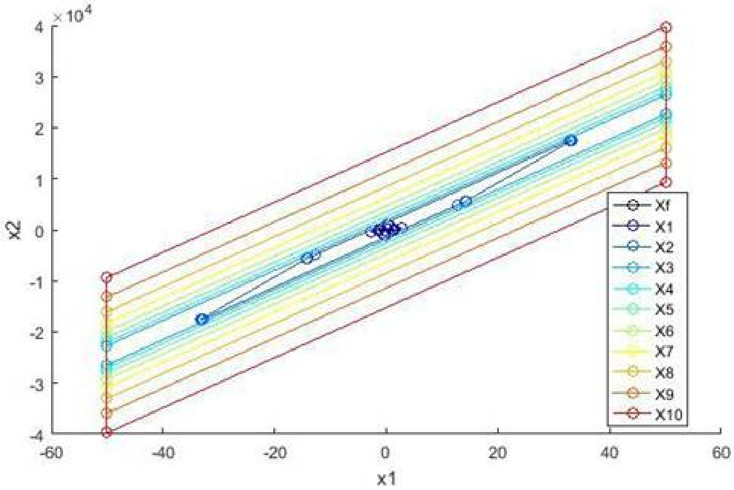

Figure 2: The terminal cost $V_f(x)$, 10 invariant tubes with $N = 10$, and the maximal set X_f.

is the associated maximal output admissible set. The regulated outputs are shown, respectively, in Figure 3 and Figure 4, it is seen that the proposed algorithm is able to steer faster the state of the uncertain CSTR to the neighborhood of the origin, and we have concluded that the use of invariant tubes based on zonotopes gives less conservative results as compared with invariant tubes based on polytope.

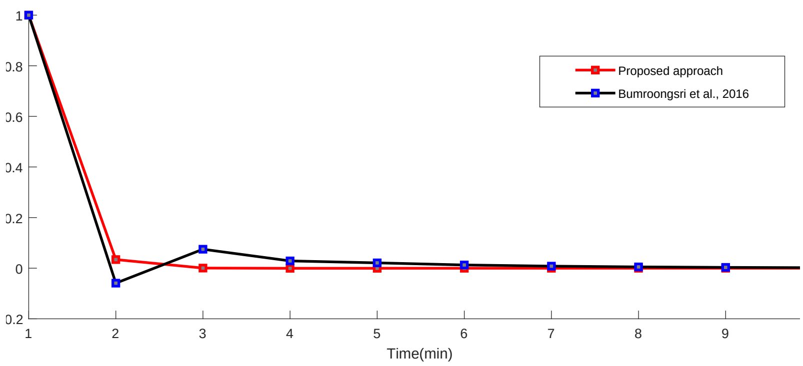

Figure 3: The concentration of A in the reactor of the regulated output.

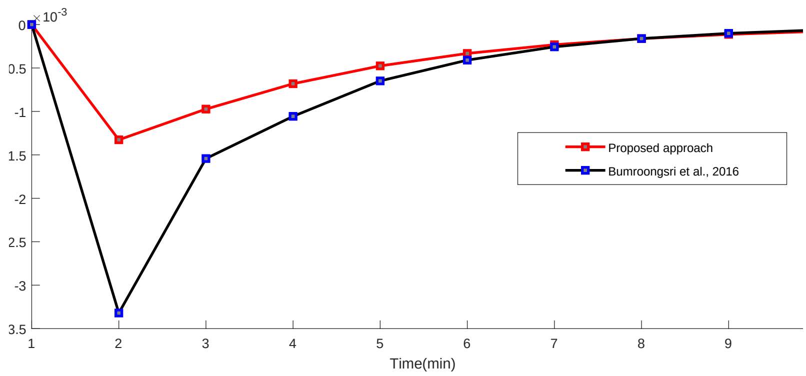

Figure 4: The reactor temperature of the regulated output.

6.2 Example 2

Consider the following LTV system with bounded disturbance:

$$x(k+1) = \begin{pmatrix} 1 & 1 \\ 0 & alpha \end{pmatrix} + \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u + w, \tag{25}$$

where $0.9 \leq \alpha \leq 1.1$. The state $x \in X$, where $X = \left\{ x \in R^2 | \, [0 \quad 1] \, x \leq 2 \right\}$, the control $u \in U$, where $U = \{ u \in R | \, |u| \leq 1 \}$, and the disturbance $w \in W$, where $W = \left\{ w \in R^2 | [-0.1 \quad 0.1]^T \leq w \right\}$.

The weighting matrices in the cost function are given as $\Theta = I$ and $R = 0.01$. The following nominal LTV system:

$$x^{'}(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & \alpha \end{bmatrix} x^{'} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u^{'} \tag{26}$$

is subject to a tighter state and control constraints, $x^{'} \in X \oplus T$ and $u^{'} \in U \oplus K_{lqr} T$. By solving the LMIs problem (6)-(10) we obtain seven feedback gains corresponding to seven invariant polytopic sets shown in Figure 5.

$K_1 = \begin{bmatrix} -0.45 & -1.02 \end{bmatrix}$, $K_2 = \begin{bmatrix} -0.47 & -1.11 \end{bmatrix}$, $K_3 = \begin{bmatrix} -0.51 & -1.24 \end{bmatrix}$, $K_4 = \begin{bmatrix} -0.47 & -1.14 \end{bmatrix}$, $K_5 = \begin{bmatrix} -0.53 & -1.26 \end{bmatrix}$,$K_6 = \begin{bmatrix} -0.53 & -1.26 \end{bmatrix}$, $K_7 = \begin{bmatrix} -0.53 & -1.26 \end{bmatrix}$.

Using these feedback gains we obtained seven zonotopes Z defined by their centers $c = \{-1.93, -0.49, 0.69, 1.93, 0, 0, 0\}$. The generators matrices are defined by

$$G_1 = \begin{bmatrix} 3.38 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 \\ 2.79 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 \\ 0 \\ 3.04 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, G_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3.28 \\ 0 \\ 0 \\ 0 \end{bmatrix}, G_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -4.63 \\ 0 \\ 0 \end{bmatrix},$$

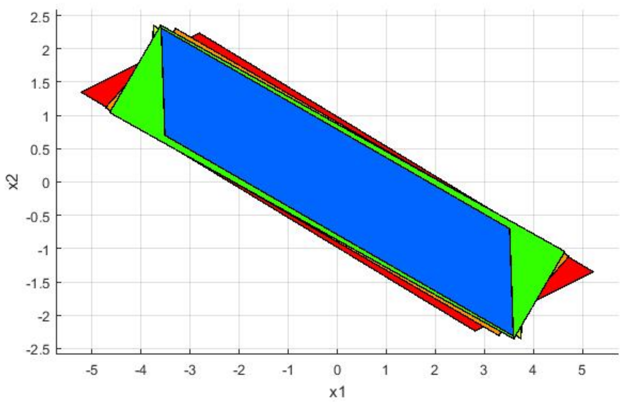

Figure 5: Polyhedral invariant sets.

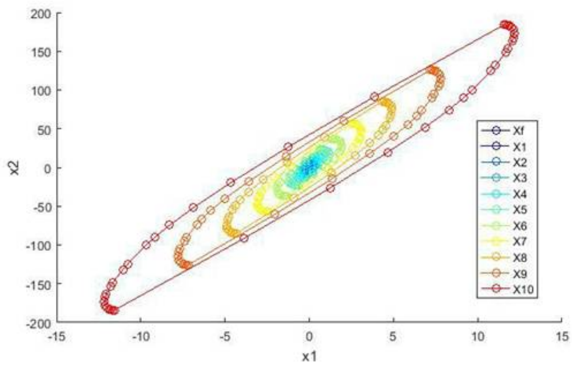

Figure 6: The terminal cost $V_f(x)$, ten invariant tubes with $N = 10$, and the maximal set X_f.

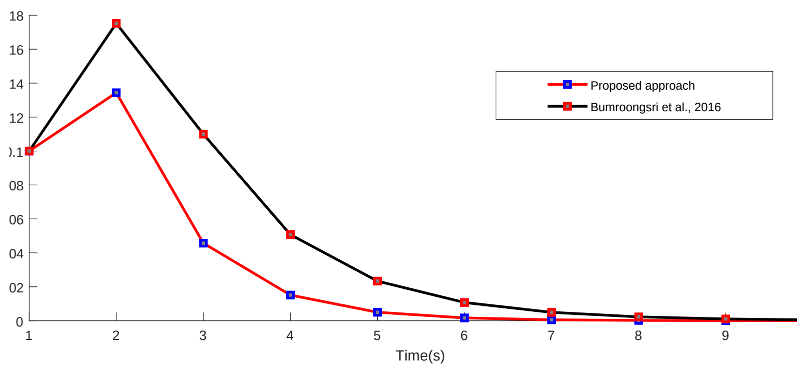

Figure 7: The regulated output using invariant tubes.

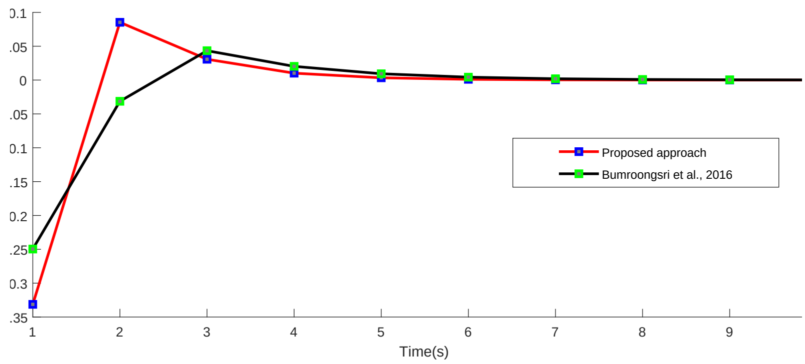

Figure 8: The control input.

$$G_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2.34 \\ 0 \end{bmatrix}, \; G_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4.63 \end{bmatrix}.$$

By solving an LQR problem to the nominal system (30) we obtain two feedback gains: $K_1 = [-0.65 - 1.25]$ and $K_2 = [-0.67 - 1.39]$. And using an optimal finite horizon $N = 10$ we obtain ten invariant tubes (Figure 6). Figure 7 and Figure 8 represent the closed-loop response of the system and the control input, respectively. The chosen horizon is $N = 10$ and the initial state $[0.10.2]^T$. It is seen that the proposed Tube MPC algorithm achieves better control performances.

7 Conclusion

In this paper, we have presented a new approach of uncertain discrete time system stabilization based on the robust tube MPC algorithm using zonotopic invariant sets. The proposed algorithm used an off-line solution of an optimal control optimization problem to determine a sequence of feedback gains for the global system. Then a sequence of feedback gains for the nominal system is computed using an LQR problem. Finally, a sequence of nested invariant tubes is constructed. At each sampling time, we determine the smallest zonotopic invariant set containing the measured state and implement the obtained global state feedback control law.

The proposed approach applied on two examples, provides better control performances and less computational cost.

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