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Analysis of an SIRS Epidemic Model for a Disease Geographic Spread

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Abstract: An SIRS epidemic model for the geographic spread is considered. The linear stability analysis is conducted to obtain the threshold condition and a supercritical instability region is found whenever the reproduction number $\mathcal{R} > 1$. An evolution equation for the leading order of infectives is derived by the long wavelength expansion method and full pattern formation analysis is carried out. The Poincaré-Lindstedt method is applied to obtain a uniformly periodic valid solution. Numerical simulations are used to present the results.

Keywords: *evolution equation; SIRS; stability; pattern formation; Poincaré-Lindstedt method.*

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1 Introduction

In epidemiology the use of mathematical models starts from the pioneering works of Kermack and McKendrick [1–4]. To describe the Great Plague of London of 1665-1666, Kermack and McKendrick use a simple basic deterministic differential equation model called the SIR model [3], [4]. Many mathematical models in the literature are built based on the modeling framework of Kermack and McKendrick.

Most of existing studies rely on different types of differential equations. For instance, first-order partial differential equations are used for modeling of age structures [5–8]; delay-differential equations or integral equations are suitable when time delay or delay factors appear [9–13]; second-order partial differential equations are more realistic when a diffusion term exists.

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Recently, the study of geographic spread of epidemics becomes of much interest. Diffusion epidemic models have been studied by many authors [14–19]. Liu and Jin [18] considered an SI model with either constant or nonlinear incidence function. The authors studied numerically the pattern formation of the model. In [19], Hadji applied the long wavelength expansion method to analyze a simple model for the geographic spread of a rabies epidemic in a population of foxes. The author studied the pattern formation of the model. In [14], the authors studied the effect of different types of animal movement on threshold conditions for disease spread by considering a simple SI diffusion model. Jawaz et. al. [10] considered a time delay HIV/AIDS reaction diffusion SIR model. The authors designed a numerical scheme to solve the model. Moreover, the proposed technique was compared with the results obtained by Euler's technique, also the results are presented by numerical simulations.

The main objective of this study is to develop and analyze a mathematical model of an epidemic incorporating with diffusion of the various epidemic sub-population within a geographical region. We conducted the linear and weakly nonlinear stability analysis of an SIRS with diffusion model. The long wavelength expansion is applied to obtain the evolution equation. Furthermore, a periodic uniformly valid solution is obtained by the Poincaré-Lindstedt method.

This paper is organized as follows, The mathematical model formulation is presented in Section 2. In Section 3, the linear stability analysis is conducted to obtain the threshold conditions. The weakly nonlinear stability is investigated in Section 4. In Section 5, a full pattern formation analysis is carried out. A uniformly periodic valid solution is obtained in Section 6. The results are concluded in Section 7.

2 Mathematical Model

We consider a population which consists of three subgroups. The susceptible, S, which can get the disease. The infected, I, those who have the disease and can transmit it. The removed, R, those who recovered, are immune, isolated or dead. The population is considered to have a constant size, N, where $N = S + I + R$. All the classes, S, I and R depend on space and time. The SIR reaction diffusion epidemic model is presented by J. D. Murray [20]:

$$\begin{aligned}
\frac{\partial \widehat{S}}{\partial \widehat{t}} &= \mathcal{D}\,\nabla^2 \widehat{S} - \beta\,\widehat{S}\widehat{I} + \gamma \widehat{R},\\
\frac{\partial \widehat{I}}{\partial \widehat{t}} &= \mathcal{D}\,\nabla^2 \widehat{I} + \beta\,\widehat{S}\widehat{I} - r\widehat{I},\\
\frac{\partial \widehat{R}}{\partial \widehat{t}} &= \mathcal{D}\,\nabla^2 \widehat{R} + r\widehat{I} - \gamma \widehat{R},
\end{aligned} \tag{1}$$

where $\mathcal{D}$ is the diffusion coefficient, β is the disease transmission coefficient, r is the recovery rate and γ is the loss of natural immunity. Upon using the following scaling $I = \widehat{I}/S_0, S = \widehat{S}/S_0, \widehat{R}/S_0, x = \widehat{x}/H$ and $t = \beta\, S_0\, \widehat{t}$, where S_0 is a reference value of the susceptible species, we obtain the dimensionless system which is described by

$$\begin{aligned}
\frac{\partial S}{\partial t} &= \nabla^2 S - SI + \delta\, R,\\
\frac{\partial I}{\partial t} &= \nabla^2 I + SI - \lambda\, I,
\end{aligned}$$

$$\frac{\partial R}{\partial t} = \nabla^2 R + \lambda\, I - \delta\, R, \tag{2}$$

where $\delta = \gamma/\beta\, S_0$ and $\lambda = r/\beta\, S_0$ is the reciprocal of the reproduction rate, $\mathcal{R}$. The corresponding boundary conditions are

$$S = 1, \qquad \frac{\partial\, I}{\partial\, z} = 0 \quad \text{and} \qquad R = 0$$

at $z = 0, 1$. The basic states of S, I and R are

$$S_B = 1, \qquad I_B = 0 \quad \text{and} \quad R_B = 0 \tag{3}$$

for any values of λ and δ. We introduce the perturbations ϕ, θ and ψ to the base state so that $S = 1 + \phi, I = \theta + 0$ and $R = \psi + 0$. Hence the system of equations (2) becomes

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= \nabla^2\phi - \theta - \phi\,\theta + \delta\,\psi, \\ \frac{\partial\theta}{\partial t} &= \nabla^2\theta + \phi\,\theta + (1-\lambda)\,\theta, \\ \frac{\partial\psi}{\partial t} &= \nabla^2\psi + \lambda\,\theta - \delta\,\psi, \end{aligned} \tag{4}$$

subject to

$$\phi = 0, \qquad \frac{\partial\,\theta}{\partial\, z} = 0 \quad \text{and} \qquad \psi = 0$$

at $z = 0, 1$.

3 Stability Threshold Condition

Following a standard procedure (see [21] and [22]), the linearized system of equations governing the convective perturbations is given by

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= \nabla^2\phi - \theta + \delta\,\psi, \\ \frac{\partial\theta}{\partial t} &= \nabla^2\theta + (1-\lambda)\,\theta, \\ \frac{\partial\psi}{\partial t} &= \nabla^2\psi + \lambda\,\theta - \delta\,\psi. \end{aligned} \tag{5}$$

We investigate the linear stability by considering the normal modes $[\Phi, \Theta, \Psi] = [\Phi(z), \Theta(z), \Psi(z)]\, \exp(i\,\mathbf{K}\cdot\,\mathbf{X} + \sigma\, t)$, where $\mathbf{X} = \langle x, y\rangle$, σ is the growth rate and $|\mathbf{K}| = k$ is the wavenumber in the system of equations (5) to obtain the following system of second order ordinary differential equations:

$$\sigma\,\Phi = (D^2 - k^2)\Phi - \Theta + \delta\,\Psi, \tag{6}$$

$$\sigma\,\Theta = (D^2 - k^2)\Theta + (1-\lambda)\,\Theta, \tag{7}$$

$$\sigma\,\Psi = (D^2 - k^2)\Psi + \lambda\,\Theta - \delta\,\Psi. \tag{8}$$

where $D = d/dz$.

Multiply both sides of equation (7) by the complex conjugate of Θ and integrate with respect to z from 0 to 1 to get

$$\sigma = \frac{-\langle |D\,\Theta|^2 \rangle - (k^2 + \lambda - 1)\langle |\Theta|^2 \rangle}{\langle |\Theta|^2 \rangle}, \tag{9}$$

where $\langle \cdot \rangle = \int_0^1 \cdot \, dz$. If $\sigma = 0$, the solution of equation (7) is $\Theta = A_1 \cosh \alpha\, z + A_2 \sinh \alpha\, z$, where $\alpha = \sqrt{k^2 + \lambda - 1}$. Apply the boundary conditions $\dfrac{d\,\Theta}{d\,z} = 0$ at $z = 0, 1$. We get $\alpha = 0$ and hence, $\lambda = 1 - k^2$. Thus, the critical λ value, $\lambda_c = 1$ when the wavenumber $k = 0$. Therefore, the reproduction number $\mathcal{R} = \dfrac{1}{\lambda} = \dfrac{1}{1 - k^2}$ and hence $\mathcal{R}_c = 1$.

Theorem 3.1 *The model (2) has a supercritical instability region whenever the reproduction number $\mathcal{R} > 1$.*

The numerical simulation of the relation between the reproduction number $\mathcal{R}$ and the wavenumber k is depicted in Figure 1. Figure 2 shows the plot of the recovered compartment R as functions of z.

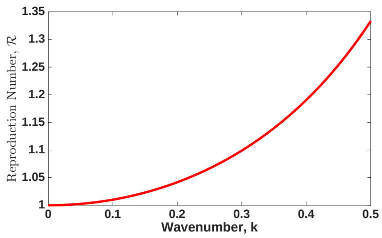

Figure 1: The plot of the reproduction number $\mathcal{R}$ as a function of the wavenumber k.

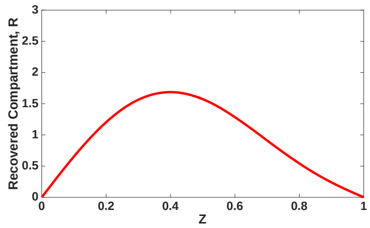

Figure 2: The plot of the recovered compartment, R, as functions of z with $r = 0.1, \beta = 0.9, \delta = 0.01$ and $S_0 = 1000$.

4 Weakly Nonlinear Stability Analysis

A nonlinear evolution equation will be derived in this section. Since the population wavenumber is zero, the long wavelength expansion can be applied to the equations (5). A small perturbation parameter ϵ, $0 < \epsilon \ll 1$, will be introduced. We scale the dimensions

$$\frac{\partial}{\partial x} = \epsilon^{1/2} \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = \epsilon^{1/2} \frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z}, \quad \tau = \epsilon^2 t$$

and we expand $\lambda = 1 - \epsilon^2 \mu^2$

$$\phi = \epsilon\,\phi_1 + \epsilon^2\,\phi_2 + \cdots,$$

$$\theta = \epsilon\,\theta_1 + \epsilon^2\,\theta_2 + \cdots,$$

$$\psi = \epsilon\,\psi_1 + \epsilon^2\,\psi_2 + \cdots\ ,$$
$$\delta = \epsilon\,\delta_1 + \epsilon^2\,\delta_2 + \cdots\ ,$$

where μ, δ_1 and δ_2 are of $O(1)$ quantities. For simplicity, we consider the one-dimensional problem. The $O(\epsilon)$ problem is described by

$$D^2\phi_1 - \theta_1 = 0,$$
$$D^2\theta_1 = 0,$$
$$D^2\psi_1 + \theta_1 = 0, \tag{10}$$

where $D = \partial/\partial\,Z$. It is subject to the boundary conditions $\psi_1 = \phi_1 = 0$ and $\partial\theta_1/\partial\,Z = 0$ at $Z = 0, 1$. Its solution is given by

$$\theta_1 = f(X, \tau),$$
$$\phi_1 = \frac{f}{2}(Z^2 - Z),$$
$$\psi_1 = -\frac{f}{2}(Z^2 - Z).$$

Proceed to the next order, the $O(\epsilon^2)$ problem is described by

$$D^2\,\phi_2 + (\phi_1)_{XX} - \theta_2 - \theta_1\,\phi_1 + \delta_1\,\psi_1 = 0,$$
$$D^2\,\theta_2 + (\theta_1)_{XX} + \theta_1\,\phi_1 = 0,$$
$$D^2\,\psi_2 + (\psi_1)_{XX} + \theta_2 - \delta_1\,\psi_1 = 0,$$

subject to the boundary conditions $\psi_2 = \phi_2 = 0$ and $\partial\theta_2/\partial\,Z = 0$ at $Z = 0, 1$. Its solution is given by

$$\phi_2 = -\frac{f_{XX}}{12}\,\left(Z^4 - Z^3\right) - \frac{f^2}{720}\,\left(Z^6 - 3\,Z^5 - 30\,Z^4 + 60\,Z^3 - 28\,Z\right),$$
$$+\frac{\delta_1\,f}{24}\,\left(Z^4 - 2\,Z^3 + Z\right) + \frac{B}{2}\,\left(Z^2 - Z\right),$$
$$\theta_2 = -\frac{f_{XX}}{2}\,Z^2 - \frac{f^2}{24}\,\left(Z^4 - 2\,Z^3\right) + B,$$
$$\psi_2 = \frac{f_{XX}}{12}\,\left(Z^4 - Z^3\right) + \frac{f^2}{720}\,\left(Z^6 - 3\,Z^5 + 2\,Z\right) - \frac{\delta_1\,f}{24}\,\left(Z^4 - 2\,Z^3 + Z\right) - \frac{B}{2}\,\left(Z^2 - Z\right),$$

where

$$B = \frac{137}{1512}\,f^2 + \frac{1}{21}\,f_{XX} + \frac{17\,\delta_1}{168}\,f.$$

Proceeding to the order $O(\epsilon^3)$, we get

$$(\theta_1)_\tau = D^2\theta_3 + (\theta_2)_{XX} - \mu^2\,\theta_1 + \theta_2\,\phi_1 + \theta_1\,\phi_2. \tag{11}$$

Application of the orthogonality conditions on equation (11) yields the sought evolution equation:

$$f_\tau = -\frac{5}{42}\,f_{XXXX} - \frac{17\,\delta_1}{168}\,f_{XX} - \mu^2\,f - \frac{199}{45360}\,f^3 - \frac{43}{5040}\,f^2 + \frac{11}{1260} f\,f_{XX} + \frac{1559}{15120}\,\left(f^2\right)_{XX}. \tag{12}$$

5 Pattern Formation

By generalizing the procedure which has been used above, the three-dimensional nonlinear evolution equation will be obtained:

$$f_\tau = -\frac{5}{42}\,\nabla_H^4\, f - \frac{17\,\delta_1}{168}\,\nabla_H^2\, f - \lambda_1\, f - \frac{199}{45360}\, f^3 - \frac{43}{5040}\, f^2 + \frac{11}{1260} f\,\nabla_H^2\, f + \frac{1559}{15120}\,\left(\nabla_H^2\, f^2\right), \tag{13}$$

where $\nabla_H = (\partial/\partial\, X, \partial/\partial\, Y)$. Upon using the following transformation in equation (13):

$$\xi = \sqrt{\frac{17\delta_1}{40}}\, X, \quad \zeta = \sqrt{\frac{17\delta_1}{40}}\, Y, \quad \eta = \frac{289\,\delta_1^2}{13440\,\tau}, \quad f = \mu^2\, F,$$

equation (13) is transferred to

$$F_\eta = -\nabla_H^4\, F - 2\,\nabla_H^2\, F - \omega\, F - A\, F^3 - B\, F^2 + C\, F\,\nabla_H^2 + E\,\nabla_H^2 F^2, \tag{14}$$

where $\nabla_H = (\partial/\partial\,\xi, \partial/\partial\,\zeta)$, $A = \dfrac{1592\mu^2}{7803\,\delta_1^2}$, $B = \dfrac{344\mu}{867\,\delta_1}$, $C = \dfrac{44\mu}{255\,\delta_1}$, $E = \dfrac{1559\mu}{765\,\delta_1}$ and $\omega = \dfrac{13440}{289\,\delta_1^2}\mu^2$.

The linear stability analysis will be applied, where the growth rate of normal modes with small amplitude can be determined by considering the linearized version of equation (14)

$$\frac{\partial\, F}{\partial\,\eta} = -\nabla_H^4\, F - 2\,\nabla_H^2\, F - \omega\, F. \tag{15}$$

Application of the normal modes $f = \exp(\sigma\,\eta + i\,\mathbf{K}\cdot\mathbf{r})$, where σ is the growth rate, $\mathbf{K}$ is the wave vector such that $|\mathbf{K}| = k$ and $\mathbf{r} = (\xi, \zeta)$, in equation (15) yields

$$\sigma = -(k^2 - 1)^2 + 1 - \omega.$$

Hence, the range of instability is $0 < \omega < 1$. That is, $0 < mu^2 < \dfrac{289\,\delta_1^2}{13440}$. The effect of nonlinear terms in the evolution equation (14) can be depicted by considering the following expansions:

$$F = \epsilon\, F_1 + \epsilon^2\, F_2 + \cdots,$$

$$\omega = 1 - \epsilon\,\omega_1 - \epsilon^2\,\omega_2 - \cdots$$

and $\eta = \epsilon^2\,\widehat{\eta}$. The solution to the $O(\epsilon)$ which is described by

$$\nabla_H^4\, F_1 + 2\,\nabla_H^2\, F_1 + F_1 = 0 \tag{16}$$

is given by

$$F_1(\xi, \zeta, \widehat{\eta}) = U(\widehat{\eta})\,\cos(\zeta) + V(\widehat{\eta})\,\cos\left(\frac{\sqrt{3}\,\xi}{3}\right)\,\cos\left(\frac{\zeta}{2}\right). \tag{17}$$

Equation (17) yields a roll structure when $V(\widehat{\eta}) = 0$ and a square structure when $U(\widehat{\eta}) = 0$. Proceed to the next order $O(\epsilon^2)$, the problem is described by

$$\nabla_H^4\, F_2 + 2\,\nabla_H^2\, F_2 + F_2 = \omega_1\, F_1 - B\, F_1^2 + C\, F_1\,\nabla_H^2\, F_1 + E\,\nabla_H^2\, F_1^2. \tag{18}$$

Upon averaging equation (18) by multiplying both sides by F_1 and integrating over the domain of ξ and ζ, we get $\omega_1 = 0$. Hence the solution of equation (18) is given by

$$F_2 = -\left(\frac{B+C}{2} + \left(\frac{B+C+4\,E}{2}\right)\cos(\zeta)\right) U^2$$

$$-2\left(E\,\cos\left(\frac{3\,\zeta}{2}\right) + (E+B+C)\,\cos(\zeta)\,\cos\left(\frac{\zeta}{2}\right)\right) \times \cos\left(\sqrt{3}\,\xi\right)\,U\,V+$$

$$\left[\left(\frac{B+C-E}{4}\right)\cos(\zeta) - \left(\frac{C+B}{4}\right) - \left(\frac{3E+C+B}{4}\right)\right.$$

$$\left.+\left(\frac{4E-B-C}{4}\right)\cos(\zeta)\cos(\sqrt{3}\,\xi)\right] V^2.$$

Proceed to the next order, $O(\epsilon^3)$, the problem is described by

$$\frac{\partial F_1}{\partial \widehat{\eta}} = -\nabla_H^4\,F_3 - 2\,\nabla_H^2\,F_3 - F_3 + \omega_1\,F_2 + \omega_2\,F_1$$

$$-\,A\,F_1^3 - B\,F_1\,F_2 + C\left(F_1\,\nabla_H^2 F_2 + F_2\,\nabla_H^2 F_1\right) + E\nabla_H^2(F_1\,F_2). \tag{19}$$

In order to obtain the amplitude equations, we will average equation (19) with F_1 to get

$$\langle\frac{\partial F_1}{\partial \widehat{\eta}}, F_1\rangle = -\langle\nabla_H^4\,F_3 + 2\,\nabla_H^2\,F_3 + F_3, F_1\rangle$$

$$+\,\langle\omega_2\,F_1 - A\,F_1^3 - B\,F_1\,F_2, F_1\rangle + \langle C\left(F_1\,\nabla_H^2 F_2 + F_2\,\nabla_H^2 F_1\right) + E\nabla_H^2(F_1\,F_2), F_1\rangle. \tag{20}$$

The following are the amplitude equations of U and V:

$$\frac{\partial U}{\partial \widehat{\eta}} = -\Gamma_1\,U^3 + \omega_2\,U + \Gamma_2\,U\,V^2, \tag{21}$$

$$\frac{\partial V}{\partial \widehat{\eta}} = -\Gamma_3\,V^3 + \omega_2\,V + \Gamma_2\,V\,U^2, \tag{22}$$

where

$$\Gamma_1 = \frac{398}{2601\,\delta_1} - \frac{1341428296}{169130025\,\delta_1^2}, \quad \Gamma_2 = \frac{2504438396}{169130025\,\delta_1^2} - \frac{796}{2601\,\delta_1}$$

and

$$\Gamma_3 = \frac{199}{1734\,\delta_1} - \frac{320488891}{56376675\,\delta_1^2}.$$

We set $V = 0$ in equation (21) to determine the stability of roll structure. Hence the equilibrium points are $(U, V) = (0, 0)$ and $(U, V) = \left(\pm\sqrt{\frac{\omega_2}{\Gamma_1}}, 0\right)$.

Theorem 5.1 *If $\Gamma_1 > 0$ and $\omega_2 > 0$, then the roll solutions exist and the bifurcation is sub-critical. If both Γ_1 and ω_2 are negative, then the roll solutions exist and the bifurcation is supercritical.*

Following the same analysis, the square structure can be determined. If we set $U = 0$ in equation (22), the equilibrium points are $(U, V) = (0, 0)$ and $(U, V) = \left(0, \pm\sqrt{\frac{\omega_2}{\Gamma_3}}\right)$.

Theorem 5.2 *The square solutions exist with sub-critical bifurcation when both Γ_3 and ω_2 are positive. But if both are negative, the square solutions exist with supercritical bifurcation.*

To determine the hexagons structure, we set $V = 2\,U$ in equation (17) and assume that $\eta = \epsilon\,\widehat{\eta}$. Hence, the $O(\epsilon^2)$ problem is described by

$$\frac{\partial F_1}{\partial \widehat{\eta}} = -\nabla_H^4 F_2 - 2\,\nabla_H^2 F_2 - F_2 + \omega_1 F_1 - B\,F_1^2 + C\,F_1\,\nabla_H^2 F_1 + E\nabla_H^2 F_1^2. \tag{23}$$

To obtain the amplitude equation for hexagons, we average equation (23) with F_1 so that

$$\langle\frac{\partial F_1}{\partial \widehat{\eta}}, F_1\rangle = \langle -\nabla_H^4 F_2 - 2\,\nabla_H^2 F_2 - F_2 + \omega_1 F_1 - B\,F_1^2 + C\,F_1\,\nabla_H^2 F_1 + E\nabla_H^2 F_1^2, F_1\rangle. \tag{24}$$

Thus, the amplitude equation for hexagons is given by

$$\frac{\partial U}{\partial \widehat{\eta}} = \left(\omega_1 + \frac{1559}{765\,\delta_1}\right) U + \Gamma\,U^2, \tag{25}$$

where $\Gamma = -\dfrac{2468}{4335\,\delta_1}$.

Theorem 5.3 *Since $\Gamma < 0$, the solution of equation (14) has sub-critical down hexagons formation.*

6 Uniformly Periodic Valid Solution

Upon applying the following scales on equation (12) $f = ah$, $\xi = bX$, $\widehat{\tau} = e\tau$, $\gamma = a\lambda_1$, $e = 1/a$, we obtain

$$\frac{\partial h}{\partial \widehat{\tau}} = -h_{\xi\xi\xi\xi} - 2\mu^2\,h_{\xi\xi} - \gamma\,h - \alpha_1\,h^2 - \alpha_2\,h^3 + \alpha_3\,h\,h_{\xi\xi} + \alpha_4\,(h_\xi)^2, \tag{26}$$

where $a = \dfrac{6}{\sqrt{65}}$, $b = \sqrt[4]{7\sqrt{\dfrac{13}{5}}}$, $\mu^2 = \dfrac{17\,\delta_1}{8\sqrt{455}} \times \sqrt[4]{\dfrac{13}{5}}$, $\alpha_1 = \dfrac{43}{9100}$, $\alpha_2 = \dfrac{199}{13650\sqrt{65}}$, $\alpha_3 = \dfrac{5}{6\sqrt{7}} \times \sqrt[4]{\dfrac{13}{5}}$ and $\alpha_4 = \dfrac{1559}{1950\sqrt{7}} \times \sqrt[4]{\dfrac{13}{5}}$.

We investigate the stability of the static solution of equation (26), when its linear part is given by

$$\frac{\partial h}{\partial \widehat{\tau}} = -h_{\xi\xi\xi\xi} - 2\mu^2\,h_{\xi\xi} - \gamma\,h. \tag{27}$$

Upon introducing the normal modes $h(\xi,\tau) = \mathrm{e}^{\sigma\tau + i\,k\,\xi}$, we obtain the following dispersion relation:

$$\sigma = -(k^2 - \mu^2)^2 - \gamma. \tag{28}$$

The static state solution is unstable whenever $\gamma < \mu^4$. To investigate the weakly nonlinear stability of the evolution equation we introduce the small parameter, $\epsilon \ll 1$, and conduct the perturbation analysis near the linear solution. We expand

$$\gamma = \mu^4 - \epsilon\,\gamma_1 - \epsilon^2\,\gamma_2, \quad \tau = \epsilon^2\,\eta,$$

$$h = \epsilon\, h_1 + \epsilon^2\, h_2 + \epsilon^3\, h_3 + \cdots$$

The order $O(\epsilon)$ of equation (26) is described by

$$(h_1)_{\xi\xi\xi\xi} + 2\mu^2\,(h_1)_{\xi\xi} + \gamma\, h_1 = 0. \tag{29}$$

The solution of equation (29) is $h_1 = \cos(\mu\xi)$. Because of the secular terms, we will apply the Poincaré - Lindstedt method [23] to obtain a uniformly valid periodic solution. Substitute $\nu = w\,\xi$ and expand $w = 1 + \epsilon\, w_1 + \epsilon^2\, w_2 + \cdots$ in equation (26) to obtain

$$w^4\, h_{\nu\nu\nu\nu} + 2\,\mu^2\, w^2\, h_{\nu\nu} + \gamma\, h = \alpha_1\, h^2 + \alpha_2\, h^3 + w^2\, \left(\alpha_3\, h\, h_{\nu\nu} + \alpha_4\,(h_\nu)^2\right). \tag{30}$$

Define: $\mathscr{L}(h) = h_{\nu\nu\nu\nu} + 2\mu^2\, h_{\nu\nu} + \gamma\, h$. The order $O(\epsilon)$ problem is described by

$$\mathscr{L}(h_1) = 0. \tag{31}$$

The solution of equation (31) is $h_1 = \cos(\mu\nu)$. Proceed to order $O(\epsilon^2)$, the problem is described by

$$\mathscr{L}(h_2) = \gamma_1\, \cos(\mu\nu) + \Gamma_1 + \Gamma_2\, \cos(2\mu\nu). \tag{32}$$

To remove the secular terms, we set $\gamma_1 = 0$, which means that there is no subcritical instability. The solution of the resulting equation is given by $h_2 = \dfrac{\Gamma_1}{\mu^4} + \dfrac{\Gamma_2}{9}\,\cos(2\,\mu\nu)$, where

$$\Gamma_1 = \frac{1}{2}\,\left(\alpha_1 - \mu^2\,\alpha_3 + \mu^2\,\alpha_4\right) \quad \text{and} \quad \Gamma_1 = \frac{1}{2}\,\left(\alpha_1 - \mu^2\,\alpha_3 - \mu^2\,\alpha_4\right).$$

Proceed to the next order $O(\epsilon^3)$, the problem is described by

$$\begin{aligned} \mathscr{L}(h_3) = \Bigg[&-4\mu^4\, w_1^2 + \gamma_2 - \frac{\Gamma_1}{\mu^4}(2\alpha_1 + \mu^2\,\alpha_3) \\ &+\frac{\Gamma_2}{36}(-4\alpha_1 - 27\alpha_2 - 10\mu^2\alpha_3 + 16\mu^2\alpha_4)\Bigg]\,\cos(\mu\nu) - \left[\frac{w_1}{18}(96\mu^4\Gamma_2 + 9\mu^2\alpha_3)\right]\,\cos(2\mu\nu) \\ &+ \left[\frac{\Gamma_2}{36}(-4\alpha_1 - 9\alpha_2 - 10\mu^2\alpha_3 + 16\mu^2\alpha_4)\right]\, cos(3\mu\nu) - \frac{w_1\mu^2\,\alpha_3}{2}. \end{aligned} \tag{33}$$

Removing the secular term by setting

$$-4\mu^4\, w_1^2 + \gamma_2 - \frac{\Gamma_1}{\mu^4}(2\alpha_1 + \mu^2\,\alpha_3) + \frac{\Gamma_2}{36}(-4\alpha_1 - 27\alpha_2 - 10\mu^2\alpha_3 + 16\mu^2\alpha_4) = 0$$

yields

$$w_1 = \pm\sqrt{\frac{\gamma_2}{4\mu^4} - \frac{\Gamma_1}{4\mu^8}\,(2\alpha_1 + \mu^2\alpha_3) + \frac{\Gamma_2}{144\mu^4}\,(-4\alpha_1 - 27\alpha_2 - 10\mu^2\alpha_3 + 16\mu^2\alpha_4)}.$$

Upon solving the rest of equation (33), we get

$$h_3 = \left[\frac{w_1}{3\mu^2}(32\mu^2\Gamma_2 + 3\alpha_3)\right]\,\cos(2\mu\nu) + \left[\frac{\Gamma_2}{2304\mu^4}(-4\alpha_1 - 9\alpha_2 - 10\mu^2\alpha_3\right.$$

$$+16\mu^2\alpha_4)\big]\,\cos(3\mu\nu)+\frac{w_1\,\alpha_3}{2\mu^2}. \tag{34}$$

Thus, a uniformly valid steady state solution to equation (26) is given by

$$\begin{aligned} h &= \cos(\mu(1+\epsilon w_1)\xi)\,\epsilon+\left(\frac{\Gamma_1}{\mu^4}+\frac{\Gamma_2}{9}\cos(2\,\mu(1+\epsilon w_1)\xi)\right)\epsilon^2 \\ &+\left(\left[\frac{w_1}{3\mu^2}(32\mu^2\Gamma_2+3\alpha_3)\right]\cos(2\mu(1+\epsilon w_1)\xi)\right) \\ &+\left[\frac{\Gamma_2}{2304\mu^4}(-4\alpha_1-9\alpha_2-10\mu^2\alpha_3+16\mu^2\alpha_4)\right]\cos(3\mu(1+\epsilon w_1)\xi))+\frac{w_1\,\alpha_3}{2\mu^2}\Big)\,\epsilon^3. \end{aligned} \tag{35}$$

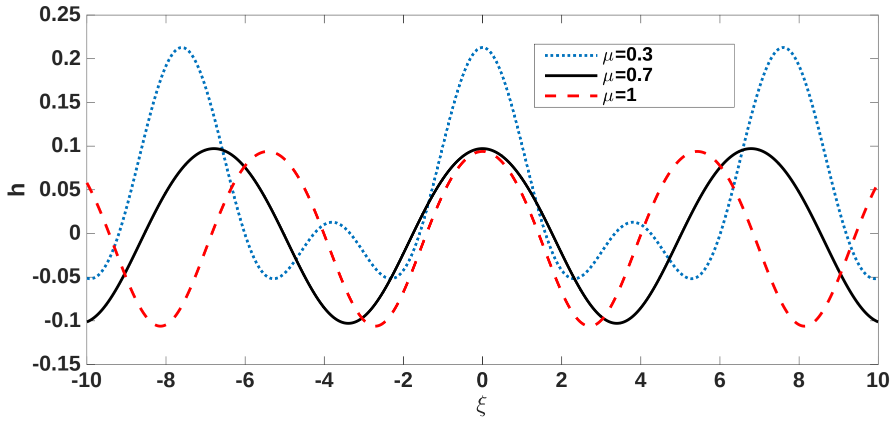

Figure 3: The plot of h as a function of ξ with $\mu = 0.3$ (dotted line), $\mu = 0.7$ (solid line), $\mu = 1$ (dashed line) and $\gamma_2 = 10$.

7 Conclusion

An SIRS model for the spread of a disease has been considered. This model involves the spatial diffusion so that the effect of landscape can be captured. The domain of the model is not infinite, it is complemented with two horizontal boundaries. The stability threshold condition is obtained so that whenever the reproduction number $\mathcal{R} > 1$, a supercritical instability region is depicted. See Theorem 3.1.

Because of the infinite wavelength, the weakly nonlinear stability analysis was conducted by applying the long wavelength asymptotic analysis method and the proposed model is reduced to a single evolution equation (12). Full pattern formation analysis of equation (12) is carried out and the subcritical down hexagons are depicted.

It is found that there exists a stable uniform solution, namely $F = 1$. Upon retrieving the original variables, we have $f = \mu\, F = \dfrac{\sqrt{1-\lambda}}{\epsilon}$, which yields the following expression of infected, susceptible and recovered:

$$I=\frac{\sqrt{1-\lambda}}{\epsilon},\quad S=1+\frac{\sqrt{1-\lambda}}{2\epsilon}\,(Z^2-Z),\quad R=\frac{\sqrt{1-\lambda}}{2\epsilon}\,(Z-Z^2).$$

Figure (4) shows the plot of I as a function of the reproducing number $\mathcal{R}$ and the space dependent functions S and R.

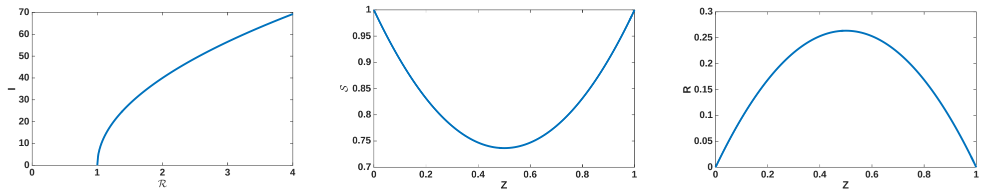

Figure 4: The plot of the infected, I, as a function of the reproduction number, $\mathcal{R}$, (left), the plot of the susceptible, S, as a function of z, (middle), and the plot of the recovered, R, as a function of z, (right).

Moreover, the Poincaré-Lindstedt method is applied to obtain a uniformly periodic valid steady state solution which is depicted in Figure 3.

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