Nonlinear Dynamics and Systems Theory, **21** (1) (2021) 27–55

Existence and Asymptotic Behavior of Unbounded Positive Solutions of a Nonlinear Degenerate Elliptic Equation

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Received: July 8, 2020;   Revised: December 29, 2020

Abstract: This paper is a contribution to the study of the elliptic equation

$$\Delta_p u + \alpha u + \beta x \cdot \nabla u + |x|^l \; u^q = 0 \quad \text{in } \mathbb{R}^N,$$

where $p > 2$, $q > 1$, $N \geq 1$, $\alpha < 0$, $\beta < 0$ and $l < 0$.
If $q \leq p-1$ or $q > p-1$ and $\dfrac{\alpha}{\beta} \neq \dfrac{l+p}{q+1-p}$ or $\dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p} \geq \dfrac{N-p}{p}$, we prove the existence of unbounded radial solutions u and we obtain their asymptotic behavior. In particular, if $\dfrac{\alpha}{\beta} < \dfrac{-l}{q-1}$, $\lim\limits_{r \to +\infty} r^{l/(q-1)} u(r) = \left(\dfrac{\beta l}{q-1} - \alpha \right)^{1/(q-1)}$.

Keywords: *nonlinear parabolic problem; nonlinear degenerate elliptic equation; self-similar solutions; nonlinear dynamical system; unbounded solutions; energy function; asymptotic behavior.*

Mathematics Subject Classification (2010): 34L30, 35J60, 35K92, 35K55.

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1 Introduction

This paper is devoted to the study of the elliptic equation

$$\Delta_p u + \alpha u + \beta x \cdot \nabla u + |x|^l \; u^q = 0 \qquad \text{in } \mathbb{R}^N, \tag{1}$$

where $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 2$, $\nabla u = \left(\dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_N}\right)$, $N \geq 1$, $q > 1$, $\alpha < 0$, $\beta < 0$ and l is a real number such that $-p < l < 0$ and $-N < l < 0$.

Equations of the above form occur in the study of self-similar solutions of the nonlinear parabolic problem

$$u_t = \Delta_p u + |x|^l \, |u|^{q-1} u \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{2}$$

A lot of work has been done concerning equation (1) when $l = 0$; discussions and bibliographies are found in [3], [4], [10], [11], [13], [15], [16] and [18]. When $p = 2$ and $-2 < l < 0$, equation (1) was studied in [8]. Note also that when $p > 2$ and $l < 0$, the equation was investigated for $\alpha > 0$ and $\beta > 0$ in [9] and was initiated in [7] by the authors for $\alpha < 0$ and $\beta < 0$.

In our paper [7], we studied radial solutions near 0 of the equation

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u + \beta r u' + r^l |u|^{q-1}u = 0, \quad r > 0.$$

Among the results obtained, we showed that for any radial solution u with $u(0) > 0$, $\lim\limits_{r \to 0} r^{(N-1)/(p-1)}u'(r)$ exists and is finite. Moreover, for any $a > 0$ and $b \in \mathbb{R}$, there exists a unique function $u \in C^0([0, +\infty[) \cap C^1(]0, +\infty[)$ such that $|u'|^{p-2}u' \in C^1(]0, +\infty[)$, and satisfying the problem

$$(\mathbf{P}) \begin{cases} \left(|u'|^{p-2}u'\right)' + \dfrac{N-1}{r}|u'|^{p-2}u' + \alpha u + \beta r u' + r^l |u|^{q-1}u = 0, \quad r > 0, \\ u(0) = a, \quad \lim\limits_{r \to 0} r^{(N-1)/(p-1)}u'(r) = b, \end{cases}$$

where $p > 2$, $q > 1$, $N \geq 1$, $-p < l < 0$, $-N < l < 0$, $\alpha < 0$ and $\beta < 0$.

It is also proved that if a is small and $b = 0$, u is strictly positive.

Our aim in this paper, is to continue our study on the problem $(\mathbf{P})$. For this we must start with the analysis of solutions of the equation

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u + \beta r u' + r^l u^q = 0, \quad r > 0. \tag{3}$$

First of all, it should be noted that when $0 < \dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p} < \dfrac{N-p}{p-1}$, we have an explicit solution $L\, r^{-\alpha/\beta}$, where

$$L = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{1/(q+1-p)} \left(\frac{\alpha}{\beta}\right)^{(p-1)/(q+1-p)}.$$

This solution is bounded near infinity but singular at the origin. However, using the theory of ODE, the equation (3), for bounded solutions whether they are singular or not, can be considered in $+\infty$ as a perturbation of the equation

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u + \beta r u' = 0,$$

whose solutions behave like the function $r^{-\alpha/\beta}$ near infinity.

On the other hand, the term $r^l u^q$ plays a key role in (3) for unbounded solutions and therefore, the perturbation theory can not be applicable. Hence the interest of focusing our study on unbounded solutions.

For this purpose, let us represent equation (3) in an equivalent but useful form. For any real c, we set

$$\upsilon_c(t) = r^c u(r) \quad \text{where } r > 0 \text{ and } t = \ln r. \tag{4}$$

Then υ_c satisfies

$$\omega_c'(t) + A_c\,\omega_c(t) + \alpha e^{K_c t}\upsilon_c(t) + \beta e^{K_c t} h_c(t) + e^{M_c t}\upsilon_c^q(t) = 0, \tag{5}$$

where

$$\omega_c(t) = |h_c|^{p-2} h_c(t), \qquad h_c(t) = \upsilon_c'(t) - c\upsilon_c(t), \tag{6}$$

$$A_c = N - p - c(p-1), \;\; K_c = c(p-2) + p \;\text{ and }\; M_c = l + p - c(q+1-p). \tag{7}$$

Five critical values of the parameter c will be involved: $\dfrac{\alpha}{\beta}$, $\dfrac{l}{q-1}$ and those which cancel A_c, K_c or M_c, that is, $c = \dfrac{-p}{p-2}$, $c = \dfrac{N-p}{p-1}$ or $c = \dfrac{l+p}{q+1-p}$.

The study of monotonicity of $r^c u(r)$ for these last five values, combined with the behavior of bounded solutions, allows us to show the existence of unbounded solutions of problem (**P**).

A fine analysis of the equation in logarithmic form, using some energy function, gives the asymptotic behavior of solutions. Our main results are given by the following theorems.

Theorem 1.1 *Assume that* $q \le p-1$ *or* $q > p-1$ *and* $\dfrac{\alpha}{\beta} \neq \dfrac{l+p}{q+1-p}$ *or* $\dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p} \ge \dfrac{N-p}{p}$. *Then any positive solution of problem* (**P**) *is unbounded.*

Theorem 1.2 *Assume* $q \ge p\left(2 + 2^{p-1}\right) - 1$ *and* $\dfrac{l}{q-1} < \min\left(\dfrac{-\alpha}{\beta}, \dfrac{N-p}{p-1}\right)$. *Let* u *be an unbounded positive solution of problem* (**P**). *Then*

$$\lim_{r\to+\infty} r^{l/(q-1)} u(r) = \Gamma$$

and

$$\lim_{r\to+\infty} r^{l/(q-1)+1} u'(r) = \frac{-l}{q-1}\Gamma,$$

where

$$\Gamma = \left(\frac{\beta l}{q-1} - \alpha\right)^{1/(q-1)}.$$

The rest of the paper is organized as follows. In the second section, we present fundamental properties of solutions of equation (3). The third section concerns the existence and the asymptotic behavior near infinity of unbounded positive solutions of problem (**P**).

2 Fundamental Properties

In this section, we give some fundamental properties that are the key stone of the main results. For this purpose, we introduce, for any real $c \neq 0$, the function

$$E_c(r) = cu(r) + ru'(r), \quad r > 0. \tag{8}$$

It is clear that

$$(r^c u(r))' = r^{c-1} E_c(r), \quad r > 0. \tag{9}$$

With the logarithmic change (4), we have

$$v_c'(t) = r^c\, E_c(r) \;\text{ and }\; h_c(t) = r^{c+1} u'(r). \tag{10}$$

The monotonicity of the function $r^c u(r)$ can be obtained by the sign of the function $E_c(r)$. Observe that for any $r > 0$ such that $u'(r) \neq 0$, we have

$$\begin{aligned}(p-1)\,|u'|^{p-2}\,(r) E_c'(r) &= \Big(p - N + c(p-1)\Big)\,|u'|^{p-2}\,u'(r) - \beta r^2 u'(r) - \\ &\quad \alpha r u(r) - r^{l+1} u^q(r) \\ &= \Big(p - N + c(p-1)\Big)\,|u'|^{p-2}\,u'(r) + |\beta| r\, E_{\alpha/\beta}(r) - \\ &\quad r^{l+1} u^q(r).\end{aligned} \tag{11}$$

Consequently, if $E_c(r_0) = 0$ for some $r_0 > 0$, equation (3) gives

$$\begin{aligned}(p-1)\,|u'|^{p-2}\,(r_0) E_c'(r_0) &= -r_0 u(r_0)\Big[\alpha - c\beta + r_0^l u^{q-1}(r_0) + \\ &\quad \Big(p - N + c(p-1)\Big)|c|^{p-2} c\, r_0^{-p} u^{p-2}(r_0)\Big] \\ &= -r_0^{l+1} u^q(r_0)\Big[1 + (\alpha - c\beta) r_0^{-l} u^{1-q}(r_0) + \\ &\quad \Big(p - N + c(p-1)\Big)|c|^{p-2} c\, r_0^{-l-p} u^{p-1-q}(r_0)\Big],\end{aligned} \tag{12}$$

from which we can study the sign of $E_c(r)$ and we use the following remarks.

Remark 2.1 If there exists r_0 such that $E_c(r_0) = 0$ and $E_c'(r_0) \neq 0$, then $E_c(r) \neq 0$ for any $r > r_0$.

Remark 2.2 If u is a bounded solution of equation (3), then, by expression (12) and Remark 2.1, we have, for any $c > 0$ such that $c\beta - \alpha \neq 0$, $E_c(r) \neq 0$ for large r.

We first give the sign of $E_{l/(q-1)}$ and $E_{-p/(p-2)}$.

Proposition 2.1 *Let u be a solution of equation (3). We put*

$$\Gamma = \left(\frac{\beta l}{q-1} - \alpha\right)^{1/(q-1)}. \tag{13}$$

The following holds:
(i) If $\dfrac{N-p}{p-1} > \dfrac{l}{q-1}$ *and* $u(r) > \Gamma r^{-l/(q-1)}$ *for large* r, *then* $E_{l/(q-1)}(r) \neq 0$ *for large*

r.
(ii) If $q \geq p-1$, $\lim_{r\to+\infty} u(r) = +\infty$ *and* $\lim_{r\to+\infty} r^{l/(q-1)}u(r) = 0$, *then* $E_{l/(q-1)}(r) < 0$ *for large* r.
(iii) If $q \geq p-1$ *and* $\lim_{r\to+\infty} r^{l/(q-1)}u(r) = +\infty$, *then* $E_{-p/(p-2)}(r) < 0$ *for large* r.

The proof requires the following result.

Lemma 2.1 *Let* u *be a solution of equation (3) such that for large* r,

$$u(r) > \Gamma r^{-l/(q-1)},$$

where Γ *is given by (13). Then* $u'(r) > 0$ *for large* r.

Proof. Suppose that there exists a large r_0 such that $u'(r_0) = 0$, then

$$\Big(|u'|^{p-2}u'\Big)'(r_0) = -\Big[\alpha + r_0^l u^{q-1}(r_0)\Big]u(r_0) < -\frac{\beta l}{q-1}u(r_0) < 0.$$

Hence, $u'(r) \neq 0$ for any $r > r_0$. Moreover, since $\lim_{r\to+\infty} u(r) = +\infty$, we have $u'(r) > 0$ for large r. □

Now, we turn to the proof of Proposition 2.1.
Proof. (of Proposition 2.1). The cases (i) and (ii) follow easily from Remark 2.1 and relation (9). Assume now that we are in the case (iii), then again Remark 2.1 gives $E_{-p/p-2}(r) \neq 0$ for large r. Suppose by contradiction that $E_{-p/(p-2)}(r) > 0$ for large r. Hence, by (9), $\lim_{r\to+\infty} r^{-p/(p-2)}u(r) \in]0, +\infty]$. Set

$$\varphi(r) = r^{N-1}|u'|^{p-2}u'(r) + \beta r^N u(r), \qquad r > 0, \tag{14}$$

then, by equation (3), we get

$$\varphi'(r) = r^{N-1}u(r)\Big[N\beta - \alpha - r^l u^{q-1}(r)\Big], \qquad \text{for any } r > 0, \tag{15}$$

so $\lim_{r\to+\infty} \varphi'(r) = -\infty$, in particular, $\varphi(r) < 0$ for large r, that is,

$$r^{N-1}|u'|^{p-2}u'(r) < |\beta| r^N u(r) \tag{16}$$

for large r. Note that by Lemma 2.1, we have $u'(r) > 0$ for large r and then a simple integration of this last inequality on (r_0, r) gives $\lim_{r\to+\infty} r^{-p/(p-2)}u(r) = d_1 > 0$.
Now we are going to the logarithmic change. First, from (5) we obtain

$$\begin{aligned}\omega'_{-p/(p-2)}(t) \quad &= -A_{-p/(p-2)}\,\omega_{-p/(p-2)}(t) - \alpha\upsilon_{-p/(p-2)}(t) - \\ &\beta h_{-p/(p-2)}(t) - e^{M_{-p/(p-2)}t}\upsilon^q_{-p/(p-2)}(t).\end{aligned} \tag{17}$$

Note that, as $E_{-p/(p-2)}(r) > 0$ for large r, then by (10) $\upsilon'_{-p/(p-2)}(t) > 0$ for large t. Then, since $\lim_{r\to+\infty} r^{-p/(p-2)}u(r) = d_1 > 0$, necessarily $\lim_{t\to+\infty} \upsilon'_{-p/(p-2)}(t) = 0$, which implies that $\lim_{t\to+\infty} h_{-p/(p-2)}(t) = \dfrac{p}{p-2}d_1$ and therefore $\lim_{t\to+\infty} \omega_{-p/(p-2)}(t) = \Big(\dfrac{p}{p-2}d_1\Big)^{p-1}$. As $M_{-p/(p-2)} > 0$, it follows by letting $t \to +\infty$ in (17) that $\lim_{t\to+\infty} \omega'_{-p/(p-2)}(t) = -\infty$. This is impossible as $\lim_{t\to+\infty} \omega_{-p/(p-2)}(t)$ is finite.

Consequently, $E_{-p/(p-2)}(r) < 0$ for large r. The proof is complete. □

Proposition 2.2 *Let u be a solution of equation (3). Then, for any* $c \geq \max\left(\frac{N-p}{p-1}, \frac{\alpha}{\beta}\right)$, $E_c(r) > 0$ *for large* r.

We need the following lemma.

Lemma 2.2 *Assume that u is a bounded solution of equation (3). Then* $u'(r) < 0$ *for large* r.

Proof. First, we claim that $u'(r)$ cannot vanish for large enough r. Assume by contradiction that there exists a large extremum r_0 of u. Then, according to equation (3), we get $\left(|u'|^{p-2}u'\right)'(r_0) = -\left[\alpha + r_0^l u^{q-1}(r_0)\right]u(r_0) > 0$, then $u'(r) > 0$ for any $r > r_0$, from which it turns out that $u'(r) \neq 0$ for large r. If $u'(r) > 0$ for large r, it follows by the boundedness of u that $\lim_{r\to+\infty} u(r) = L > 0$ and from (8), $E_N(r) > 0$ for large r. On the other hand, by (11), we have that for large r

$$\begin{aligned}(p-1)\left|u'\right|^{p-2}(r)E'_N(r) &= (p+N(p-2))\left|u'\right|^{p-1} + |\beta| r^2 u'(r)\\ &+ \left[|\alpha| - r^l u^{q-1}\right] r u(r)\\ &> \left[|\alpha| - r^l u^{q-1}\right] r u(r) > 0. \end{aligned} \tag{18}$$

So, $\lim_{r\to+\infty} E_N(r) \in]0, +\infty]$.

Note that if $\lim_{r\to+\infty} E_N(r) = +\infty$, then by (8), $\lim_{r\to+\infty} ru'(r) = +\infty$, which contradicts the boundedness of u. On the other hand, if $\lim_{r\to+\infty} E_N(r)$ is finite and strictly positive, necessarily $\lim_{r\to+\infty} ru'(r) = 0$ and by letting r to $+\infty$ in equation (3), we obtain $\lim_{r\to+\infty}(|u'|^{p-2}u')'(r) = -\alpha L > 0$, which implies that $\lim_{r\to+\infty} u'(r) = +\infty$ and we have again a contradiction. Consequently, $u'(r) < 0$ for large r. □

Now we prove Proposition 2.2.

Proof. (of Proposition 2.2). We distinguish two cases.

Case 1: u is bounded.

Assume $\frac{\alpha}{\beta} \geq \frac{N-p}{p-1}$. We have easily from Remark 2.1, $E_{\alpha/\beta}(r) \neq 0$ for large r. Suppose that $E_{\alpha/\beta}(r) < 0$ for large r, it turns out by (9) that the function $r^{\alpha/\beta}u(r)$ is decreasing for large r, hence $\lim_{r\to+\infty} r^{\alpha/\beta}u(r)$ exists and is finite. But $\frac{\alpha}{\beta} > 0$, then necessarily $\lim_{r\to+\infty} u(r) = 0$. On the other hand, using the equation (11), Lemma 2.2, the fact that $E_{\alpha/\beta}(r) < 0$ and $\frac{\alpha}{\beta} \geq \frac{N-p}{p-1}$, we get $E'_{\alpha/\beta}(r) < 0$ for large r, hence $\lim_{r\to+\infty} E_{\alpha/\beta}(r) \in [-\infty, 0[$. Combining this with $\lim_{r\to+\infty} u(r) = 0$, we obtain from (8), $\lim_{r\to+\infty} ru'(r) \in [-\infty, 0[$. But this contradicts the fact that u is positive. Consequently, $E_{\alpha/\beta}(r) > 0$ for large r and $E_c(r) > 0$ for $c \geq \frac{\alpha}{\beta}$.

Assume now $\frac{\alpha}{\beta} < \frac{N-p}{p-1}$. Then, from Remark 2.2, $E_{(N-p)/(p-1)}(r) \neq 0$ for large r. Suppose on the contrary that $E_{(N-p)/(p-1)}(r) < 0$ for large r. Since $\frac{\alpha}{\beta} < \frac{N-p}{p-1}$, we have $E_{\alpha/\beta}(r) < 0$ for large r. On the other hand, by (11), we have $E'_{(N-p)/(p-1)}(r) < 0$ and thereby $\lim_{r\to+\infty} E_{(N-p)/(p-1)}(r) \in [-\infty, 0[$. Using similar arguments as before, we get a contradiction. In fact, as the function $r^{(N-p)/(p-1)}u(r)$ is decreasing, then it admits a finite limit, this implies that $\lim_{r\to+\infty} u(r) = 0$. So, by (8), $\lim_{r\to+\infty} ru'(r) \in [-\infty, 0[$, which is impossible. Consequently, $E_{(N-p)/(p-1)}(r) > 0$ for large r and $E_c(r) > 0$ for $c \geq \dfrac{N-p}{p-1}$.

Case 2: u is unbounded.

It is easy to see by Remark 2.1 that for $c \geq \max\left(\dfrac{N-p}{p-1}, \dfrac{\alpha}{\beta}\right)$, $E_c(r) \neq 0$ for large r, that is, by (9), $r^c u(r)$ is strictly monotone. Since u is unbounded, necessarily $\lim_{r\to+\infty} r^c u(r) = +\infty$. Consequently, by (9), $E_c(r) > 0$ for large r. □

3 Unbounded Solutions

In this section we study the unbounded positive solutions of problem (**P**) and we give their asymptotic behavior near infinity.

Theorem 3.1 *Assume that* $q \leq p-1$ *or* $q > p-1$ *and* $\dfrac{\alpha}{\beta} \neq \dfrac{l+p}{q+1-p}$ *or* $\dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p} \geq \dfrac{N-p}{p}$. *Then any positive solution of problem* (**P**) *is unbounded.*

Before giving the proof, we need the behavior of bounded solutions near infinity. For this purpose we start with the following result.

Proposition 3.1 *Assume that u is a bounded solution of equation (3). Then*

$$\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} ru'(r) = 0. \tag{19}$$

Proof. Recall Lemma 2.2 and Proposition 2.2, we deduce that $\lim_{r\to+\infty} u(r) = L$ exists and for any $c \geq \max\left(\dfrac{N-p}{p-1}, \dfrac{\alpha}{\beta}\right)$,

$$-\,cu(r) < r\,u'(r) < 0 \quad \text{for large } r. \tag{20}$$

Thus $ru'(r)$ is bounded for large r. Assume by contradiction that $L > 0$.

First, suppose that $ru'(r)$ is monotone for large r, then necessarily $\lim_{r\to+\infty} ru'(r) = 0$ and we get from equation (3)

$$\lim_{r\to+\infty} \left(|u'|^{p-2}u'\right)'(r) = -\alpha L > 0.$$

This is a contradiction with $u'(r) < 0$ for large r.

Next, suppose that $ru'(r)$ is oscillating for large r. Since u is positive and strictly decreasing, one has $\limsup_{r\to+\infty} ru'(r) = 0$. Otherwise, there exists a constant $C > 0$ such that $ru'(r) < -C$ for large r. This contradicts $u(r) > 0$. Consequently, there exists a sequence $\{\xi_i\}$ going to $+\infty$ as $i \to +\infty$ such that the function $ru'(r)$ has a local maximum in ξ_i satisfying $\lim_{i\to+\infty} \xi_i u'(\xi_i) = \limsup_{r\to+\infty} ru'(r) = 0$ and $u'(\xi_i) + \xi_i u''(\xi_i) = 0$ (u'' exists because $u' < 0$). Therefore

$$\lim_{i\to+\infty} u'(\xi_i) = \lim_{i\to+\infty} u''(\xi_i) = 0. \tag{21}$$

On the other hand, take $r = \xi_i$ in equation (3), we obtain

$$\lim_{i\to+\infty} \Big(|u'|^{p-2}u'\Big)'(\xi_i) = -\alpha L > 0,$$

that is,

$$\lim_{i\to+\infty} |u'|^{p-2}u''(\xi_i) = -\frac{\alpha L}{p-1} > 0.$$

This contradicts (21).

It follows from both cases that $\lim_{r\to+\infty} u(r) = 0$. Hence, by inequality (20), we have $\lim_{r\to+\infty} ru'(r) = 0$ and the proof is complete. □

Proposition 3.2 *Assume that u is a bounded solution of equation (3). Then*

(*i*) *If* $0 < c < \dfrac{\alpha}{\beta} < N$, $\lim_{r\to+\infty} r^c u(r) = 0$.

(*ii*) *If* $\dfrac{\alpha}{\beta} < c < N$, $\lim_{r\to+\infty} r^c u(r) = +\infty$.

Proof. According to Remark 2.2 and expression (9), $r^c u(r)$ is monotone for large r for any $c \neq \dfrac{\alpha}{\beta}$. Hence $\lim_{r\to+\infty} r^c u(r) \in [0, +\infty]$.

(*i*) Assume by contradiction that $\lim_{r\to+\infty} r^c u(r) \in]0, +\infty]$. Then, for $0 < c < \dfrac{\alpha}{\beta} < N$, $\lim_{r\to+\infty} r^N u(r) = +\infty$ and by Remark 2.2, $E_N(r) > 0$ for large r. Hence, using the fact that u is bounded, $u'(r) < 0$ for large r and expression of E_N, we find

$$\lim_{r\to+\infty} \frac{|u'(r)|^{p-1}}{ru} = 0. \tag{22}$$

Then by (22), (14) and (15), we have

$$\varphi(r) \underset{+\infty}{\sim} \beta r^N u(r) < 0 \tag{23}$$

and

$$\varphi'(r) \underset{+\infty}{\sim} (N\beta - \alpha) r^{N-1} u(r). \tag{24}$$

Combining these two estimates, we get

$$r(r^{c-N}\varphi)' \underset{+\infty}{\sim} (c\beta - \alpha) r^c u(r).$$

Since $\lim_{r\to+\infty} r^c u(r) \in]0, +\infty]$ and $c\beta - \alpha > 0$, there exists some $C_1 > 0$ such that

$$r(r^{c-N}\varphi)'(r) > C_1 \quad \text{for large } r.$$

Whence $\lim_{r\to+\infty}\varphi(r) = +\infty$. This is a contradiction with (23). We deduce that $\lim_{r\to+\infty} r^c u(r) = 0$.
(*ii*) Assume by contradiction that $\lim_{r\to+\infty} r^c u(r) = K \in [0, +\infty[$. We distinguish two cases:
• $K = 0$. Then necessarily $E_c(r) < 0$ for large r, this means that

$$\frac{u(r)}{r|u'(r)|} < \frac{1}{c} \quad \text{for large}\, r. \tag{25}$$

Using equation (3) and the fact that $u > 0, u' < 0$ and $\frac{\alpha}{\beta} < c$, we obtain

$$\Big(|u'|^{p-2}u'\Big)'(r) < r|u'(r)|\Big[\beta + \frac{|\alpha|}{c} + \frac{N-1}{r^2}|u'(r)|^{p-2}\Big] < 0 \quad \text{ for large } r.$$

Thus $u'(r) > 0$ for large r, which is a contradiction with Lemma 2.2.
• $K > 0$. Since $c < N$, $\lim_{r\to+\infty} r^N u(r) = +\infty$ and $E_N(r) > 0$ for large r. Therefore, (22) is satisfied and thereby from (23) and (24), we get

$$\lim_{r\to+\infty}\varphi(r) = -\infty \text{ and } \lim_{r\to+\infty} r^{c+1-N}\varphi'(r) = K(N\beta - \alpha). \tag{26}$$

Then Hopital's rule implies

$$\lim_{r\to+\infty} r^{c-N}\varphi(r) = \frac{K(N\beta-\alpha)}{N-c}. \tag{27}$$

But from (23), this limit is exactly $K\beta$. This contradicts the fact that $c > \frac{\alpha}{\beta}$. Consequently, $\lim_{r\to+\infty} r^c u(r) = +\infty$ and the proof of the proposition is complete. □

Proposition 3.3 *Assume that u is a bounded solution of equation (3). Then the function $r^{\alpha/\beta}u(r)$ is not strictly monotone for large r.*

Proof. We argue by contradiction and assume that $r^{\alpha/\beta}u(r)$ is strictly monotone for large r. Therefore, according to (9), $E_{\alpha/\beta}(r) \neq 0$ for large r. We distinguish two cases.
Case 1: $E_{\alpha/\beta}(r) > 0$ for large r.
We set

$$J_1(r) = u(r) - r^{p-1}|u'|^{p-1}. \tag{28}$$

Then for large r,

$$J_1'(r) = r^{p-1}u\Big[-\alpha - r^l u^{q-1}\Big] - r^p|u'|\Big[-\beta + r^{-p} + (p-N)r^{-2}|u'|^{p-2}\Big]. \tag{29}$$

Using now Proposition 3.1 and $E_{\alpha/\beta}(r) > 0$ for large r, we get

$$\lim_{r\to+\infty} J_1(r) = 0,$$

$$J_1(r) > u(r)\left[1 - \left(\frac{\alpha}{\beta}\right)^{p-1} u^{p-2}(r)\right] > 0 \quad \text{ for large } r$$

and

$$J_1'(r) \underset{+\infty}{\sim} -\alpha r^{p-1}u(r) + \beta r^p |u'(r)| = -\beta r^{p-1}E_{\alpha/\beta}(r) > 0 \quad \text{for large } r.$$

This is a contradiction.

Case 2: $E_{\alpha/\beta}(r) < 0$ for large r.

Then necessarily $\frac{\alpha}{\beta} < \frac{N-p}{p-1}$, by Proposition 2.2.

Now, we set

$$J_2(r) = r^k u^p(r) - r^{p-1}|u'|^{p-1}, \tag{30}$$

with $0 < k < \min\left(\frac{\alpha}{\beta}, p\right)$. Then, for large r,

$$\begin{aligned} J_2'(r) \quad &= r^{p-1}u\Big[-\alpha - r^l u^{q-1} + kr^{k-p}u^{p-1}\Big] - r^p|u'|\Big[-\beta + \\ &+ pr^{k-p}u^{p-1} + (p-N)r^{-2}|u'|^{p-2}\Big]. \end{aligned} \tag{31}$$

As $k < p$, $k < \frac{\alpha}{\beta} < \frac{N-p}{p-1} < N$, $E_{\alpha/\beta}(r) < 0$ for large r, by Proposition 3.1 and Proposition 3.2, we obtain

$$\lim_{r\to+\infty} J_2(r) = 0,$$

$$J_2(r) < u^{p-1}(r)\left[-\left(\frac{\alpha}{\beta}\right)^{p-1} + r^k u(r)\right] < 0 \quad \text{for large } r$$

and

$$J_2'(r) \underset{+\infty}{\sim} -\alpha r^{p-1}u(r) + \beta r^p |u'(r)| = -\beta r^{p-1}E_{\alpha/\beta}(r) < 0 \quad \text{for large } r.$$

Again, we have a contradiction. Consequently, the function $r^{\alpha/\beta}u(r)$ is not strictly monotone for large r. The proof of the proposition is complete. □

Proposition 3.4 *Assume that u is a bounded solution of equation (3). If $\frac{\alpha}{\beta} = \frac{l+p}{q+1-p} < \frac{N-p}{p-1}$, then*

$$\lim_{r\to+\infty} r^{\alpha/\beta}u(r) = L \tag{32}$$

and

$$\lim_{r\to+\infty} r^{\alpha/\beta+1}u'(r) = \frac{-\alpha}{\beta}L, \tag{33}$$

where

$$L = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{1/(q+1-p)} \left(\frac{\alpha}{\beta}\right)^{(p-1)/(q+1-p)}. \tag{34}$$

Proof. Define the following function:

$$I(r) = r^{\alpha/\beta}u(r)\left[\frac{|u'|^{p-2}u'(r)}{ru(r)} + \beta\right]. \tag{35}$$

We have $I(r) < 0$ for large r. Its derivative is given by

$$I'(r) = \left(\frac{\alpha}{\beta} - N\right) r^{\alpha/\beta-2}|u'|^{p-2}u'(r) - r^{\alpha/\beta+l-1}u^q(r). \tag{36}$$

The proof will be done in five steps.

Step 1: $I(r) \underset{+\infty}{\sim} \beta r^{\alpha/\beta} u(r)$.

Since $\dfrac{\alpha}{\beta} < \dfrac{N-p}{p-1}$, Proposition 2.2 implies that $E_{(N-p)/(p-1)}(r) > 0$ for large r and thus, from the boundedness of u, $\displaystyle\lim_{r\to+\infty} \frac{|u'|^{p-2}u'(r)}{ru} = 0$ and one deduces that $I(r) \underset{+\infty}{\sim} \beta r^{\alpha/\beta} u(r)$.

Step 2: $\displaystyle\lim_{r\to+\infty} r^{\alpha/\beta} u(r)$ exists and is finite.

According to Step 1, it suffices to prove that $\displaystyle\lim_{r\to+\infty} I(r)$ exists and is finite. For this purpose, consider a real σ such that

$$0 < \sigma < \min\left(\frac{\alpha}{\beta}, \frac{1}{q}\left(\frac{\alpha}{\beta}(q-1) - l\right), \frac{1}{p-1}\left(\frac{\alpha}{\beta}(p-2) + p\right)\right).$$

So, by Proposition 3.2, $\displaystyle\lim_{r\to+\infty} r^{\alpha/\beta-\sigma} u(r) = 0$. In particular, there exists a constant $C > 0$ such that

$$u(r) \le C r^{\sigma-\alpha/\beta} \quad \text{for large } r. \tag{37}$$

Recall the positivity of $E_{(N-p)/(p-1)}(r)$ for large r, we get that there exists $C_1 > 0$ such that

$$r^{\alpha/\beta-2}|u'|^{p-1} < C_1^{p-1} r^{\gamma} \quad \text{for large } r, \tag{38}$$

where $\gamma = \dfrac{\alpha}{\beta}(2-p) + \sigma(p-1) - p - 1$.

By the choice of σ, the functions $r \to r^{\alpha/\beta+l-1}u^q(r)$ and $r \to r^{\alpha/\beta-2}|u'|^{p-1}$ are integrable near $+\infty$, therefore, $I'(r)$ is also integrable near $+\infty$. To conclude, we observe that for any $r_0 > 0$,

$$\lim_{r\to+\infty} I(r) = \int_{r_0}^{+\infty} I'(s)ds + I(r_0)$$

exists and is finite. Therefore, $\displaystyle\lim_{r\to+\infty} r^{\alpha/\beta} u(r)$ exists and is finite.

Set $\displaystyle\lim_{r\to+\infty} r^{\alpha/\beta} u(r) = L \ge 0$.

Step 3: $\displaystyle\lim_{r\to+\infty} r^{\alpha/\beta} u(r) = L > 0$.

Assume that $\displaystyle\lim_{r\to+\infty} r^{\alpha/\beta} u(r) = 0$. Then $\displaystyle\lim_{r\to+\infty} I(r) = 0$. Therefore, applying Hopital's rule and using the first step, we obtain

$$\lim_{r\to+\infty} \frac{I'(r)}{\left(r^{\alpha/\beta}u(r)\right)'} = \lim_{r\to+\infty} \frac{I(r)}{r^{\alpha/\beta}u} = \beta. \tag{39}$$

On the other hand, using (36), we have

$$I'(r) = r^{\alpha/\beta-2}|u'|^{p-1}\left[N - \frac{\alpha}{\beta} - \frac{r^{l+1}u^q(r)}{|u'|^{p-1}}\right]. \tag{40}$$

Let $0 < c < \dfrac{\alpha}{\beta} < N$, then $\displaystyle\lim_{r\to+\infty} r^c u(r) = 0$ and according to Remark 2.2, we have $E_c(r) \ne 0$ for large r. Therefore, necessarily by (9), $E_c(r) < 0$ for large r. Hence,

$$0 < \frac{r^{l+1}u^q(r)}{|u'|^{p-1}} < c^{1-p} r^{l+p} u^{q+1-p}(r). \tag{41}$$

Since $\dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p}$, then $\lim\limits_{r\to+\infty} r^{l+p}u^{q+1-p}(r) = 0$ and therefore $\lim\limits_{r\to+\infty} \dfrac{r^{l+1}u^q(r)}{|u'|^{p-1}} = 0$. Hence, as $\dfrac{\alpha}{\beta} < N$ and $|u'(r)| > 0$ for large r, we have by (40), $I'(r) > 0$ for large r. Consequently, by (39) and the fact that $\beta < 0$, $\left(r^{\alpha/\beta}u(r)\right)' < 0$ for large r. But this contradicts Proposition 3.3.

Consequently, $\lim\limits_{r\to+\infty} r^{\alpha/\beta}u(r) = L > 0$.

Step 4: $\lim\limits_{r\to+\infty} r^{\alpha/\beta+1}u'(r) = \dfrac{-\alpha}{\beta}L$.
Since $\lim\limits_{r\to+\infty} u(r) = 0$, then applying Hopital's rule, we obtain

$$\lim_{r\to+\infty} r^{\alpha/\beta+1}u'(r) = \frac{-\alpha}{\beta}\lim_{r\to+\infty} r^{\alpha/\beta}u(r) = \frac{-\alpha}{\beta}L.$$

Step 5: $L = \left(N-p-\dfrac{\alpha}{\beta}(p-1)\right)^{1/(q+1-p)}\left(\dfrac{\alpha}{\beta}\right)^{(p-1)/(q+1-p)}$.
According to (11), we have

$$\begin{aligned} -\beta r E_{\alpha/\beta}(r) &= |u'|^{p-2}u'(r)\Bigg[\left(N-p-\frac{\alpha}{\beta}(p-1)\right) \\ &\quad + (p-1)\frac{E'_{\alpha/\beta}(r)}{u'(r)} + \frac{r^{l+1}u^q(r)}{|u'|^{p-2}u'(r)}\Bigg]. \end{aligned} \tag{42}$$

Using Step 3 and Step 4 and applying Hopital's rule, we get

$$\lim_{r\to+\infty}\frac{E'_{\alpha/\beta}(r)}{u'(r)} = \lim_{r\to+\infty}\frac{E_{\alpha/\beta}(r)}{u(r)} = \lim_{r\to+\infty}\left(\frac{\alpha}{\beta} + \frac{ru'(r)}{u}\right) = 0 \tag{43}$$

and

$$\lim_{r\to+\infty}\frac{r^{l+1}u^q(r)}{|u'|^{p-2}u'(r)} = \frac{-L^{q+1-p}}{\left(\dfrac{\alpha}{\beta}\right)^{p-1}}, \tag{44}$$

when $\dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p}$. Suppose by contradiction

$$N-p-\frac{\alpha}{\beta}(p-1) - \frac{L^{q+1-p}}{\left(\dfrac{\alpha}{\beta}\right)^{p-1}} \neq 0.$$

After combining these estimates, equation (42) gives

$$-\beta r E_{\alpha/\beta}(r) \underset{+\infty}{\sim} \left[N-p-\frac{\alpha}{\beta}(p-1) - \frac{L^{q+1-p}}{\left(\dfrac{\alpha}{\beta}\right)^{p-1}}\right]|u'|^{p-2}u'(r).$$

So, $E_{\alpha/\beta}(r) \neq 0$ for large r, that is, $r^{\alpha/\beta}u(r)$ is strictly monotone for large r. Again, this contradicts Proposition 3.3.

Consequently,

$$L = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{1/(q+1-p)} \left(\frac{\alpha}{\beta}\right)^{(p-1)/(q+1-p)}.$$

The proof is complete. □

The following figure illustrates the behavior of the bounded solution.

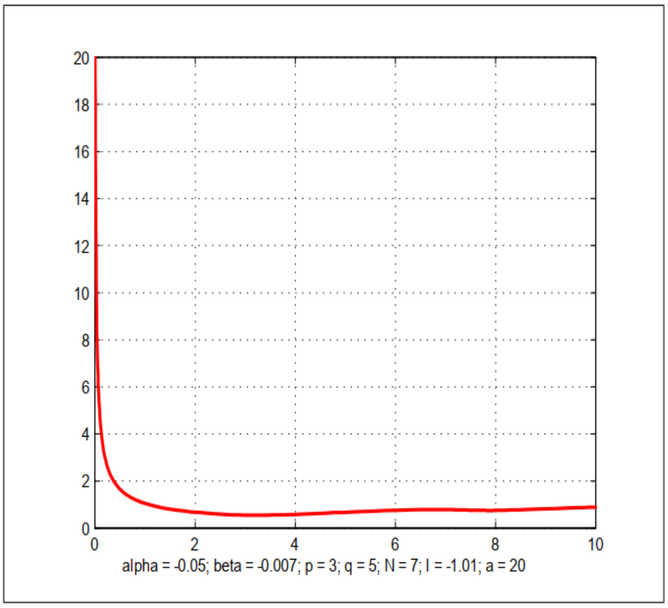

Figure 1: Bounded solution.

Now, we turn to the proof of Theorem 3.1.

Proof. (of Theorem 3.1). We argue by contradiction and assume that u is bounded. We claim that if one of the first two cases holds, then $E_{\alpha/\beta}(r) \neq 0$ for large r and this contradicts Proposition 3.3. In fact, if $\dfrac{\alpha}{\beta} \geq \dfrac{N-p}{p-1}$, we have $E_{\alpha/\beta}(r) > 0$ for large r, from Proposition 2.2.

Now we consider the case where $\dfrac{\alpha}{\beta} < \dfrac{N-p}{p-1}$.

Assume that there exists a large r_0 such that $E_{\alpha/\beta}(r_0) = 0$. Recall formula (12) with $c = \dfrac{\alpha}{\beta}$, we have in the case $q \leq p-1$, $E'_{\alpha/\beta}(r_0) \neq 0$. In the case $q > p-1$ and $\dfrac{\alpha}{\beta} \neq \dfrac{l+p}{q+1-p}$, we have by Proposition 3.2, $\lim\limits_{r\to+\infty} r^{l+p}u^{q+1-p}(r) = 0$ or $\lim\limits_{r\to+\infty} r^{l+p}u^{q+1-p}(r) = +\infty$ (because $\dfrac{\alpha}{\beta} < \dfrac{N-p}{p-1} < N$). Then we have also $E'_{\alpha/\beta}(r_0) \neq 0$. Consequently, Remark 2.1 gives $E_{\alpha/\beta}(r) \neq 0$ for any $r > r_0$.

Suppose now that we are in the third case $\dfrac{N-p}{p} \leq \dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p} < \dfrac{N-p}{p-1}$. Recall the logarithmic change with $c = \dfrac{\alpha}{\beta}$, then

$$\omega'_{\alpha/\beta}(t) + A_{\alpha/\beta}\,\omega_{\alpha/\beta}(t) + \alpha e^{K_{\alpha/\beta}t}\upsilon_{\alpha/\beta}(t) + \beta e^{K_{\alpha/\beta}t}h_{\alpha/\beta}(t) + \upsilon^q_{\alpha/\beta}(t) = 0 \tag{45}$$

and Proposition 3.4 gives

$$\lim_{t\to+\infty} \upsilon_{\alpha/\beta}(t) = L \quad \text{and} \quad \lim_{t\to+\infty} h_{\alpha/\beta}(t) = \frac{-\alpha}{\beta}L. \tag{46}$$

Define the following energy function:

$$Z(t) = \frac{p-1}{p}\left|h_{\alpha/\beta}(t)\right|^p + \frac{\alpha}{\beta}\omega_{\alpha/\beta}(t)\upsilon_{\alpha/\beta}(t) + \frac{\upsilon^{q+1}_{\alpha/\beta}(t)}{q+1} + \frac{\varrho}{p}\left(\frac{\alpha}{\beta}\right)^{p-1}\upsilon^p_{\alpha/\beta}(t), \tag{47}$$

where

$$\varrho = \frac{\alpha}{\beta} - A_{\alpha/\beta} = \frac{\alpha}{\beta}p - (N-p) \geq 0. \tag{48}$$

According to [7], we have $\lim\limits_{r\to 0} ru'(r) = 0$. Therefore

$$\lim_{r\to 0} r^{\alpha/\beta}u(r) = \lim_{r\to 0} r^{1+\alpha/\beta}u'(r) = 0.$$

It gives

$$\lim_{t\to-\infty} \upsilon_{\alpha/\beta}(t) = \lim_{t\to-\infty} h_{\alpha/\beta}(t) = 0.$$

This implies by (47) that $\lim\limits_{t\to-\infty} Z(t) = 0$.
On the other hand, by a straightforward calculation, the function Z satisfies

$$Z'(t) = \varrho Y(t) - \beta e^{K_{\alpha/\beta}t}\left(h_{\alpha/\beta}(t) + \frac{\alpha}{\beta}\upsilon_{\alpha/\beta}(t)\right)^2, \tag{49}$$

where

$$Y(t) = \left(\left|h_{\alpha/\beta}(t)\right|^{p-2}h_{\alpha/\beta}(t) + \left(\frac{\alpha}{\beta}\right)^{p-1}\upsilon^{p-1}_{\alpha/\beta}(t)\right)\left(h_{\alpha/\beta}(t) + \frac{\alpha}{\beta}\upsilon_{\alpha/\beta}(t)\right). \tag{50}$$

As $\varrho \geq 0$, $\beta < 0$ and the function $s \to |s|^{p-2}s$ is increasing, then $Z'(t) \geq 0$ for any $t \in (-\infty, +\infty)$, that is, Z is increasing on $(-\infty, +\infty)$. Therefore, $Z(t) \geq 0$ for any $t \in (-\infty, +\infty)$. But letting $t \to +\infty$ in (47), we get

$$\lim_{t\to+\infty} Z(t) = A_{\alpha/\beta}\,L^p\left(\frac{\alpha}{\beta}\right)^{p-1}\left(\frac{p-q-1}{p(q+1)}\right) < 0.$$

This is a contradiction. Consequently, u is unbounded. The proof of Theorem 3.1 is complete. □

In the following, we are concerned with the asymptotic behavior of unbounded solutions near infinity.

Theorem 3.2 *Assume* $q \geq p-1$ *and* $\dfrac{N-p}{p-1} > \dfrac{l}{q-1}$. *Let* u *be an unbounded solution of equation (3). Then*

$$\lim_{r\to+\infty} r^{-p/(p-2)}u(r) = \lim_{r\to+\infty} r^{-2/(p-2)}u'(r) = 0. \tag{51}$$

The proof requires some preliminary results.

Proposition 3.5 *Assume* $q \geq p-1$ *and* $\dfrac{N-p}{p-1} > \dfrac{l}{q-1}$. *Then there is no solution of equation (3) such that*

$$u(r) > \Gamma r^{-l/(q-1)} \qquad \textit{for large } r, \tag{52}$$

where Γ *is given by (13).*

Proof. We argue by contradiction and assume that u satisfies (52). Then, according to Lemma 2.1 and Proposition 2.1, we have $u'(r) > 0$ and $E_{l/(q-1)}(r) \neq 0$ for large r. This gives, with logarithmic change (4), that $\omega_{l/(q-1)}(t) > 0$ and $\upsilon'_{l/(q-1)}(t) \neq 0$ for large t. We distinguish two cases.

Case 1: $\upsilon'_{l/(q-1)}(t) < 0$ for large t. Then $\lim\limits_{t\to+\infty} \upsilon_{l/(q-1)}(t) = d \in [\Gamma, +\infty[$.

From equation (5), we have

$$\begin{aligned} \omega'_{l/(q-1)}(t) + A_{l/(q-1)}\,\omega_{l/(q-1)}(t) \quad &= -e^{K_{l/(q-1)}t}\upsilon_{l/(q-1)}(t)\Bigg[\alpha + \\ &\beta\frac{h_{l/(q-1)}(t)}{\upsilon_{l/(q-1)}(t)} + \upsilon^{q-1}_{l/(q-1)}(t)\Bigg]. \end{aligned} \tag{53}$$

Note that

$$\frac{h_{l/(q-1)}(t)}{\upsilon_{l/(q-1)}(t)} = \frac{\upsilon'_{l/(q-1)}(t)}{\upsilon_{l/(q-1)}(t)} - \frac{l}{q-1} < -\frac{l}{q-1}. \tag{54}$$

Then we deduce from (52) and (53) that

$$\omega'_{l/(q-1)}(t) + A_{l/(q-1)}\,\omega_{l/(q-1)}(t) < 0 \quad \text{for large } t.$$

This means that the function $e^{A_{l/(q-1)}t}\omega_{l/(q-1)}(t)$ is decreasing for large t. As $\omega_{l/(q-1)}(t) > 0$ for large t, then $e^{A_{l/(q-1)}t}\omega_{l/(q-1)}(t)$ has a finite limit. Since $A_{l/(q-1)} > 0$, necessarily $\lim\limits_{t\to+\infty} \omega_{l/(q-1)}(t) = 0$. Now, recalling (6), we obtain $\lim\limits_{t\to+\infty} h_{l/(q-1)}(t) = 0$ and $\lim\limits_{t\to+\infty} \upsilon'_{l/(q-1)}(t) = \dfrac{l}{q-1}d < 0$. But this contradicts the fact that $\upsilon_{l/(q-1)}$ is positive.

Case 2: $\upsilon'_{l/(q-1)}(t) > 0$ for large t. Then $\lim\limits_{t\to+\infty} \upsilon_{l/(q-1)}(t) \in]\Gamma, +\infty]$.

(a) Assume that $\lim\limits_{t\to+\infty} \upsilon_{l/(q-1)}(t) = d < +\infty$, then necessarily $\lim\limits_{t\to+\infty} \upsilon'_{l/(q-1)}(t) = 0$. This implies by (6) that $\lim\limits_{t\to+\infty} \omega_{l/(q-1)}(t) = \Big(-\dfrac{l}{q-1}d\Big)^{p-1}$. Therefore,

$$\lim_{t\to+\infty}\Big[\alpha + \beta\frac{h_{l/(q-1)}(t)}{\upsilon_{l/(q-1)}(t)} + \upsilon^{q-1}_{l/(q-1)}(t)\Big] = \alpha - \frac{\beta l}{q-1} + d^{q-1} > 0,$$

by letting $t \to +\infty$ in (53), we obtain $\lim_{t\to+\infty} \omega'_{l/(q-1)}(t) = -\infty$. This is a contradiction with $\omega_{l/(q-1)}$ being positive.
(b) Assume that $\lim_{t\to+\infty} v_{l/(q-1)}(t) = +\infty$, that is, $\lim_{r\to+\infty} r^{l/(q-1)}u(r) = +\infty$. Then, according to Proposition 2.1, $E_{-p/(p-2)}(r) < 0$ for large r. Hence, according to (8) and the fact that $u'(r) > 0$ for large r, we have

$$0 < \frac{ru'(r)}{u} < \frac{p}{p-2} \quad \text{for large } r.$$

On the other hand, we have by equation (3),

$$\Big(|u'|^{p-2}u'\Big)'(r) < -u\Big[\alpha + \beta\frac{ru'}{u} + r^l u^{q-1}\Big].$$

So, $\lim_{r\to+\infty} \left(|u'|^{p-2}u'\right)'(r) = -\infty$, which means that $\lim_{r\to+\infty} u'(r) = -\infty$. This is impossible.
In conclusion, the two cases can not hold, so there is no solution satisfying (52). The proof is complete. □

As a consequence of the previous proposition, we have the following result.

Corollary 3.1 *Assume $q \ge p-1$ and $\dfrac{N-p}{p-1} > \dfrac{l}{q-1}$. Let u be a solution of equation (3). Then $\liminf_{r\to+\infty} r^c u(r) = 0$ for any $c < \dfrac{l}{q-1}$ and $\liminf_{r\to+\infty} r^{l/(q-1)}u(r) \le \Gamma$.*

Before giving the proof of Theorem 3.2, we need a comparison between the solutions of equation (3) and their derivatives.

Proposition 3.6 *Assume $q \ge p-1$ and $\dfrac{N-p}{p-1} > \dfrac{l}{q-1}$. Let u be an unbounded solution of equation (3). Then*

$$|u'|^{p-2}u'(r) < \max\left(\frac{|\alpha|}{N}, |\beta|\right) ru(r) \quad \textit{for large } r. \tag{55}$$

Proof. Let $\lambda = \min\left(\dfrac{\alpha}{N}, \beta\right) < 0$ and set

$$G(r) = r^{N-1}\Big[|u'|^{p-2}u'(r) + \lambda r\, u(r)\Big]. \tag{56}$$

From equation (3), we have

$$G'(r) = r^{N-1}u(r)\Big[\lambda N - \alpha - r^l|u|^{q-1}\Big] + (\lambda - \beta)r^N u'(r). \tag{57}$$

We will show that $G(r) < 0$ for large r.

Suppose that there exists a large r_0 such that $G(r_0) = 0$, then $G'(r_0)$ can be written in the following form:

$$G'(r_0) = r_0^{N-1}u(r_0)\Big[\lambda N - \alpha - r_0^l u^{q-1}(r_0)\Big] + |\lambda|^{1/(p-1)}(\lambda - \beta)r_0^{N+1/(p-1)}u^{1/(p-1)}(r_0). \tag{58}$$

Therefore, according to the choice of λ, we have $G'(r_0) < 0$. Hence, $G(r) \neq 0$ for any $r > r_0$.
If $G(r) > 0$ for large r, then $u'(r) > 0$ for large r and by a simple integration, we deduce that there exists a constant $c > 0$ such that $r^{-p/(p-2)}u(r) \geq c$ for large r. But this is a contradiction with Proposition 3.5.

In conclusion, $G(r) < 0$ for large r and the proof is complete. □

Proposition 3.7 *Assume* $q \geq p-1$ *and* $\dfrac{N-p}{p-1} > \dfrac{l}{q-1}$. *Let* u *be an unbounded solution of equation (3). Then the functions* $r^{-p/(p-2)}u(r)$ *and* $r^{-2/(p-2)}u'(r)$ *are bounded for large* r.

Proof. The proof will be done in two steps.
Step 1: The function $r^{-p/(p-2)}u(r)$ is bounded for large r.
We argue by contradiction and assume that $\upsilon_{-p/(p-2)}(t) = r^{-p/(p-2)}u(r)$ is unbounded for large t, where we use the notation (4). Since $\dfrac{l}{q-1} > \dfrac{-p}{p-2}$, Proposition 3.5 ensures that $\upsilon_{-p/(p-2)}(t)$ can not converge to $+\infty$. Hence, it must necessarily oscillate. Then there exists a sequence $\{\xi_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\upsilon_{-p/(p-2)}$ has a local maximum in ξ_i satisfying $\lim_{i\to+\infty} \upsilon_{-p/(p-2)}(\xi_i) = +\infty$. Since $\upsilon'_{-p/(p-2)}(\xi_i) = 0$, we have

$$h_{-p/(p-2)}(\xi_i) = \frac{p}{p-2}\upsilon_{-p/(p-2)}(\xi_i) > 0,$$

therefore

$$\omega_{-p/(p-2)}(\xi_i) = \left(\frac{p}{p-2}\right)^{p-1} \upsilon^{p-1}_{-p/(p-2)}(\xi_i).$$

On the other hand, estimate (55) can be written in the following form:

$$\omega_{-p/(p-2)}(t) < |\lambda|\, \upsilon_{-p/(p-2)}(t) \quad \text{for large } t, \tag{59}$$

where $|\lambda| = \max\left(\dfrac{|\alpha|}{N}, |\beta|\right)$. In particular, for $t = \xi_i$, we obtain

$$\upsilon^{p-2}_{-p/(p-2)}(\xi_i) < |\lambda| \left(\frac{p-2}{p}\right)^{p-1} \quad \text{for large } i.$$

But this contradicts the fact that $\lim_{i\to+\infty} \upsilon_{-p/(p-2)}(\xi_i) = +\infty$. Consequently, $\upsilon_{-p/(p-2)}(t)$ is bounded for large t.
Step 2: The function $r^{-2/(p-2)}u'(r)$ is bounded for large r, that means that $\omega_{-p/(p-2)}(t)$ is bounded for large t.

Observe preliminary that if $u(r)$ is monotone for large r, necessarily $u'(r) \geq 0$ for large r (because u is unbounded) and then $\omega_{-p/(p-2)}(t) \geq 0$ for large t. Therefore, by (59) and Step 1, $\omega_{-p/(p-2)}(t)$ is bounded for large t.

So, we only have to deal with the case where $u(r)$ is not monotone for large r and then the idea of the proof is the same as for Step 1.

Suppose by contradiction that $\omega_{-p/(p-2)}(t)$ is not bounded and let a sequence $\{k_i\}$ go to $+\infty$ as $i \to +\infty$ such that $\omega'_{-p/(p-2)}(k_i) = 0$ and $\lim_{i\to+\infty} \omega_{-p/(p-2)}(k_i) = +\infty$ or $-\infty$.

First, note that as $\upsilon_{-p/(p-2)}(t)$ is bounded for large t, then (59) implies that we can not have $\lim_{i\to+\infty}\omega_{-p/(p-2)}(k_i)=+\infty$. Secondly, take some constant $\delta>\max\left(\frac{N-p}{p-1},\frac{\alpha}{\beta}\right)>0$, then by Proposition 2.2, $E_\delta(r)>0$ for large r. So,

$$h_{-p/(p-2)}(t)>-\delta\upsilon_{-p/(p-2)}(t)\quad\text{for large } t. \tag{60}$$

In particular,

$$h_{-p/(p-2)}(k_i)>-\delta\upsilon_{-p/(p-2)}(k_i)\quad\text{for large } i. \tag{61}$$

Again, as $\upsilon_{-p/(p-2)}(t)$ is bounded for large t, we deduce that $\omega_{-p/(p-2)}(k_i)$ is bounded for large i. This is a contradiction. It follows that $\omega_{-p/(p-2)}(t)$ is bounded for large t. Hence $r^{-2/(p-2)}u'(r)$ is bounded for large r. □

Now, we are ready to give the proof of Theorem 3.2.

Proof. (of Theorem 3.2). Using the logarithmic change (4) and Proposition 3.7, we deduce that the functions $\upsilon_{-p/(p-2)}(t)$ and $\omega_{-p/(p-2)}(t)$ are bounded for large t.
We proceed in two steps.
Step 1: $\lim_{t\to+\infty}\omega_{-p/(p-2)}(t)=0$.
First, we claim that $\omega_{-p/(p-2)}(t)$ converges when $t\to+\infty$. Assume by contradiction that $\omega_{-p/(p-2)}(t)$ oscillates, that is, there exist two sequences $\{s_i\}$ and $\{k_i\}$ going to $+\infty$ as $i\to+\infty$ such that $\omega_{-p/(p-2)}(t)$ has a local minimum in s_i and a local maximum in k_i satisfying $s_i<k_i<s_{i+1}$ and

$$\liminf_{t\to+\infty}\omega_{-p/(p-2)}(t)=\lim_{i\to+\infty}\omega_{-p/(p-2)}(s_i)<\limsup_{t\to+\infty}\omega_{-p/(p-2)}(t)=\lim_{i\to+\infty}\omega_{-p/(p-2)}(k_i). \tag{62}$$

Applying equation (17) at the point $t=k_i$, we get $e^{M_{-p/(p-2)}k_i}\upsilon^q_{-p/(p-2)}(k_i)$ is bounded. Therefore, since $M_{-p/(p-2)}>0$, $\lim_{i\to+\infty}\upsilon_{-p/(p-2)}(k_i)=0$ and thanks to (59), we deduce

$$\lim_{i\to+\infty}\omega_{-p/(p-2)}(k_i)=\limsup_{t\to+\infty}\omega_{-p/(p-2)}(t)\le 0.$$

As u is unbounded, then u cannot be decreasing and by (6) and (10) necessarily $\limsup_{t\to+\infty}\omega_{-p/(p-2)}(t)=0$, so, by (62)

$$\lim_{i\to+\infty}\omega_{-p/(p-2)}(s_i)=\liminf_{t\to+\infty}\omega_{-p/(p-2)}(t)<0.$$

This implies that $\lim_{i\to+\infty}h_{-p/(p-2)}(s_i)<0$ and therefore, by (60), we find $\lim_{i\to+\infty}\upsilon_{-p/(p-2)}(s_i)>0$. On the other hand, $\omega'_{-p/(p-2)}(s_i)=0$, then equation (17) implies that $\lim_{i\to+\infty}\upsilon_{-p/(p-2)}(s_i)=0$. This is a contradiction. So, $\omega_{-p/(p-2)}(t)$ is monotone and then it has a finite limit when $t\to+\infty$.

As u is positive and unbounded, then necessarily $\lim_{t\to+\infty}\omega_{-p/(p-2)}(t)\ge 0$. If $\lim_{t\to+\infty}\omega_{-p/(p-2)}(t)>0$, then, by (59), there exists a constant $C>0$ such that $\upsilon_{-p/(p-2)}(t)>C$ for large t. Therefore, we obtain by equation (17) that $\lim_{t\to+\infty}\omega'_{-p/(p-2)}(t)=-\infty$, which is a contradiction with the boundedness of $\omega_{-p/(p-2)}$. Consequently, $\lim_{t\to+\infty}\omega_{-p/(p-2)}(t)=0$. Therefore, $\lim_{r\to+\infty}r^{-2/(p-2)}u'(r)=0$.

Step 2: $\lim\limits_{t\to+\infty} \upsilon_{-p/(p-2)}(t) = 0$.
Knowing that $\upsilon_{-p/(p-2)}(t)$ is bounded, assume by contradiction that it oscillates, that is, there exist two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\upsilon_{-p/(p-2)}$ has a local minimum in η_i and a local maximum in ξ_i satisfying $\eta_i < \xi_i < \eta_{i+1}$ and

$$\liminf_{t\to+\infty} \upsilon_{-p/(p-2)}(t) = \lim_{i\to+\infty} \upsilon_{-p/(p-2)}(\eta_i) < \limsup_{t\to+\infty} \upsilon_{-p/(p-2)}(t) = \lim_{i\to+\infty} \upsilon_{-p/(p-2)}(\xi_i). \tag{63}$$

Since $\upsilon'_{-p/(p-2)}(\eta_i) = \upsilon'_{-p/(p-2)}(\xi_i) = 0$, then we have by (6)

$$h_{-p/(p-2)}(\eta_i) = \frac{p}{p-2}\upsilon_{-p/(p-2)}(\eta_i) \text{ and } h_{-p/(p-2)}(\xi_i) = \frac{p}{p-2}\upsilon_{-p/(p-2)}(\xi_i).$$

Since $\lim\limits_{t\to+\infty} h_{-p/(p-2)}(t) = 0$,

$$\lim_{i\to+\infty} h_{-p/(p-2)}(\eta_i) = \lim_{i\to+\infty} h_{-p/(p-2)}(\xi_i) = 0.$$

This implies that

$$\lim_{i\to+\infty} \upsilon_{-p/(p-2)}(\eta_i) = \lim_{i\to+\infty} \upsilon_{-p/(p-2)}(\xi_i) = 0.$$

But this contradicts (63). Therefore, $\upsilon_{-p/(p-2)}$ converges. Hence, by (6), $\upsilon'_{-p/(p-2)}$ converges necessarily to 0. Consequently, $\lim\limits_{t\to+\infty} \upsilon_{-p/(p-2)}(t) = 0$, i.e., $\lim\limits_{r\to+\infty} r^{-p/(p-2)}u(r) = 0$. The proof is complete. □

Theorem 3.3 *Assume* $q \geq p\left(2 + 2^{p-1}\right) - 1$ *and* $\frac{l}{q-1} < \min\left(\frac{-\alpha}{\beta}, \frac{N-p}{p-1}\right)$. *Let* u *be an unbounded positive solution of problem* $(\mathbf{P})$. *Then*

$$\lim_{r\to+\infty} r^{l/(q-1)}u(r) = \Gamma \tag{64}$$

and

$$\lim_{r\to+\infty} r^{l/(q-1)+1}u'(r) = \frac{-l}{q-1}\Gamma, \tag{65}$$

where Γ *is given by (13).*

To prove this theorem we will need the following results.

Proposition 3.8 *Assume* $q \geq p - 1$ *and* $\frac{N-p}{p-1} > \frac{l}{q-1}$. *Let* u *be an unbounded positive solution of problem* $(\mathbf{P})$. *Then, for any* $c > \left(\frac{\alpha}{\beta}\right)^{p-1}$,

$$|u'|^{p-2}u'(r) > -cr^{1-p}u^{p-1}(r) \quad \textit{for large } r. \tag{66}$$

Proof. Let $c > \left(\frac{\alpha}{\beta}\right)^{p-1}$ and

$$F(r) = r^{N-1}|u'|^{p-2}u'(r) + cr^{N-p}u^{p-1}(r). \tag{67}$$

Then, according to equation (3),

$$F'(r)=r^{N-1}u\Big[-\alpha-r^lu^{q-1}+c(N-p)r^{-p}u^{p-2}\Big]+r^Nu'\Big[-\beta+c(p-1)r^{-p}u^{p-2}\Big]. \quad (68)$$

We will show that $F(r)>0$ for large r.

First, we prove that $F(r)\neq 0$ for large r. Suppose that there exists a large r_0 such that $F(r_0)=0$. Then, according to (67) and (68), we have

$$F'(r_0)=r_0^{N-1}u\Big[-\alpha+\beta c^{1/(p-1)}-r_0^lu^{q-1}+\Big(c(N-p)-c^{p/(p-1)}(p-1)\Big)r_0^{-p}u^{p-2}\Big]. \quad (69)$$

Since $-\alpha+\beta c^{1/(p-1)}<0$, $r^lu^{q-1}(r)>0$ and $\lim\limits_{r\to+\infty} r^{-p}u^{p-2}(r)=0$ (by Theorem 3.2), we get $F'(r_0)<0$. Hence, $F(r)\neq 0$ for any $r>r_0$ and then, as u is unbounded, $F(r)$ cannot be negative for large r. The proof is complete. □

Proposition 3.9 *Assume* $q\geq p\Big(2+2^{p-1}\Big)-1$ *and* $\dfrac{l}{q-1}<\min\left(\dfrac{-\alpha}{\beta},\dfrac{N-p}{p-1}\right)$. *Let u be an unbounded positive solution of problem* (**P**). *Then the functions* $r^{l/(q-1)}u(r)$ *and* $r^{l/(q-1)+1}u'(r)$ *are bounded for large* r.

Proof. (i) We first show that $r^{l/(q-1)}u(r)$ is bounded for large r.
We make the change (4) for $c=\dfrac{l}{q-1}$ and we set

$$D=\frac{K_{l/(q-1)}(p-1)}{p}-\frac{l}{q-1}+A_{l/(q-1)}>0, \quad (70)$$

We define, for any real $\theta>0$, the following energy function:

$$\begin{aligned}F_\theta(t)\ &=\frac{p-1}{p}e^{-K_{l/(q-1)}t}\left|h_{l/(q-1)}(t)\right|^p-\frac{\Gamma^{q-1}}{2}\upsilon^2_{l/(q-1)}(t)+\frac{\upsilon^{q+1}_{l/(q-1)}(t)}{q+1}+\\ &\frac{l}{q-1}e^{-K_{l/(q-1)}t}\omega_{l/(q-1)}(t)\upsilon_{l/(q-1)}(t)-\frac{\theta}{p}\left(\frac{-l}{q-1}\right)^{p-1}e^{-K_{l/(q-1)}t}\upsilon^p_{l/(q-1)}(t).\end{aligned} \quad (71)$$

Using equation (5), a straightforward calculation gives

$$\begin{aligned}F'_\theta(t)\ &=\ \theta e^{-K_{l/(q-1)}t}X(t)-\beta\left(h_{l/(q-1)}(t)+\frac{l}{q-1}\upsilon_{l/(q-1)}(t)\right)^2+\\ &D_1e^{-K_{l/(q-1)}t}\left|h_{l/(q-1)}(t)\right|^p+D_2e^{-K_{l/(q-1)}t}\omega_{l/(q-1)}(t)\upsilon_{l/(q-1)}(t)+\\ &\frac{K_{l/(q-1)}\theta}{p}\left(\frac{-l}{q-1}\right)^{p-1}e^{-K_{l/(q-1)}t}\upsilon^p_{l/(q-1)}(t),\end{aligned} \quad (72)$$

where

$$\begin{aligned}X(t)\ &=\ \left[\left|h_{l/(q-1)}(t)\right|^{p-2}h_{l/(q-1)}(t)-\left(\frac{-l}{q-1}\right)^{p-1}\upsilon^{p-1}_{l/(q-1)}(t)\right]\Big[h_{l/(q-1)}(t)+\\ &\frac{l}{q-1}\upsilon_{l/(q-1)}(t)\Big],\end{aligned} \quad (73)$$

$$D_1 = -(D+\theta) < 0 \tag{74}$$

and

$$D_2 = \frac{-l}{q-1}\left(D+\theta+\frac{K_{l/(q-1)}}{p}\right) > 0. \tag{75}$$

We show in two steps that $\upsilon_{l/(q-1)}(t)$ is bounded for large t.

Step 1: $F'_\theta(t) > 0$ for large t by choosing a suitable θ.

We know that for any $\rho \geq 2$ and for any $a, b \in \mathbb{R}$, we have

$$|a-b|^\rho \leq 2^{\rho-1}\left(|a|^\rho + |b|^\rho\right).$$

Therefore, particulary for $\rho = p$, $a = \upsilon'_{l/(q-1)}(t)$ and $b = \dfrac{l}{q-1}\upsilon_{l/(q-1)}(t)$, we obtain

$$\begin{aligned}\left|h_{l/(q-1)}(t)\right|^p &= \left|\upsilon'_{l/(q-1)}(t) - \frac{l}{q-1}\upsilon_{l/(q-1)}(t)\right|^p \\ &\leq 2^{p-1}\left(\left|\upsilon'_{l/(q-1)}(t)\right|^p + \left(\frac{|l|}{q-1}\right)^p \upsilon^p_{l/(q-1)}(t)\right).\end{aligned}$$

Since $D_1 < 0$,

$$\begin{aligned}F'_\theta(t) \quad \geq \quad & \theta e^{-K_{l/(q-1)}t}X(t) + \upsilon'^2_{l/(q-1)}(t)\left[-\beta + 2^{p-1}D_1 e^{-K_{l/(q-1)}t}\left|\upsilon'_{l/(q-1)}\right|^{p-2}\right] + \\ & e^{-K_{l/(q-1)}t}\upsilon^p_{l/(q-1)}(t)\Bigg[2^{p-1}D_1\left(\frac{|l|}{q-1}\right)^p + \frac{K_{l/(q-1)}\theta}{p}\left(\frac{|l|}{q-1}\right)^{p-1} + \\ & D_2\,\omega_{l/(q-1)}\upsilon^{1-p}_{l/(q-1)}\Bigg].\end{aligned} \tag{76}$$

As $\dfrac{|l|}{q-1} > \dfrac{\alpha}{\beta}$, we have by Proposition 3.8,

$$\omega_{l/(q-1)}(t)\upsilon^{1-p}_{l/(q-1)}(t) > -\left(\frac{|l|}{q-1}\right)^{p-1} \qquad \text{for large } t. \tag{77}$$

Therefore, according to (76) we get

$$\begin{aligned}F'_\theta(t) \quad \geq \quad & \theta e^{-K_{l/(q-1)}t}X(t) + \upsilon'^2_{l/(q-1)}(t)\left[-\beta + 2^{p-1}D_1 e^{-K_{l/(q-1)}t}\left|\upsilon'_{l/(q-1)}\right|^{p-2}\right] + \\ & e^{-K_{l/(q-1)}t}\upsilon^p_{l/(q-1)}\Bigg[2^{p-1}D_1\left(\frac{|l|}{q-1}\right)^p + \frac{K_{l/(q-1)}\theta}{p}\left(\frac{|l|}{q-1}\right)^{p-1} - \\ & D_2\left(\frac{|l|}{q-1}\right)^{p-1}\Bigg].\end{aligned} \tag{78}$$

Now, we choose in (71)

$$\theta = \frac{2^p|l|pD + 2|l|pD + 2|l|K_{l/(q-1)}}{K_{l/(q-1)}(q-1) + lp(2^{p-1}+1)} > 0. \tag{79}$$

Note that, since $q \geq p\Big(2+2^{p-1}\Big)-1$, we have $K_{l/(q-1)}(q-1)+lp(2^{p-1}+1)>0$, and the expression (79) gives

$$2^{p-1}D_1\left(\frac{|l|}{q-1}\right)^p+\frac{K_{l/(q-1)}\theta}{p}\left(\frac{|l|}{q-1}\right)^{p-1}-D_2\left(\frac{|l|}{q-1}\right)^{p-1}>0. \qquad (80)$$

Moreover,

$$e^{-K_{l/(q-1)}t/(p-2)}\upsilon'_{l/(q-1)}(t)=e^{-K_{l/(q-1)}t/(p-2)}\left(h_{l/(q-1)}(t)+\frac{l}{q-1}\upsilon_{l/(q-1)}(t)\right), \qquad (81)$$

and from Theorem 3.2,

$$\lim_{t\to+\infty}e^{-K_{l/(q-1)}t/(p-2)}\upsilon_{l/(q-1)}(t)=\lim_{r\to+\infty}r^{-p/(p-2)}u(r)=0 \qquad (82)$$

and

$$\lim_{t\to+\infty}e^{-K_{l/(q-1)}t/(p-2)}h_{l/(q-1)}(t)=\lim_{r\to+\infty}r^{-2/(p-2)}u'(r)=0, \qquad (83)$$

then

$$\lim_{t\to+\infty}e^{-K_{l/(q-1)}t/(p-2)}\upsilon'_{l/(q-1)}(t)=0. \qquad (84)$$

Using the last equality, the estimate (80), the fact that $X(t)\geq 0$ (because the function $s\to|s|^{p-2}s$ is increasing), $\upsilon_{l/(q-1)}(t)>0$, $\theta>0$ and $\beta<0$, we deduce from (78) that $F'_\theta(t)>0$ for large t.

Step 2: The function $\upsilon_{l/(q-1)}(t)$ cannot oscillate about a constant B such that $B>(q+1)^{1/(q-1)}\Gamma>\Gamma$, where Γ is given by (13).

We argue by contradiction and assume that there exist two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to $+\infty$ as $i\to+\infty$ such that $\upsilon_{l/(q-1)}$ has a local minimum in η_i and a local maximum in ξ_i satisfying $\eta_i<\xi_i<\eta_{i+1}$ and $\upsilon_{l/(q-1)}(\xi_i)>B$.

Since $h_{l/(q-1)}(\eta_i)=-\dfrac{l}{q-1}\upsilon_{l/(q-1)}(\eta_i)>0$, we see that $h'_{l/(q-1)}(\eta_i)$ exists and, more exactly, $h'_{l/(q-1)}(\eta_i)=\upsilon''_{l/(q-1)}(\eta_i)\geq 0$ (because $\upsilon'_{l/(q-1)}(\eta_i)=0$), which implies that $\omega'_{l/(q-1)}(\eta_i)\geq 0$. Taking $c=\dfrac{l}{q-1}$ in (4), we obtain

$$\begin{aligned}e^{-K_{l/(q-1)}t}\omega'_{l/(q-1)}(t) &= -A_{l/(q-1)}e^{-K_{l/(q-1)}t}\omega_{l/(q-1)}(t)-\alpha\upsilon_{l/(q-1)}(t)\\ &\quad -\beta h_{l/(q-1)}(t)-\upsilon^q_{l/(q-1)}(t).\end{aligned} \qquad (85)$$

So, for $t=\eta_i$

$$e^{-K_{l/(q-1)}\eta_i}\omega'_{l/(q-1)}(\eta_i)<-\psi(\upsilon_{l/(q-1)}(\eta_i)),$$

where

$$\psi(s)=\left[s^{q-1}-\Gamma^{q-1}\right]s,\quad s\geq 0. \qquad (86)$$

Since $\omega'_{l/(q-1)}(\eta_i)\geq 0$, one has $\psi(\upsilon_{l/(q-1)}(\eta_i))<0$ and therefore, necessarily $\upsilon_{l/(q-1)}(\eta_i)<\Gamma$. On the other hand, according to (71), we have

$$F_\theta(\eta_i)=\frac{\upsilon^{q+1}_{l/(q-1)}(\eta_i)}{q+1}+\upsilon^2_{l/(q-1)}(\eta_i)\left[-\frac{\Gamma^{q-1}}{2}+C_2e^{-K_{l/(q-1)}\eta_i}\upsilon^{p-2}_{l/(q-1)}(\eta_i)\right], \qquad (87)$$

where $C_2 = \dfrac{1}{p}\left(\dfrac{-l}{q-1}\right)^{p-1}\left(\dfrac{l}{q-1}-\theta\right) < 0$. In particular, we obtain $F_\theta(\eta_i) < \phi_1(\upsilon_{l/(q-1)}(\eta_i))$, where

$$\phi_1(s) = \frac{s^{q+1}}{q+1} - \frac{\Gamma^{q-1}}{2}s^2, \quad s \geq 0. \tag{88}$$

Therefore, since $0 < \upsilon_{l/(q-1)}(\eta_i) < \Gamma$, a simple study of the function ϕ_1 gives $\phi_1(\upsilon_{l/(q-1)}(\eta_i)) < 0$. Consequently, $F_\theta(\eta_i) < 0$ for large i.

In the same way, since $\upsilon'_{l/(q-1)}(\xi_i) = 0$,

$$F_\theta(\xi_i) = \frac{\upsilon^{q+1}_{l/(q-1)}(\xi_i)}{q+1} + \upsilon^2_{l/(q-1)}(\xi_i)\left[-\frac{\Gamma^{q-1}}{2} + C_2 e^{-K_{l/(q-1)}\xi_i}\upsilon^{p-2}_{l/(q-1)}(\xi_i)\right]. \tag{89}$$

Since $\lim\limits_{i\to+\infty} e^{-K_{l/(q-1)}\xi_i}\upsilon^{p-2}_{l/(q-1)}(\xi_i) = 0$ by (82),

$$F_\theta(\xi_i) > \frac{\upsilon^{q+1}_{l/(q-1)}(\xi_i)}{q+1} - \Gamma^{q-1}\upsilon^2_{l/(q-1)}(\xi_i) \quad \text{for large } i. \tag{90}$$

Set

$$\phi_2(s) = \frac{s^{q+1}}{q+1} - \Gamma^{q-1}s^2, \quad s \geq 0. \tag{91}$$

Therefore, by (90), $F_\theta(\xi_i) > \phi_2(\upsilon_{l/(q-1)}(\xi_i))$ for large i. Since $\upsilon_{l/(q-1)}(\xi_i) > B > (q+1)^{1/(q-1)}\Gamma$, we have $\phi_2(\upsilon_{l/(q-1)}(\xi_i)) > 0$, which implies that $F_\theta(\xi_i) > 0$ for large i. Hence, $F_\theta(\eta_i) < 0$ and $F_\theta(\xi_i) > 0$, for large i, which clearly contradicts the monotonicity of $F_\theta(t)$ for large t. It follows that $\upsilon_{l/(q-1)}(t)$ cannot oscillate about the constant B. Moreover, since $\upsilon_{l/(q-1)}(t)$ cannot stay above B (from Proposition 3.5), one has $\upsilon_{l/(q-1)}(t) \leq B$ for large t. Consequently, $\upsilon_{l/(q-1)}(t)$ is bounded for large t. That is, $r^{l/(q-1)}u(r)$ is bounded for large r.

(ii) Now, we show that $r^{l/(q-1)+1}u'(r)$ is bounded for large r, i.e., by (6) and (10), $\omega_{l/(q-1)}(t)$ is bounded for large t.

We argue by contradiction. As u is positive and unbounded, then we have two possibilities.

- $\lim\limits_{t\to+\infty}\omega_{l/(q-1)}(t) = +\infty$, then $\lim\limits_{t\to+\infty} h_{l/(q-1)}(t) = +\infty$. Using (6) and the fact that $\upsilon_{l/(q-1)}(t)$ is bounded for large t, we obtain $\lim\limits_{t\to+\infty}\upsilon'_{l/(q-1)}(t) = +\infty$ and therefore, $\lim\limits_{t\to+\infty}\upsilon_{l/(q-1)}(t) = +\infty$, which is impossible.
- There exists a sequence $\{k_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\omega_{l/(q-1)}$ has a local extremum in k_i satisfying $\lim\limits_{i\to+\infty}\omega_{l/(q-1)}(k_i) = +\infty$ or $-\infty$.

We use expression (6), equation (85) and the fact that $\omega'_{l/(q-1)}(k_i) = 0$, then

$$\alpha\upsilon_{l/(q-1)}(k_i) + \upsilon^q_{l/(q-1)}(k_i) = h_{l/(q-1)}(k_i)\left[-\beta - A_{l/(q-1)}e^{-K_{l/(q-1)}k_i}|h_{l/(q-1)}(k_i)|^{p-2}\right]. \tag{92}$$

Since $\lim\limits_{t\to+\infty} e^{-K_{l/(q-1)}t}|h_{l/(q-1)}(t)|^{p-2} = 0$ by (83), $\lim\limits_{i\to+\infty} h_{l/(q-1)}(k_i) = +\infty$ or $-\infty$ and $\beta < 0$, we have

$$\lim_{i\to+\infty}\alpha\upsilon_{l/(q-1)}(k_i) + \upsilon^q_{l/(q-1)}(k_i) = +\infty$$

or

$$\lim_{i\to+\infty} \alpha\upsilon_{l/(q-1)}(k_i) + \upsilon^q_{l/(q-1)}(k_i) = -\infty.$$

But this contradicts the fact that $\upsilon_{l/(q-1)}(t)$ is bounded for large t.
Hence $\omega_{l/(q-1)}(t)$ is bounded for large t and therefore $r^{l/(q-1)+1}u'(r)$ is bounded for large r. □

We can now give the proof of Theorem 3.3.

Proof. (of Theorem 3.3). We use the logarithmic change, we have by Proposition 3.9, $\upsilon_{l/(q-1)}(t)$ and $h_{l/(q-1)}(t)$ are bounded for large t. The proof will be done in three steps.

Step 1: The function $\upsilon_{l/(q-1)}(t)$ converges.
We argue by contradiction and assume that $\upsilon_{l/(q-1)}$ oscillates, that is, there exist two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to $+\infty$ as $i\to+\infty$ such that $\upsilon_{l/(q-1)}$ has a local minimum in η_i and a local maximum in ξ_i satisfying $\eta_i < \xi_i < \eta_{i+1}$ and

$$\liminf_{t\to+\infty} \upsilon_{l/(q-1)}(t) = \lim_{i\to+\infty} \upsilon_{l/(q-1)}(\eta_i) = m_1 < \limsup_{t\to+\infty} \upsilon_{l/(q-1)}(t) = \lim_{i\to+\infty} \upsilon_{l/(q-1)}(\xi_i) = M_1. \tag{93}$$

On the other hand, we know by the proof of Proposition 3.9 that $F'_\theta(t) > 0$ for large t where θ and F_θ are given, respectively, by (79) and (71). Then $F_\theta(t) \neq 0$ for large t. We show that $F_\theta(t) < 0$ for large t.
If $F_\theta(t) > 0$ for large t, then, since $F'_\theta(t) > 0$ for large t, $\lim_{t\to+\infty} F_\theta(t) \in]0,+\infty]$.
Using the fact that

$$\lim_{i\to+\infty} F_\theta(\eta_i) = \phi_1(m_1) < +\infty \tag{94}$$

and

$$\lim_{i\to+\infty} F_\theta(\xi_i) = \phi_1(M_1) < +\infty, \tag{95}$$

where ϕ_1 is given by (88), we see that $\lim_{t\to+\infty} F_\theta(t)$ is finite and strictly positive. More exactly, we have

$$\lim_{t\to+\infty} F_\theta(t) = \phi_1(m_1) = \phi_1(M_1) > 0.$$

But this contradicts the fact that $\phi_1(m_1) \leq 0$ because by Corollary 3.1, we have $0 \leq \liminf_{t\to+\infty} \upsilon_{l/(q-1)}(t) = m_1 \leq \Gamma$. We deduce that $F_\theta(t) < 0$ for large t.

Since $F'_\theta(t) > 0$ for large t, one has $\lim_{t\to+\infty} F_\theta(t)$ is finite and negative. Therefore, according to (94) and (95), we have

$$\lim_{t\to+\infty} F_\theta(t) = \phi_1(m_1) = \phi_1(M_1) \leq 0. \tag{96}$$

Set $L_1 = \lim_{t\to+\infty} F_\theta(t)$. Then

$$L_1 = \phi_1(m_1) = \phi_1(M_1).$$

Therefore, there exist $\gamma \in (m_1, M_1)$ and $t_i \in (\eta_i, \xi_i)$ such that $\upsilon_{l/(q-1)}(t_i) = \gamma$, $\phi'_1(\gamma) = 0$ and $\phi_1(\gamma) \neq L_1$.

On the other hand, $\upsilon_{l/(q-1)}(t)$, $h_{l/(q-1)}(t)$, $\omega_{l/(q-1)}(t)$ are bounded for large t and $\upsilon_{l/(q-1)}(t_i) = \gamma$, we get by (71), $\lim_{i\to+\infty} F_\theta(t_i) = \phi_1(\gamma)$, hence $\phi_1(\gamma) = L_1$. But this

contradicts the fact that $\phi_1(\gamma) \neq L_1$.
Consequently, $\upsilon_{l/(q-1)}$ converges. Set $\lim_{t\to+\infty} \upsilon_{l/(q-1)}(t) = d \geq 0$.

Step 2: The function $h_{l/(q-1)}(t)$ converges.
According to (6), it suffices to show that $\omega_{l/(q-1)}$ converges. Since $h_{l/(q-1)}(t)$ is bounded for large t, we see that $\omega_{l/(q-1)}(t)$ is also bounded for large t. Assume by contradiction that it oscillates, that is, there exist two sequences $\{s_i\}$ and $\{k_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\omega_{l/(q-1)}$ has a local minimum in s_i and a local maximum in k_i satisfying $s_i < k_i < s_{i+1}$ and

$$\liminf_{t\to+\infty} \omega_{l/(q-1)}(t) = \lim_{i\to+\infty} \omega_{l/(q-1)}(s_i) < \limsup_{t\to+\infty} \omega_{l/(q-1)}(t) = \lim_{i\to+\infty} \omega_{l/(q-1)}(k_i). \quad (97)$$

Using equation (85), the fact that $\omega_{l/(q-1)}(t)$ is bounded for large t, $\lim_{t\to+\infty} \upsilon_{l/(q-1)}(t) = d$ and $\omega'_{l/(q-1)}(s_i) = \omega'_{l/(q-1)}(k_i) = 0$, we obtain

$$\lim_{i\to+\infty} -\beta h_{l/(q-1)}(s_i) = \lim_{i\to+\infty} -\beta h_{l/(q-1)}(k_i) = \alpha d + d^q.$$

Since $\beta < 0$,

$$\lim_{i\to+\infty} h_{l/(q-1)}(s_i) = \lim_{i\to+\infty} h_{l/(q-1)}(k_i),$$

which gives

$$\lim_{i\to+\infty} \omega_{l/(q-1)}(s_i) = \lim_{i\to+\infty} \omega_{l/(q-1)}(k_i).$$

But this contradicts (97). Hence, $\omega_{l/(q-1)}$ converges and therefore $h_{l/(q-1)}$ converges.
According to (6), $\lim_{t\to+\infty} \upsilon'_{l/(q-1)}(t)$ exists and must be 0. Hence,

$$\lim_{t\to+\infty} h_{l/(q-1)}(t) = \frac{-l}{q-1} d. \quad (98)$$

Step 3: $\lim_{t\to+\infty} \upsilon_{l/(q-1)}(t) = \Gamma$.
Combining equation (85), expression of Γ given by (13), Step 1 and Step 2, we get

$$\lim_{t\to+\infty} e^{-K_{l/(q-1)}t} \omega'_{l/(q-1)}(t) = d\left(\Gamma^{q-1} - d^{q-1}\right).$$

Since $\omega_{l/(q-1)}$ converges and $\lim_{t\to+\infty} e^{-K_{l/(q-1)}t} \omega'_{l/(q-1)}(t)$ exists, necessarily $\lim_{t\to+\infty} e^{-K_{l/(q-1)}t} \omega'_{l/(q-1)}(t) = 0$. Therefore, $d\left(\Gamma^{q-1} - d^{q-1}\right) = 0$. We claim that $d = \Gamma$.

Assume by contradiction that $d = \lim_{t\to+\infty} \upsilon_{l/(q-1)}(t) = 0$. First, we prove that $\omega_{l/(q-1)}(t) \neq 0$ for large t.

For any large T such that $\omega_{l/(q-1)}(T) = 0$, we have $h_{l/(q-1)}(T) = 0$ and by equation (85)

$$e^{-K_{l/(q-1)}T} \omega'_{l/(q-1)}(T) = \upsilon_{l/(q-1)}(T)\left[-\alpha - \upsilon^{q-1}_{l/(q-1)}(T)\right].$$

Since $\alpha < 0$, $\upsilon_{l/(q-1)}(t) > 0$ and $\lim_{t\to+\infty} \upsilon^{q-1}_{l/(q-1)}(t) = 0$, one has $\omega'_{l/(q-1)}(T) > 0$. Hence, $\omega_{l/(q-1)}(t) \neq 0$ for any $t > T$.

As u is positive and unbounded, then $\omega_{l/(q-1)}(t) > 0$ for large t.
On the other hand, we have by (85)

$$e^{-K_{l/(q-1)}t}\omega'_{l/(q-1)}(t) = h_{l/(q-1)}(t)\left[-\beta - A_{l/(q-1)}e^{-K_{l/(q-1)}t}\left|h_{l/(q-1)}(t)\right|^{p-2}\right] + \upsilon_{l/(q-1)}(t)\left[-\alpha - \upsilon^{q-1}_{l/(q-1)}(t)\right]. \quad (99)$$

Using the fact that $\beta < 0$, $\alpha < 0$, $\lim\limits_{t\to+\infty} e^{-K_{l/(q-1)}t}\left|h_{l/(q-1)}(t)\right|^{p-2} = 0$ (by (83)), $\lim\limits_{t\to+\infty} \upsilon^{q-1}_{l/(q-1)}(t) = 0$, $\upsilon_{l/(q-1)}(t) > 0$ and $h_{l/(q-1)}(t) > 0$ for large t, we have $\omega'_{l/(q-1)}(t) > 0$ for large t. This implies, since $\omega_{l/(q-1)}(t) > 0$ for large t, that $\lim\limits_{t\to+\infty}\omega_{l/(q-1)}(t) \in]0, +\infty]$. But this contradicts the fact that $\lim\limits_{t\to+\infty}\omega_{l/(q-1)}(t) = 0$ by (6) and (98).
Consequently, $d = \Gamma$. It follows that

$$\lim_{t\to+\infty}\upsilon_{l/(q-1)}(t) = \Gamma \quad \text{and} \quad \lim_{t\to+\infty} h_{l/(q-1)}(t) = \frac{-l}{q-1}\Gamma.$$

The proof is complete. □

The behavior of unbounded solution is illustrated by the following figure.

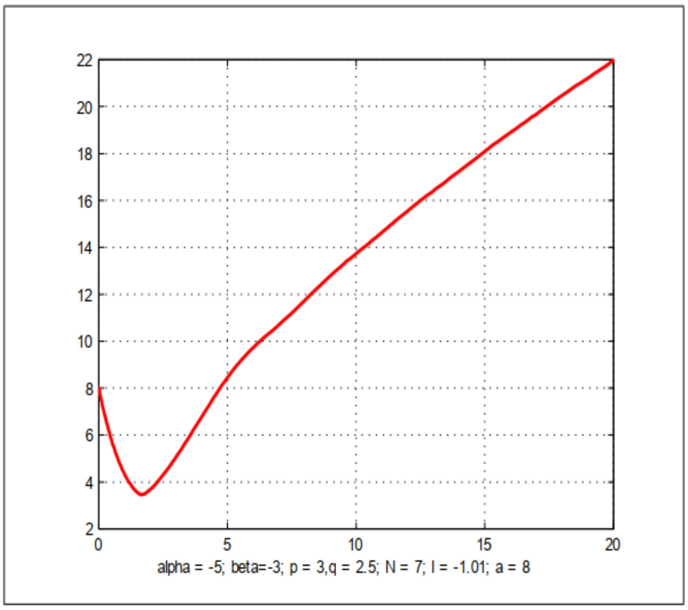

Figure 2: Unbounded solution.

To finish this work, we note that if we have the monotonicity of u, then the asymptotic behavior (64) and (65) is validated by reducing the assumptions of Theorem 3.3. More precisely, we have the following result.

Proposition 3.10 *Assume* $q \geq p\Big(1 + 2^{p-1}\Big) - 1$ *and* $\dfrac{N-p}{p-1} > \dfrac{l}{q-1}$. *Let* u *be an unbounded positive solution of problem* (**P**). *If* u *is an increasing function for large* r, *then it satisfies (64) and (65).*

Proof. First of all we note that the idea of the proof is similar to that of Theorem 3.3. So, we follow the same steps and we only change the step where we use the monotonicity of u.

According to the proof of Propostition 3.9, we choose a suitable θ to show that $F_\theta(t)$ is strictly increasing for large t, where F_θ is given by (71). In fact, we choose in (71)

$$\theta = \frac{2^p|l|pD}{K_{l/(q-1)}(q-1) + lp2^{p-1}} > 0. \tag{100}$$

Note that, since $q \geq p\Big(1 + 2^{p-1}\Big) - 1$, one has $K_{l/(q-1)}(q-1) + lp2^{p-1} > 0$ and by expression (100),

$$2^{p-1}D_1\left(\frac{|l|}{q-1}\right)^p + \frac{K_{l/(q-1)}\theta}{p}\left(\frac{|l|}{q-1}\right)^{p-1} = \left(\frac{|l|}{q-1}\right)^{p-1}\frac{2^{p-1}|l|D}{q-1} > 0.$$

According to inequality (76) and using expressions (100) and (75) and the fact that $\omega_{l/(q-1)}(t) \geq 0$ for large t (because $u'(r) \geq 0$ for large r), we deduce that

$$\begin{aligned} F'_\theta(t) \geq{}& \theta e^{-K_{l/(q-1)}t}X(t) + \upsilon'^2_{l/(q-1)}(t)\left[-\beta + 2^{p-1}D_1 e^{-K_{l/(q-1)}t}\left|\upsilon'_{l/(q-1)}\right|^{p-2}\right] + \\ & \left(\frac{|l|}{q-1}\right)^{p-1}\frac{2^{p-1}|l|D}{q-1}e^{-K_{l/(q-1)}t}\upsilon^p_{l/(q-1)}. \end{aligned} \tag{101}$$

Using (84), the fact that $X(t) \geq 0$, $\upsilon_{l/(q-1)}(t) > 0$, $\theta > 0$ and $\beta < 0$, we deduce from estimate (101) that $F'_\theta(t) > 0$ for large t. We complete the proof in the same way as that of Theorem 3.3. □

4 Conclusion

We consider equation (1) as a natural generalization of the pure Laplacian case ($p = 2$) already studied by Filippas and Tertikas in [8]. Its appears in studying the self-similar solution of the parabolic equation (2). This equation admits a family of radial self-similar solutions defined in the form

$$v(x,t) = t^{-\alpha}u(t^{-\beta}\,|x|),$$

where u is the solution of equation (3) with

$$\alpha = \frac{l+p}{p(q-1)+l(p-2)}, \quad \beta = \frac{q+1-p}{p(q-1)+l(p-2)}.$$

There is an extensive literature on equation (2) in the case $l = 0$. The study of equation (1) in this case is a reasonable first step towards the understanding of the behavior of blowing up solutions of (2).

Taking the case $l < 0$, we have proven in [7] the existence of an entire solution u of (3). In this paper we have proven that if $q \leq p-1$ or $q > p-1$ and $\dfrac{\alpha}{\beta} \neq \dfrac{l+p}{q+1-p}$ or $\dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p} \geq \dfrac{N-p}{p}$, this solution is unbounded. The study of its asymptotic behavior depends strongly on the study of the following nonlinear dynamical system by using the logarithmic change (4), $\upsilon_{l/(q-1)}(t) = r^{l/(q-1)}u(r)$,

$$\begin{cases} \upsilon'_{l/(q-1)}(t) = & |\omega_{l/(q-1)}|^{(2-p)/(p-1)}\omega_{l/(q-1)}(t) + \dfrac{l}{q-1}\upsilon_{l/(q-1)}(t), \\ \omega'_{l/(q-1)}(t) = & -A_{l/(q-1)}\omega_{l/(q-1)}(t) - \alpha e^{K_{l/(q-1)}t}\upsilon_{l/(q-1)}(t) - \beta e^{K_{l/(q-1)}t}h_{l/(q-1)}(t) - \\ & e^{K_{l/(q-1)}t}\upsilon^q_{l/(q-1)}(t). \end{cases}$$

It is shown that under some assumptions, the solution $(\upsilon_{l/(q-1)}, \omega_{l/(q-1)})$ of the above system tends to the equilibrium point $\left(\Gamma, \left(\dfrac{-l}{q-1}\Gamma\right)^{p-1}\right)$, where Γ is given by (13). This result can be translated in terms of u and u' by

$$u(r) \underset{+\infty}{\sim} \Gamma\, r^{-l/(q-1)}$$

and

$$|u'|^{p-2}u'(r) \underset{+\infty}{\sim} \left(\frac{-l}{q-1}\Gamma\right)^{p-1} r^{-(l/(q-1)+1)(p-1)}.$$

The more complicated case $\dfrac{\alpha}{\beta} = \dfrac{l+p}{q+1-p} < \dfrac{N-p}{p}$ has not been completely investigated.

Acknowledgment

The authors thank the reviewer for several remarks on a preliminary version of the paper.

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