Nonlinear Dynamics and Systems Theory, **21** (1) (2021) 1–12

Upper and Lower Solutions for Fractional q-Difference Inclusions

S. Abbas [1], M. Benchohra [2] and J. R. Graef [3*]

[1] *Department of Mathematics, Tahar Moulay University of Saïda, P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria.*
[2] *Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria, and Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.*
[3] *Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA.*

Received: October 29, 2019; Revised: December 24, 2020

Abstract: This paper deals with some existence results for a class of boundary value problems for Caputo fractional q-difference inclusions by using set-valued analysis, fixed point theory, and the method of upper and lower solutions.

Keywords: *fractional q-difference inclusion; upper solution; lower solution; boundary condition; fixed point.*

Mathematics Subject Classification (2010): 26A33, 34A08, 34A60, 34B15, 39A13.

1 Introduction

Fractional differential equations and inclusions have been applied in various areas of engineering, mathematics, physics, and other applied sciences. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with Caputo fractional derivatives. The method of upper and lower solutions has been successfully applied to study the existence of solutions for differential equations and inclusions; see [1–5, 11, 12] and the references therein.

The study of fractional q-difference equations was initiated early in the 20-th century [6, 14] and has received significant attention in recent years [10, 16]. Some interesting details about initial and boundary value problems for q-difference and fractional

* Corresponding author: mailto:John-Graef@utc.edu

q-difference equations can be found in [8, 9, 15–17] and the included references. In this paper, we discuss the existence of solutions to the fractional q-difference inclusion

$$({}^cD_q^\alpha u)(t) \in F(t, u(t)),\ t \in I := [0, T], \tag{1}$$

with the boundary condition

$$L(u(0), u(T)) = 0, \tag{2}$$

where $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α, and $L : \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function.

This paper initiates the application of the method of upper and lower solutions to Caputo q-fractional difference equations.

2 Preliminaries

Consider the Banach space $C(I) := C(I, \mathbb{R})$ of continuous functions from I into $\mathbb{R}$ equipped with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} |u(t)|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

Let us recall some definitions and properties of the fractional q-calculus. For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q analogue of the power $(a-b)^n$ is

$$(a-b)^{(0)} = 1,\ (a-b)^{(n)} = \Pi_{k=0}^{n-1}(a - bq^k),\ a, b \in \mathbb{R},\ n \in \mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right),\ a, b, \alpha \in \mathbb{R}.$$

Definition 2.1 ([19]) The q-gamma function is defined by

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}} \text{ for } \xi \in \mathbb{R} - \{0, -1, -2, \ldots\}.$$

Notice that the q-gamma function satisfies $\Gamma_q(1+\xi) = [\xi]_q\Gamma_q(\xi)$.

Next, we give definitions of different types of q-derivatives and q-integrals and indicate some of their properties.

Definition 2.2 ([19]) The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1-q)t},\ t \neq 0,\quad (D_q u)(0) = \lim_{t\to 0}(D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t),\ t \in I,\ n \in \{1, 2, \ldots\}.$$

We set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3 ([19]) The q-integral of a function $u : I_t \to \mathbb{R}$ is defined by

$$(I_q u)(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 2.4 ([7]) The Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $u : I \to \mathbb{R}$ is defined by $(I_q^0 u)(t) = u(t)$, and

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s, \ t \in I.$$

Lemma 2.1 ([20]) *For $\alpha \in \mathbb{R}_+ := [0, \infty)$ and $\lambda \in (-1, \infty)$, we have*

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)} (t-a)^{(\lambda+\alpha)}, \ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 2.5 ([21]) The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$, and

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t), \ t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 2.6 ([21]) The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $({}^C D_q^0 u)(t) = u(t)$ and

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t), \ t \in I.$$

Lemma 2.2 ([21]) *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha \ {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha \ {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

For a given Banach space $(X, \|\cdot\|)$, we define the following subsets of $\mathcal{P}(X)$:

$$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}, \qquad P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\},$$

$$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}, \quad P_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex}\},$$

$$P_{cp,cv}(X) = P_{cp}(X) \Cap P_{cv}(X).$$

The following properties of multivalued maps will be needed.

Definition 2.7 A multivalued map $G : X \to \mathcal{P}(X)$ is said to be convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}$ exists).

Definition 2.8 A multivalued map $G : X \to \mathcal{P}(X)$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set $N \subset X$ containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. Moreover, G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.

Definition 2.9 Let $G : X \to \mathcal{P}(X)$ be completely continuous with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). We say that G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

We denote by $Fix\, G$ the set of fixed points of the multivalued operator G.

Definition 2.10 A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

The following relationship between upper semi-continuous maps and closed graphs is well known.

Lemma 2.3 ([18]) *Let G be a completely continuous multivalued map with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph.*

Definition 2.11 A multivalued map $F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

(1) $t \to F(t, u)$ is measurable for each $u \in \mathbb{R}$;

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in I$.

Moreover, F is said to be L^1-Carathéodory if (1), (2), and the following condition hold:

(3) For each $q > 0$, there exists $\varphi_q \in L^1(I, \mathbb{R}_+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t, u)\} \leq \varphi_q \text{ for all } |u| \leq q \text{ and for a.e. } t \in I.$$

For each $u \in C(I, \mathbb{R})$, we define the set of selections of F by

$$S_{F\circ u} = \{v \in L^1(I, \mathbb{R}) : v(t) \in F(t, u(t)) \text{ a.e. } t \in I\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. The function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$$

is known as the Hausdorff–Pompeiu metric. For more details on multivalued maps see the books of Hu and Papageorgiou [18].

In the sequel, we need the following fixed point theorem.

Theorem 2.1 (Bohnenblust-Karlin [13]) *Let X be a Banach space and $K \in \mathcal{P}_{cl,cv}(X)$, and suppose that the operator $G : K \to \mathcal{P}_{cl,cv}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

3 Main Results

We begin by defining what we mean by a solution, an upper solution, and a lower solution to our problem.

Definition 3.1 A function $u \in C(I)$ is said to be a solution of (1)–(2) if there exists a function $f \in S_{F\circ u}$ such that ${}^C D_q^\alpha u(t) = f(t)$ a.e. $t \in I$ and the boundary condition $L(u(0), u(T)) = 0$ is satisfied.

Definition 3.2 A function $w \in C(I)$ is said to be an upper solution of (1)–(2) if $L(w(0), w(T)) \geq 0$, and there exists a function $v_1 \in S_{F\circ w}$ such that ${}^C D_q^\alpha w(t) \geq v_1(t)$ a.e. $t \in I$. Similarly, a function $v \in C(I)$ is said to be a lower solution of (1)–(2) if $L(v(0), v(T)) \leq 0$, and there exists a function $v_2 \in S_{F\circ v}$ such that ${}^C D^\alpha v(t) \leq v_2(t)$ a.e. $t \in I$.

We now present the main result in this paper.

Theorem 3.1 *Assume that the following conditions hold:*

(H1) *$F : I \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is Carathéodory;*

(H2) *There exist $v, w \in C(I)$, which are the lower and upper solutions, respectively, for problem (1)–(2) such that $v \leq w$;*

(H3) *The function $L(\cdot, \cdot)$ is continuous on $[u(0), w(0)] \times [u(T), w(T)]$ and is nonincreasing in each of its arguments;*

(H4) *There exists $l \in L^1(I, \mathbb{R}^+)$ such that*

$$H_d(F(t, u), F(t, \bar{u})) \leq l(t)|u - \bar{u}| \textit{ for every } u, \bar{u} \in \mathbb{R},$$

and

$$d(0, F(t, 0)) \leq l(t) \textit{ a.e. } t \in I.$$

Then the problem (1)–(2) has at least one solution u defined on I such that

$$v \leq u \leq w.$$

Proof. Consider the following modified problem

$$^{C}D_q^{\alpha} u(t) \in F(t, \tau(u(t))), \text{ for a.e. } t \in I, \tag{3}$$

$$u(0) = \tau(u(0)) - L(\overline{u}(0), \overline{u}(T)), \tag{4}$$

where

$$\tau(u(t)) = \max\{v(t), \min\{u(t), w(t)\}\},$$

and

$$\overline{u}(t) = \tau(u(t)).$$

A solution to (3)–(4) is a fixed point of the operator $N : C(I) \to \mathcal{P}(C(I))$ defined by

$$N(u) = \{h \in C(I) \; : \; h(t) = u(0) + (I_q^{\alpha}\nu)(t)\},$$

where

$$\nu \in \{x \in \widetilde{S}^1_{F\circ\tau(u)} : x(t) \geq v_1(t) \text{ on } A_1 \text{ and } x(t) \leq v_2(t) \text{ on } A_2\},$$

$$S^1_{F\circ\tau(y)} = \{x \in L^1(I) \; : x(t) \in F(t, (\tau u)(t)), \; a.e. \; t \in I\},$$

$$A_1 = \{t \in I : u(t) < v(t) \leq w(t)\}, \; A_2 = \{t \in I \; : v(t) \leq w(t) < u(t)\}.$$

Remark 3.1 (1) For each $u \in C(I)$, the set $\widetilde{S}^1_{F\circ\tau(u)}$ is nonempty. In fact, $(H1)$ implies that there exists $v_3 \in S^1_{F\circ\tau(u)}$, so we set

$$v = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3},$$

where

$$A_3 = \{t \in I : v(t) \leq u(t) \leq w(t)\}.$$

Then, by decomposability, $x \in \widetilde{S}^1_{F\circ\tau(u)}$.

(2) From the definition of τ, it is clear that $F(\cdot, \tau u(\cdot))$ is an L^1-Carathéodory multi-valued map with compact convex values and there exists $\phi_1 \in C(I, \mathbb{R}^+)$ such that

$$\|F(t, \tau u(t))\|_{\mathcal{P}} \leq \phi_1(t) \text{ for each } u \in \mathbb{R}.$$

(3) Since $\tau(u(t)) = v(t)$ for $t \in A_1$, and $\tau(u(t)) = w(t)$ for $t \in A_2$, in view of (H3), equation (4) implies that

$$|u(0)| \leq |v(0)| + |L(v(0), v(T)| \leq |v(0)| + |L(u(0), u(T))| = |v(0)| \; on \; A_1,$$

and

$$u(1) = w(0) - L(w(0), w(T) \leq w(0) - L(u(0), u(T)) = w(0) \; on \; A_2.$$

Thus,

$$|u(0)| \leq \min\{|v(0)|, |w(0)|\}.$$

Now set

$$L := \sup_{t\in I} \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s,$$

let

$$R := \min\{|v(0)|, |w(0)|\} + L\|\phi_1\|_\infty,$$

and consider the closed and convex subset of $C(I)$ given by

$$B = \{u \in C(I) : \|u\|_\infty \leq R\}.$$

We shall show that the operator $N : B \to \mathcal{P}_{cl,cv}(B)$ satisfies all the assumptions of Theorem 2.1. The proof will be given in steps.

Step 1: *$N(u)$ is convex for each $y \in B$.*
Let h_1, h_2 belong to $N(u)$; then there exist ν_1, $\nu_2 \in \widetilde{S}^1_{F\circ\tau(u)}$ such that, for each $t \in I$ and any $i = 1, 2$, we have

$$h_i(t) = u(0) + (I_q^\alpha \nu_i)(t).$$

Let $0 \leq d \leq 1$. Then, for each $t \in I$, we have

$$(dh_1 + (1-d)h_2)(t) = u(0) + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [d\nu_1(s) + (1-d)\nu_2(s)] d_q s.$$

Since $S_{F\circ\tau(u)}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(u).$$

Step 2: *N maps bounded sets into bounded sets in B.*
For each $h \in N(u)$, there exists $\nu \in \widetilde{S}^1_{F\circ\tau(u)}$ such that

$$h(t) = u(0) + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \nu(s) d_q s.$$

From conditions (H1)–(H3), for each $t \in I$, we have

$$\begin{aligned} |h(t)| &\leq |u(0)| + \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |\nu(s)| d_q s \right| \\ &\leq \min\{|v(0)|, |w(0)|\} + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |\nu(s)| d_q s \\ &\leq \min\{|v(0)|, |w(0)|\} + L\|\phi_1\|_\infty. \end{aligned}$$

Thus,

$$\|h\|_\infty \leq R.$$

Step 3: *N maps bounded sets into equicontinuous sets of B.*

Let t_1, $t_2 \in I$ with $t_1 < t_2$, and let $u \in B$ and $h \in N(u)$. Then

$$\begin{aligned}
|h(t_2) - h(t_1)| &= \Big| \int_0^{t_1} \frac{|(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} \nu(s) d_q s \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \nu(s) d_q s \Big| \\
&\le \int_0^{t_1} \frac{|(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |\nu(s)| d_q s \\
&\quad + \int_{t_1}^{t_2} \frac{|(t_2 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |\nu(s)| d_q s \\
&\le \|\phi_1\|_\infty \int_0^{t_1} \frac{|(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\
&\quad + \|\phi_1\|_\infty \int_{t_1}^{t_2} \frac{|(t_2 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\
&\to 0 \ as \ t_1 \to t_2.
\end{aligned}$$

As a consequence of the three steps above, we can conclude from the Arzelà-Ascoli theorem that $N : C(I) \to \mathcal{P}(C(I))$ is continuous and completely continuous.

Step 4: *N has a closed graph.*
Let $u_n \to u_*$, $h_n \in N(u_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$. Now $h_n \in N(u_n)$ implies there exists $\nu_n \in \widetilde{S}^1_{F\circ\tau(u_n)}$ such that, for each $t \in I$,

$$h_n(t) = u(0) + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \nu_n(s) d_q s.$$

We must show that there exists $\nu_* \in \widetilde{S}^1_{F\circ\tau(u_*)}$ such that, for each $t \in I$,

$$h_*(t) = u(0) + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \nu_*(s) d_q s.$$

Since $F(t, \cdot)$ is upper semi-continuous, for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n \ge n_0(\epsilon)$, we have

$$\nu_n(t) \in F(t, \tau u_n(t)) \subset F(t, u_*(t)) + \epsilon B(0,1) \quad \text{a.e. } t \in I.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $\nu_{n_m}(\cdot)$ such that

$$\nu_{n_m}(\cdot) \to \nu_*(\cdot) \quad \text{as} \quad m \to \infty,$$

and

$$\nu_*(t) \in F(t, \tau u_*(t)) \quad \text{a.e. } t \in I.$$

For every $w \in F(t, \tau u_*(t))$, we have

$$|\nu_{n_m}(t) - \nu_*(t)| \le |\nu_{n_m}(t) - w| + |w - \nu_*(t)|.$$

Hence,

$$|\nu_{n_m}(t) - \nu_*(t)| \le d(\nu_{n_m}(t), F(t, \tau u_*(t))).$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_* to obtain

$$|\nu_{n_m}(t) - \nu_*(t)| \leq H_d(F(t, \tau u_n(t)), F(t, \tau u_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Thus,

$$\begin{aligned}|h_{n_m}(t) - h_*(t)| &\leq \int_0^t \frac{|(t-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |\nu_{n_m}(s) - \nu_*(s)| d_q s \\ &\leq \|u_{n_m} - u_*\|_\infty \int_0^t \frac{|(t-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} l(s) d_q s.\end{aligned}$$

Therefore,

$$\|h_{n_m} - h_*\|_\infty \leq \|u_{n_m} - u_*\|_\infty \int_0^t \frac{|(t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} l(s) d_q s \to 0 \quad \text{as } m \to \infty,$$

so Lemma 2.3 implies that N is upper semicontinuous.

Step 5: *Every solution u of (3)–(4) satisfies $v(t) \leq u(t) \leq w(t)$ for all $t \in I$.*
Let u be a solution of (3)–(4). To prove that $v(t) \leq u(t)$ for all $t \in I$, suppose this is not the case. Then there exist t_1, t_2, with $t_1 < t_2$, such that $v(t_1) = u(t_1)$ and $v(t) > u(t)$ for all $t \in (t_1, t_2)$. In view of the definition of τ,

$${}^C D_q^\alpha u(t) \in F(t, v(t)) \text{ for all } t \in (t_1, t_2).$$

Thus, there exists $y \in S_{F \circ \tau(v)}$ with $y(t) \geq v_1(t)$ a.e. on (t_1, t_2) such that

$${}^C D_q^\alpha u(t) = y(t) \text{ for all } t \in (t_1, t_2).$$

An integration on $(t_1, t]$, with $t \in (t_1, t_2)$, yields

$$u(t) - y(t_1) = \int_{t_1}^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \nu(s) d_q.$$

Since v is a lower solution of (1)–(2),

$$v(t) - v(t_1) \leq \int_{t_1}^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_1(s) d_q, \ t \in (t_1, t_2).$$

From the facts that $u(t_0) = v(t_0)$ and $\nu(t) \geq v_1(t)$, it follows that

$$v(t) \leq u(t) \text{ for all } t \in (t_1, t_2).$$

This is a contradiction, since $v(t) > u(t)$ for all $t \in (t_1, t_2)$. Consequently,

$$v(t) \leq u(t) \text{ for all } t \in I.$$

Similarly, we can prove that

$$u(t) \leq w(t) \text{ for all } t \in I.$$

This shows that

$$v(t) \leq u(t) \leq w(t) \text{ for all } t \in I.$$

Therefore, the problem (3)–(4) has a solution u satisfying $v \leq u \leq w$.

Step 6: *Every solution of problem (3)–(4) is a solution of (1)–(2).* Suppose that u is a solution of the problem (3)–(4). Then, we have

$$^{C}D_q^{\alpha}u(t) \in F(t,\tau(u(t))) \text{ for a.e. } t \in I,$$

and

$$u(0) = \tau(u(0)) - L(\overline{u}(0), \overline{u}(T)).$$

Since, for all $t \in I$, we have $v(t) \leq u(t) \leq w(t)$, it follows that $\tau(u(t)) = u(t)$. Thus, we have

$$^{C}D_q^{\alpha}u(t) \in F(t,u(t)) \text{ for a.e. } t \in I,$$

and $\quad L(u(0), u(T)) = 0$. We only need to prove that

$$v(0) \leq u(0) - L(u(0), u(T)) \leq w(1),$$

so suppose that

$$u(0) - L(u(0), u(T)) < u(0).$$

Since $L(v(0), v(T)) \leq 0$, we have

$$u(0) \leq u(1) - L(v(0), v(T)),$$

and since $L(\cdot,\cdot)$ is nonincreasing with respect to both of its arguments,

$$u(0) \leq u(0) - L(v(0), v(T)) \leq u(0) - L(u(0), u(T)) < v(0).$$

Hence, $u(0) < v(0)$, which is a contradiction. Similarly, we can prove that

$$u(0) - L(u(0), u(T)) \leq w(1).$$

Thus, u is a solution of (1)–(2).

This shows that the problem (1)–(2) has a solution u satisfying $v \leq u \leq w$, and completes the proof of the theorem.

Remark 3.2 In the case where $L(x,y) = ax - by - c$, Theorem 3.1 yields existence results to the problem

$$^{C}D_q^{\alpha}u(t) \in F(t,u(t)) \text{ for a.e. } \quad t \in I, \tag{5}$$

$$ay(1) - by(T) = c, \tag{6}$$

where $-b < a \leq 0 \leq b$, $c \in \mathbb{R}$, which includes the anti-periodic problem $b = -a$, $c = 0$, the initial value problem, and the terminal value problem.

4 An Example

Consider the following problem of a Caputo fractional $\frac{1}{4}-$difference inclusion of order $\alpha = \frac{1}{2}$,

$$\begin{cases} \left({}^{c}D_{\frac{1}{4}}^{\frac{1}{2}}u\right)(t) \in \frac{7t^2}{27(1+|u(t)|)}[u(t), 33(1+u(t))], \ t \in [0,1], \\ \\ u(0) + u(1) = 1. \end{cases} \tag{7}$$

Set

$$F(t,u(t)) = \frac{7t^2}{27(1+|u(t)|)}[u(t), 33(1+u(t))],\ t \in [0,1],$$

and $L(x,y) = -x - y + 1$ for x, $y \in \mathbb{R}$. It is easy to see that $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is Carathéodory.

In order to see that $(H2)$ holds, let v, $w \in C([0,1], \mathbb{R})$ be given by $v(t) = t^{\frac{5}{2}}$ and $w(t) = t^{\frac{3}{2}}$. Now $L(v(0), v(1)) = 0 \leq 0$ and

$$\left({}^cD_{\frac{1}{4}}^{\frac{1}{2}} v\right)(t) = \frac{217}{27}t^2 \leq \frac{7t^2}{27(1+v(t))}(31 + 31v(t)) \in F(t, v(t)).$$

Also, $L(w(0), w(1)) = 0 \geq 0$ and

$$\left({}^cD_{\frac{1}{4}}^{\frac{1}{2}} w\right)(t) = \frac{7}{9}t \geq \frac{7}{9}t^2 = \frac{7t^2}{27(1+w(t))}(3 + 3w(t)) \in F(t, w(t)).$$

Therefore, v and w are lower and upper solutions, respectively, for problem (7) with $v \leq w$. To see that condition $(H3)$ is satisfied, note that L is continuous and

$$\frac{\partial L(x,y)}{\partial x} = \frac{\partial L(x,y)}{\partial y} = -1 < 0.$$

Finally, for each u, $\bar{u} \in \mathbb{R}$ and $t \in [0,1]$, we have

$$H_d(F(t,u), F(t,\bar{u})) \leq \frac{7}{27}t^2|u - \bar{u}| \quad \text{and} \quad d(0, F(t,0)) = \|F(t,0)\|_{\mathcal{P}} \leq \frac{7}{27}t^2,$$

so $(H4)$ is satisfied with $l(t) = \frac{7}{27}t^2$.

Consequently, all conditions of Theorem 3.1 are satisfied, and so problem (7) has at least one solution u defined on $[0,1]$ with $t^2\sqrt{t} \leq u(t) \leq t\sqrt{t}$.

5 Concluding Remarks

In this paper the authors study the existence of solutions to a boundary value problem for a fractional q-difference inclusion involving the Caputo fractional derivative. This topic fits well in the scope of problems covered by the journal *Nonlinear Dyamics and Systems Theory*.

This paper is the first attempt at using the method of upper and lower solutions to study problems of this type. In order to illustrate the applicability of the results, an example is given detailing how the various hypotheses are satisfied.

References

[1] S. Abbas and M. Benchohra. Upper and lower solutions method for Darboux problem for fractional order implicit impulsive partial hyperbolic differential equations. *Acta Univ. Palacki. Olomuc.* **51** (2012) 5–18.

[2] S. Abbas and M. Benchohra. The method of upper and lower solutions for partial hyperbolic fractional order differential inclusions with impulses. *Discuss. Math. Differ. Incl. Control Optim.* **30** (2010) 141–161.

[3] S. Abbas, M. Benchohra, S. Hamani and J. Henderson. Upper and lower solutions method for Caputo–Hadamard fractional differential inclusions. *Math. Moravica.* **23** (2019) 107–118.

[4] S. Abbas, M. Benchohra and A. Hammoudi. Upper, lower solutions method and extremal solutions for impulsive discontinuous partial fractional differential inclusions. *Panamer. Math. J.* **24** (2014) 31–52.

[5] S. Abbas, M. Benchohra and J. J. Trujillo. Upper and lower solutions method for partial fractional differential inclusions with not instantaneous impulses. *Prog. Frac. Diff. Appl.* **1** (2015) 11–22.

[6] C. R. Adams. On the linear ordinary q-difference equation. *Annals Math.* **30** (1928) 195–205.

[7] R. Agarwal. Certain fractional q-integrals and q-derivatives. *Proc. Camb. Philos. Soc.* **66** (1969) 365–370.

[8] B. Ahmad. Boundary value problem for nonlinear third order q-difference equations. *Electron. J. Differential Equations* **2011** (94) (2011) 1–7.

[9] B. Ahmad, S. K. Ntouyas and L. K. Purnaras. Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. *Adv. Difference Equ.* **2012**, 2012:140.

[10] M. H. Annaby and Z. S. Mansour. *q-fractional Calculus and Equations.* Lecture Notes in Mathematics, vol. 2056. Springer, Heidelberg, 2012.

[11] M. Benchohra and S. Hamani. The method of upper and lower solution and impulsive fractional differential inclusions. *Nonlinear Anal. Hybrid Syst.* **3** (2009) 433–440.

[12] M. Benchohra and S. K. Ntouyas. The lower and upper method for first order differential inclusions with nonlinear boundary condition. *J. Inequ. Pure Appl. Math.* **3** (2002), Article 14, 1–20.

[13] H. F. Bohnenblust and S. Karlin. On a theorem of Ville. Contribution on the theory of games. In: *Annals of Mathematics Studies*, Vol. 24. Princeton University Press, Princeton, 1950, 155–160.

[14] R. D. Carmichael. The general theory of linear q-difference equations. *American J. Math.* **34** (1912) 147–168.

[15] M. El-Shahed and H. A. Hassan. Positive solutions of q-difference equation. *Proc. Amer. Math. Soc.* **138** (2010) 1733–1738.

[16] T. Ernst. *A Comprehensive Treatment of q-Calculus.* Birkhäuser, Basel, 2012.

[17] S. Etemad, S. K. Ntouyas and B. Ahmad. Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders. *Mathematics* **7** 659 (2019) 1–15.

[18] Sh. Hu and N. Papageorgiou. *Handbook of Multivalued Analysis, Volume I: Theory.* Kluwer, Dordrecht, 1997.

[19] V. Kac and P. Cheung. *Quantum Calculus.* Springer, New York, 2002.

[20] P. M. Rajkovic, S. D. Marinkovic and M. S. Stankovic. Fractional integrals and derivatives in q-calculus. *Appl. Anal. Discrete Math.* **1** (2007) 311–323.

[21] P. M. Rajkovic, S. D. Marinkovic and M. S. Stankovic. On q-analogues of Caputo derivative and Mittag-Leffler function. *Fract. Calc. Appl. Anal.* **10** (2007) 359–373.