Nonlinear Dynamics and Systems Theory, **21** (1) (2021) 68–75

Boundedness and Dynamics of a Modified Discrete Chaotic System with Rational Fraction

N. Djafri [1], T. Hamaizia [2*] and F. Derouiche [3]

[1] *1EPSECSG Constantine*
[2] *Department of Mathematics, Faculty of Exact Sciences, University of Constantine 1, Algeria*
[3] *Department of Mathematics, Faculty of Exact Sciences, University of Oum El-Bouaghi, Algeria*

Received: October 12, 2020; Revised: December 25, 2020

Abstract: A modified 2-D discrete chaotic system with rational fraction is introduced in this paper, it has more complicated dynamical structures than the Hénon map and Lozi map. Some dynamical behaviors, value domain, fixed point, period-doubling bifurcation, the route to chaos, and Lyapunov exponents spectrum are further investigated using both theoretical analysis and numerical simulation. In particular, the map under consideration is a simple rational discrete bounded map capable of generating multi-fold strange attractors via period-doubling bifurcation routes to chaos. This new discrete chaotic system has extensive application in many fields such as optimization chaos and secure communication.

Keywords: *2-D rational chaotic map; new chaotic attractor; coexisting attractors.*

Mathematics Subject Classification (2010): 34D45, 70K55, 34D20.

1 Introduction

A discrete-time dynamical system is given by a map $T : X \to X$ from a space X into itself; we are interested in the asymptotic behavior of sequences $(x(n))$ defined by $x(n+1) = T(x(n))$, depending on the initial condition $x(0)$. Interest in dynamical systems sprang up in the 1960s-70s when it was shown that: (a) very simple dynamical systems can have an extremely complex "chaotic" behavior, which appears to be "random"; (b) such "chaotic" behavior can paradoxically be "stable"; (c) the behavior of some dynamical systems is so "chaotic" and "random" that it is best studied statistically. One of these models is the Lozi map [1, 2]. Moreover, it is possible to change the form of

* Corresponding author: mailto:el.tayyeb@umc.edu.dz

the Lozi map for obtaining others chaotic attractors [1, 3–6]. In [7], a one-dimensional discrete chaotic system with rational fraction was proposed. In [8], the authors extended the first one-dimensional discrete chaotic system with two-dimensional and in a recent work given in [9] the dynamics of a new simple 2-D rational discrete mapping was studied. In particular, an example of coexistence of several chaotic attractors was presented and discussed. In this paper, we propose the new discrete chaotic system with rational fraction given by

$$f(x,y) = \begin{pmatrix} y + 1 - a.(\frac{1}{0.1+x^2}) \\ b.x \end{pmatrix}. \tag{1}$$

The map (1) is obtained by changing the term $|x|$ in the nonlinear Lozi mapping by the fraction $(\frac{1}{0.1+x^2})$, the discrete iterative systems with rational fraction was discovered in the study of evolutionary algorithm, this type of applications is used in secure communications using the notions of chaos [10, 11].

2 Analytical Results

The new chaotic attractors described by map (1) have several important properties such as: (i) The map (1) is defined for all points in the plane. (ii) The associated function $f(x,y)$ of the map (1) is of class $C^\infty(\mathbb{R}^2)$, and it has no vanishing denominator. (iii) The system (1) and the Lozi system are not topologically equivalent, because the Lozi system is a piecewise linear, but the model (1) is a nonlinear system.

3 Fixed Points and Their Stability

In this section, we begin by studying the existence of fixed points of the f mapping and determine their stability type. Indeed, we have

$$\begin{cases} x = bx + 1 - a(\dfrac{1}{0.1+x^2}), \\ y = bx. \end{cases} \tag{2}$$

Hence, $x = bx + 1 - a(\dfrac{1}{0.1+x^2})$, then $[(1-b)x - 1](0.1+x^2) + a = 0$, so

$$(1-b)x^3 - x^2 + 0.1(1-b)x - 0.1 + a = 0. \tag{3}$$

First, eliminate the term x^2 by substituting $x = X - (-\frac{1}{3(1-b)})$ which yields the reduced cubic equation $X^3 + PX + q = 0$, where

$$P = 0.1 - \frac{1}{3(1-b)^2} \tag{4}$$

and

$$q = -\frac{2}{27(1-b)} + \frac{a-0.1}{1-b} + \frac{0.1}{3(1-b)^2}. \tag{5}$$

The reduced cubic equations with the negative discriminate $27q^2 + 4p^3$ will have 3 real roots only if $P < 0$.

Proposition 3.1 *The f mapping will have 3 fixed points only if* $b \in]1 - \sqrt{\frac{10}{3}}, 1[\cup]1, 1 + \sqrt{\frac{10}{3}}[$.

Proof. Let

$$\begin{cases} x = b.x + 1 - a.(\frac{1}{0.1+x^2}), \\ y = b.x. \end{cases}$$

Then we have $x = b.x + 1 - a(\frac{1}{0.1+x^2})$. Hence, $[(1-b)x - 1](0.1 + x^2) + a = 0$. That is, $(1-b)x^3 - x^2 + 0.1(1-b)x - 0.1 + a = 0$. In this case, we have $q^2 < -\frac{4}{27}p^3 \Leftrightarrow q^2 < -\frac{4}{27}p \times p^2$ only if $p < 0$. That is, $0.1 - \frac{1}{3(1-b)^2} < 0; (b - 1 \neq 0)$. so $\frac{1}{3(1-b)^2} > 0.1$, then $(1-b)^2 < \frac{10}{3}$. Thus $1 - \sqrt{\frac{10}{3}} < b < 1 + \sqrt{\frac{10}{3}}$.

The Jacobian matrix of the map (1) is

$$J = \begin{pmatrix} \frac{2ax}{(0.1+x^2)^2} & 1 \\ b & 0 \end{pmatrix}.$$

We have $|J| = -b$. So, if $b > -1$, then the system is dissipative. J has the following characteristic polynomial:

$$P(\lambda) = \lambda^2 - \frac{2ax}{(0.1 + x^2)^2}\lambda - b,$$

so, the eigenvalues are

$$\lambda_{1,2} = \frac{ax}{(0.1 + x^2)^2} \pm \sqrt{\frac{(ax)^2}{(0.1 + x^2)^4} + b}.$$

It is easy to check that the smallest absolute values are always less than 1. Then we deduce that the fixed points are of saddle type.

4 Determination of Bounded and Unbounded Orbits

We remark that the variations of the right-hand side of system (1) depend mainly on the fraction which is a smooth function. In what follows, we shall prove the boundedness of system (1) using a comparison criterion. It is possible to rewrite system (1) in the form (6) below:

$$x_{n+1} = 1 - (\frac{a}{0.1 + x_n^2}) + bx_{n-1}. \tag{6}$$

Now, by successive substitution of the terms of the sequence $(x_n)_n$ we can prove that this sequence is bounded for all $b < 1$ as shown by the following result.

Theorem 4.1 *For every $n > 1$, and all values of a and b, and for all values of the initial conditions $(x_0, x_1) \in \mathbb{R}^2$, the sequence $(x_n)_n$ satisfies the following conditions:*

(a) *If $b \neq 1$, then*

$$x_n = \begin{cases} \frac{b^{\frac{n-1}{2}}-1}{b-1} + b^{\frac{n-1}{2}}x_1 - a\sum\limits_{m=1}^{m=\frac{n-1}{2}} \frac{b^{m-1}}{0.1+x^2_{n-(2m-1)}} & \textit{if } n \textit{ is odd,} \\ \frac{b^{\frac{n}{2}}-1}{b-1} + b^{\frac{n}{2}}x_0 - a\sum\limits_{m=1}^{m=\frac{n}{2}} \frac{b^{m-1}}{0.1+x^2_{n-(2m-1)}} & \textit{if } n \textit{ is even;} \end{cases} \tag{7}$$

(b) *If b=1, then*

$$x_n = \begin{cases} \frac{n-1}{2} + x_1 - a \sum_{m=1}^{m=\frac{n-1}{2}} \frac{1}{0.1+x^2_{n-(2m-1)}} & \text{if } n \text{ is odd,} \\ \frac{n}{2} + x_0 - a \sum_{m=1}^{m=\frac{n}{2}} \frac{1}{0.1+x^2_{n-(2m-1)}} & \text{if } n \text{ is even.} \end{cases} \tag{8}$$

.

Theorem 4.2 *The sequence $(x_n)_n$ given in (1) satisfies the following inequality:*

$$\forall a, b \in \mathbb{R}, \forall n > 1, |1 - x_n + bx_{n-2}| \leq |10a|. \tag{9}$$

Proof. We have for every $n > 1$: $x_n = 1 - a.(\frac{1}{0.1+x^2_{n-1}}) + b.x_{n-2}$, then one has

$$| - x_n + 1 + bx_{n-2}| = |\frac{a}{0.1 + x^2_{n-1}}| \leq |10a|. \tag{10}$$

Since $x_n^2 > 0$, one has $0.1 + x_n^2 > 0.1$, so $\frac{1}{0.1+x_n^2} < 10$, then we get $|\frac{a}{0.1+x_n^2}| < |10a|$.

5 Existence of Bounded and Unbounded Orbits

In the following theorem, we give sufficient conditions for bounded and unbounded orbits of the system (1).

Theorem 5.1 *For all $a \in \mathbb{R}$ and all initial conditions $(x_0; x_1) \in \mathbb{R}^2$:*

i. *The orbits of the map (1) are bounded in the following subregions of $\mathbb{R}^4$:*

$$\Gamma_1 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 / |b| < 1\right\}. \tag{11}$$

ii. *The map (1) possesses unbounded orbits in the following subregions of $\mathbb{R}^4$:*

$$\Gamma_2 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 / |b| > 1, \textit{and both } |x_0|, |x_1| > \frac{|10a| + 1}{|b| - 1}\right\} \tag{12}$$

and

$$\Gamma_3 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 / |b| = 1, \textit{and } |10a| < 1\right\}. \tag{13}$$

Proof. **I)** From equation (1) and the fact that $(\frac{1}{0.1+x_n^2})$ is a bounded function for all $x \in \mathbb{R}$, one has the following inequalities for all $n > 1$:

$$|x_n| \leq 1 + |10a| + |bx_{n-2}|. \tag{14}$$

If we replace the successive terms $x_{n-2}, x_{n-4}, x_{n-6}...$, in the term x_n, then the last term is obtained:

$$|x_n| \leq (1 + |10a|) + |b|(1 + |10a|) + |b|^2(1 + |10a|) + |b|^3|x_{n-6}|. \tag{15}$$

Since $|b| < 1$, the use of (15) and induction about some integer k using the sum of a geometric growth formula permits us to obtain the following inequalities for every $n > 1$, $k > 0$:

$$|x_n| \leq (1 + |10a|)(\frac{1 - |b|^k}{1 - |b|}) + |b|^k|x_{n-2k}|. \tag{16}$$

Thus, one has the following two cases:
1) if $n = 2m + 1$, then $(x_n)_n$ satisfies the following inequalities:

$$|x_{2m+1}| \leq (1 + |10a|)(\frac{1 - |b|^m}{1 - |b|}) + |b|^m |x_1| = w_m; \tag{17}$$

2) if $n = 2m$, then $(x_n)_n$ satisfies the following inequalities:

$$|x_{2m}| \leq (1 + |10a|)(\frac{1 - |b|^m}{1 - |b|}) + |b|^m |x_0| = v_m. \tag{18}$$

Thus, since $|b| < 1$, the sequences $(w_m)_m$ and $(v_m)_m$ are bounded, and one has

$$\begin{cases} w_n \leq \frac{1+|10a|}{1-|b|} + ||x_1| - \frac{1+|10a|}{1-|b|}| & \text{for all } m \in \mathbb{N}, \\ v_n \leq \frac{1+|10a|}{1-|b|} + ||x_0| - \frac{1+|10a|}{1-|b|}| & \text{for all } m \in \mathbb{N}. \end{cases} \tag{19}$$

Thus, the previous formulas give the following bounds for the sequence $(x_n)_n$:

$$|x_m| \leq max(\frac{1 + |10a|}{1 - |b|} + ||x_1| - \frac{1 + |10a|}{1 - |b|}|, \frac{1 + |10a|}{1 - |b|} + ||x_0| - \frac{1 + |10a|}{1 - |b|}|). \tag{20}$$

Finally, for all values of a and all values of b satisfying $|b| < 1$ and all initial conditions $(x_0; x_1) \in \mathbb{R}^2$, one concludes that all orbits of the map (1) are bounded, i.e., in the sub-region of $\mathbb{R}^4$

$$\Gamma_1 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 / |b| < 1\right\}.$$

Hence the proof (i) is completed.

II) (a) For every $n > 1$ we have $x_n = 1 - a.(\frac{1}{0.1+x_{n-1}^2}) + b.x_{n-2}$, then $|b.x_{n-2} - a.(\frac{1}{0.1+x_{n-1}^2})| = |x_n - 1|$ and $||b.x_{n-2}| - |a.(\frac{1}{0.1+x_{n-1}^2})|| \leq |x_n - 1|$, (we use the triangular inequality), this implies that

$$|b.x_{n-2}| - |a.(\frac{1}{0.1 + x_{n-1}^2})| \leq |x_n| + 1. \tag{21}$$

Since $|(\frac{1}{0.1+x_{n-1}^2})| \leq 10$, this implies that $|a(\frac{1}{0.1+x_{n-1}^2})| \leq 10|a|$.
$|b.x_{n-2}| - |a(\frac{1}{0.1+x_{n-1}^2})| \geq |b.x_{n-2}| - 10|a|$. Finally, one has from (21) that

$$|b.x_{n-2}| - (10|a| + 1) \leq |x_n|. \tag{22}$$

Then, by induction, as in the previous section, one has

$$|x_n| \geq \begin{cases} (\frac{|10a|+1}{|b|-1} + |x_1|)|b|^{\frac{n-1}{2}} + \frac{|10a|+1}{|b|-1} & \text{if } n \text{ is odd,} \\ (\frac{|10a|+1}{|b|-1} + |x_0|)|b|^{\frac{n}{2}} + \frac{|10a|+1}{|b|-1} & \text{if } n \text{ is even.} \end{cases} \tag{23}$$

Thus, if $|b| > 1$ and both $|x_0|, |x_1| > (\frac{|10a|+1}{|b|-1})$, one has $lim_{n \to +\infty} |x_n| = +\infty$.
(b) For $b = 1$, one has

$$|x_n| \geq \begin{cases} (1 - |10a|)(\frac{n-1}{2}) + |x_1| & \text{if } n \text{ is odd,} \\ (1 - |10a|)(\frac{n}{2}) + |x_0| & \text{if } n \text{ is even.} \end{cases} \tag{24}$$

Hence, if $|10a| < 1$, then one has $lim_{n\to+\infty}|x_n| = +\infty$.

For $b = -1$, one has from Theorem 1 the following inequalities:

$$x_n \leq \begin{cases} -(\frac{n-1}{2}) + x_1 + |a \sum\limits_{m=1}^{m=\frac{n-1}{2}} \frac{(-1)^{m-1}}{0.1+x^2_{n-(2m-1)}}| & \text{if } n \text{ is odd,} \\ -(\frac{n}{2}) + x_0 + |a \sum\limits_{m=1}^{m=\frac{n}{2}} \frac{(-1)^{m-1}}{0.1+x^2_{n-(2m-1)}}| & \text{if } n \text{ is even.} \end{cases} \tag{25}$$

Because $|(\frac{a(-1)^{m-1}}{0.1+x^2_{n-(2m-1)}})| \leq |10a|$, then one has

$$x_n \leq \begin{cases} (|10a|-1)(\frac{n-1}{2}) + |x_1| & \text{if } n \text{ is odd} \\ (|10a|-1)(\frac{n}{2}) + |x_0| & \text{if } n \text{ is even.} \end{cases} \tag{26}$$

Thus, if $|10a| < 1$, then one has $lim_{n\to+\infty}|x_n| = +\infty$. Note that there is no similar proof for the following subregions of $\mathbb{R}^4$ defined by

$$\Gamma_4 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4/|b| > 1, \text{and both } |x_0|, |x_1| \leq \frac{|10a|+1}{|b|-1}\right\} \tag{27}$$

and

$$\Gamma_5 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4/|b| = 1, \text{and } |a| \geq 1\right\}. \tag{28}$$

Hence, the proof (ii) is completed.

6 Some Observed New Attractors

As already mentioned in the introduction, we study the two-dimensional discrete chaotic system with two parameters a and b, obtained via a direct modification in the Lozi map, where the absolute value term is replaced by the rational fraction defined on all $\mathbb{R}$ and being differentiable continuous. This fact is the central idea which makes the solutions of system (1) bounded for some values of b. This new map generates chaotic attractors with multiple "multifold" that evolves around three points as shown in Fig.1.

7 Numerical Simulations and Route to Chaos

In this section, the dynamic behavior of the map (1) is studied numerically. We shall illustrate some observed chaotic attractors. The bifurcation diagram is a way for a discrete dynamical system to make a transition from regular behavior to chaos [12]. To demonstrate the chaotic dynamics, the largest Lyapunov exponent should be the first thing to be considered, because any system containing at least one positive Lyapunov exponent is defined to be chaotic. From Fig.2 and Fig.3, it is clear that the bifurcation diagram well coincides with the spectrum of Lyapunov exponents. Fig.2 shows that the system (1) can evolve into periodic and chaotic behaviors. Indeed, when a varies from 0.108 to 0.347, it can be seen that there is a positive Lyapunov exponent over a wide range of parameters, implying that the system is chaotic over this range. When a increases in the region [0.118, 0.220], the system (1) converges to a stable fixed point. In the interval section (0.108, 0.112), the trajectory of the system will turn to a stable limit cycle. In addition, the system shows a periodic motion of some windows in the chaotic region, i.e., [0.245, 0.255]. Finally, it is clear that the system is chaotic for $a \in (0.220, 0.245) \cup (0.255, 0.347)$.

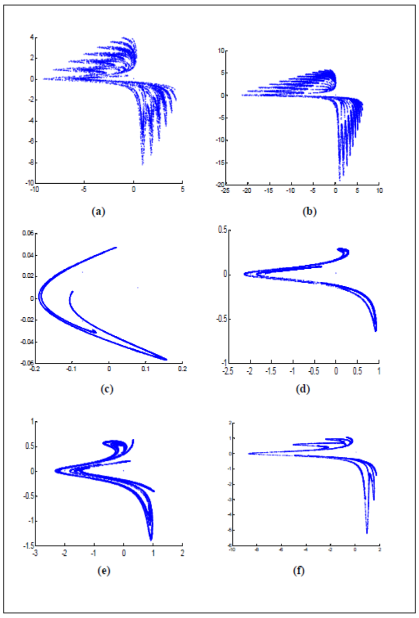

Figure 1: Attractors of the map (1) with (a) $a = 0.9$, $b = 0.9$, (b) $a = 0.9$, $b = 2$, (c) $a = 0.3$, $b = 0.115$, (d) $a = 0.3$, $b = 0.3$, (e) $a = 0.6$, $b = 0.4$, (f) $a = 0.6$, $b = 0.9$.

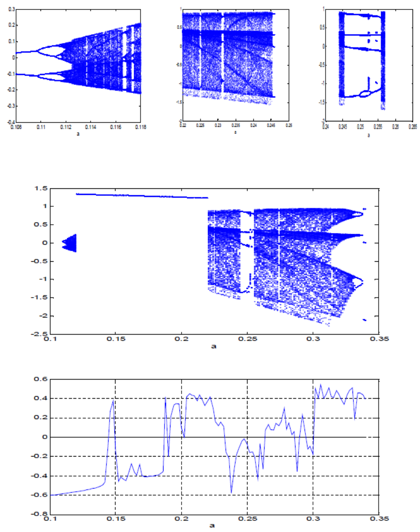

Figure 2: (a) The bifurcation diagram for the map (1) obtained for $b = 0.3$ and $0,108 < a < 0.347$. (b) Variation of the Lyapunov exponents of map (1) versus the parameter $0,108 < a < 0.347$ with $b = 0.3$.

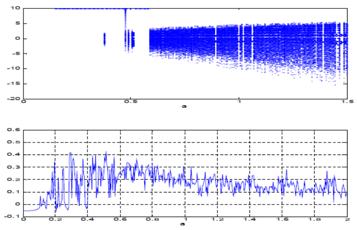

Figure 3: (a) The bifurcation diagram for the map (3) obtained for $b = 0.9$ and $0,13 < a < 2$. (b) Variation of the Lyapunov exponents of map (3) versus the parameter $0,13 < a < 2$ with $b = 0.9$.

8 Conclusion

In this paper we have presented a modified two-dimensional discrete chaotic system with rational fraction, obtained via direct modification of the Hénon mapping. The detailed dynamical behaviors of this map (which is useful for the evolutionary algorithm and secure communication) are further investigated using both theoretical analysis and numerical simulation.

References

[1] R. Lozi. Un attracteur etrange (?) du type attracteur de Hénon. *J. hys.* **39** (C5) (1978) 9–10.

[2] M. Misiurewicz. Strange attractors for the Lozi mappings. *Nonlinear Dyn.* **357** (1) (1980) 348–358.

[3] M. Hénon. A two dimensional mapping with a strange attractor. *Communications in Mathematical Physics* **50** (1976) 69–77.

[4] M. Benedicks and L. Carleson. The dynamics of the Hénon maps. *Annals of Mathematics* **133** (1) (1991) 73–169.

[5] Y. Cao and Z. Liu. Oriontation-preserving Lozi map. *Chaos, Solutions and Fractals* **9** (11) (1998) 1857–1863.

[6] A. Aziz-Alaoui, C. Robert and C. Grebogi. Dynamics of a Hénon-Lozi map. Chaos, Solitions and Fractals **12** (11) (2001) 2323–2341.

[7] J.A. Lu, X. Wu, J. L and L. Kang. A new discrete chaotic system with rational fraction and its dynamical behaviors. *Chaos, Solitons and Fractals* **22** (2) (2004) 311–319.

[8] L. Chang, J. Lu and X. Deng. A new two-dimensional discrete chaotic system with rational fraction and its tracking and synchronization. *Chaos, Solitons and Fractals* **24** (4) (2005) 1135–1143.

[9] E. Zeraoulia and J.C. Sprott. On the dynamics of a new simple 2-D rational discrete mapping. *Int. J. Bifurc. Chaos.* **21** (1) (2011) 1–6.

[10] G. Chen and X. Dong. *From Chaos to Order: Methodology, Perspectives and Applications.* World Scientific, 1998.

[11] B.R. Hunt, J.A. Kennedy, T.Y. Li and H. Nusse. *The Theory of Chaotic Attractorss.* Springer, New-York. 2003.

[12] V. Avrutin, M. Schanz and S. Banerjee. Occurrence of multiple attractor bifurcations in the two-dimensional piecewise linear normal form map. *Nonlinear Dyn.* **67** (2012) 293–307. DOI 10.1007/s11071-011-9978-5.