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On Approximate Controllability of Impulsive Fractional Semilinear Systems with Deviated Argument in Hilbert Spaces

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Abstract: In this paper we apply a fixed-point theorem to study the existence and uniqueness of a mild solution and the approximate controllability of a fractional order impulsive differential equation with deviated argument in Hilbert spaces. An example is provided to show the effectiveness of the theory.

Keywords: *controllability; differential equations with impulses; deviated arguments; fractional derivatives and integrals; semigroup theory; fixed-point theorems.*

Mathematics Subject Classification (2010): 93B05, 34A37, 34K30, 26A33, 47H10.

1 Introduction

Differential equations with deviated arguments have received considerable attention in recent years due to their ability to generalize differential equations that show an unknown quantity and their derivatives in different values of their arguments. It is an ideal model for the study of automatic control theory, self-oscillating systems theory, long-term planning problems in economics, etc. For more details about differential equations with deviated arguments, we refer to the papers [8, 11, 15] and the references therein.

Interestingly, in this paper, we will enhance the study of differential equations with deviated arguments by fractional calculus, which, in turn, is currently attracting considerable interest from researchers, due to its wide range of applications in various scientific and technological fields such as thermal engineering, electromagnetism, control, robotics,

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viscoelasticity, edge detection, signal processing, and many other physical and biological processes. Fractional differential equations have also been applied in the modeling of many physical and engineering problems. For more details, the reader is kindly requested to go through [12, 14, 17] and the references therein.

On the other hand, impulsive differential equations have become the target of several authors, mainly because of their ability to model processes that undergo sudden changes of their states. They appear in nano-electronics, population dynamics, heat propagation, electromagnetic wave radiation, control theory and pharmacology. See [1, 3, 4, 10, 18–20] and the references therein.

Controllability of linear and nonlinear systems for various type of differential equations and inclusions was studied, on a large scale, using fixed point and semigroup theories, for more details the reader is kindly requested to go through [2, 5, 7, 9, 13] in order to know more details about these results. However, the controllability of fractional impulsive systems with deviated arguments requires a lot of attention since it has not yet received a careful study, and many aspects of this field have not been discovered yet.

In this paper, we consider the approximate controllability of fractional impulsive differential equations with deviated argument of the form

$$\begin{cases} {}^CD^\alpha x(t) = Ax(t) + Bu(t) + f(t, x(t), x(\varphi(x(t), t))), \quad t \in [0, b],\ 0 < \alpha < 1, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = \overline{1, m},\ 0 < t_1 < t_2 < \cdots < t_m < b, \\ x(0) = x_0, \end{cases} \tag{1}$$

where the state function $\varphi(\cdot)$ takes values in a Hilbert space E. ${}^CD^\alpha$ is the Caputo fractional derivative of order α. The control function $u(\cdot)$ is given in $L^2([0, b], U)$, where U is a Hilbert space. B is a bounded linear operator from U into E. The linear operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E. f and φ are suitably defined functions satisfying certain conditions to be specified in Section 3. $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k^+) - x(t_k)$.

This paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3 we prove the existence of mild solutions for control systems and we establish its approximate controllability. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Let $J = [0, b], 0 < t_1 < t_2 < \cdots < t_m < b$, and $J' = [0, b]\backslash\{t_1, t_2, \cdots, t_m\} \subset J$. $(E, \|\cdot\|)$ is a Hilbert space and $C(J, E)$ is the Hilbert space of all E-valued continuous functions from J into E,

$$PC(J, E) = \left\{x : [0, b] \to E;\ x \in C(J', E), x(t_k^+) \text{ and } x(t_k^-) \text{ exist},\ x(t_k^-) = x(t_k), 1 \leq k \leq m\right\},$$

$PC(J, E)$ is a Banach space with norm $\|x\| = \sup_{t\in J} \|x(t)\|$,

$$D = C_L(J, E) = \left\{x \in PC(J, E) : \|x(t) - x(s)\| \leq L|t - s|,\ \forall t, s \in J'\right\}, \tag{2}$$

where L is a positive constant. It is clear that D is a Banach space with the sup-norm $\|x\| = \sup_{t\in J} \|x(t)\|$.

Definition 2.1 [17] The fractional (arbitrary) order integral of the function $f \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where Γ is the Gamma function, when $a = 0$, we write $I_a^\alpha f(t) = I^\alpha f(t)$.

Definition 2.2 [17] For a function f given on the interval $[a,b]$, the Riemann-Liouville fractional-order derivative of order α of f is defined by

$${}^{(R-L)}D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int\limits_a^t (t-s)^{n-\alpha-1} f(s)\, ds,$$

here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α, when $a = 0$, $D_a^\alpha f(t) = D^\alpha f(t)$.

Definition 2.3 [17] For a function f given on the interval $[a,b]$, the Caputo fractional-order derivative of order α of f is defined by

$${}^C D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Definition 2.4 [16] A one parameter family $(T(t))_{t\geq 0}$ of bounded linear operators from E into E is a semi group of bounded linear operators on E if

(1) $T(0) = I$ (I is the identity operator in E).

(2) $T(t+s) = T(t)T(s)$, for every $t \geq 0$, $s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \left\{ x \in E : \lim_{t\to 0} \frac{T(t)x - x}{t} \ exists \right\}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \left. \frac{d^+T(t)x}{dt} \right|_{t=0}, \quad for \quad x \in D(A),$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

Definition 2.5 A mild solution of problem (1) is defined as a function $x(.) \in D$ that satisfies:

(i) $x(0) = x_0$.

(ii) $\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m$.

(iii) The restriction of $x(t)$ to the interval J' is continuous and the following integral equation is satisfied:

$$\begin{aligned}x(t) = T(t)x_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) B u_x(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} T(t-s) B u_x(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) f(s, x(s), x(\varphi(x(s), s))) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} T(t-s) f(s, x(s), x(\varphi(x(s), s))) ds \\ &+ \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k)), \quad 0 \le s < t < b, \text{ and } t \neq t_k. \end{aligned} \tag{3}$$

Let $x_b(x_0; u)$ be the state value of (1) at terminal time b corresponding to the control u and the initial value x_0. Introduce the set

$$\mathfrak{R}(b; x_0) = \{x_b(x_0; u) : u \in L^2(J, U)\}$$

which is called the reachable set of system (1) at terminal time b, its closure in E is denoted by $\overline{\mathfrak{R}(b, x_0)}$.

Definition 2.6 The system (1) is said to be approximately controllable on J if $\overline{\mathfrak{R}(b; x_0)} = E$, that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the point x_0 to within a distance ϵ from all point in the state space E at time b.

Consider the linear fractional differential system

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + Bu(t), \quad t \in J = [0, b], \ 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{4}$$

is approximately controllable. It is convenient at this position to introduce the controllability operator associated with (4), thus

$$\Gamma_0^b x = \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} T(b-s) B B^* T^*(b-s) x ds, \text{ for } x \in E.$$

For $\lambda > 0$, we consider the relevant operator $\mathcal{R}(\lambda; \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1}$. It is convenient at this point to define the operators

$$\Gamma_{t_k}^b = \frac{1}{\Gamma(\alpha)} \int_{t_k}^b (b-s)^{\alpha-1} T(b-s) B B^* T^*(b-s) ds,$$

$$\Gamma_{t_{k-1}}^{t_k} = \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} T(t_k-s) B B^* T^*(t_k-s) ds,$$

$$\mathcal{R}(\lambda; \Gamma_{t_{k-1}}^{t_k}) = (\lambda I + \Gamma_{t_{k-1}}^{t_k})^{-1}, \text{for } \lambda > 0, k = 1, \cdots, m,$$

where B^* denotes the adjoint of B and $T^*(t)$ is the adjoint of $T(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

3 Main Result

This section deals with the existence and uniqueness of mild solutions and approximate controllability of the problem (1). Before stating and proving the main results, we introduce the following hypotheses:

$(H1)$ A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ in the Hilbert space E ($T(t)$ is compact for $t \geq 0$) and there exists a constant $\widehat{M} \geq 1$ such that

$$\|T(t)\| \leq \widehat{M} \quad \text{for every} \quad t \geq 0$$

has an inverse operator $T^{-1}(t)$ and there exists a positive constant $\widehat{M_1}$ such that

$$\|T^{-1}(t)\| \leq \widehat{M_1} \quad \text{for every} \quad t \geq 0.$$

$(H2)$ The nonlinear map $f : J \times E \times E \to E$ satisfies the Lipschitz condition such that there exist constants $M_1 = M_1(t, x, y, r) > 0$ and $M_2 = M_2(t, 0, x, r) > 0$, we have for all $x_i,\ y_i \in B_r,\ i = 1, 2.$

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq M_1 \left\{\|x_1 - x_2\| + \|y_1 - y_2\|\right\} \text{ for each } t \in J$$

and $\max\limits_{t\in J} \|f(t, 0, x(0))\| = M_2$.

$(H3)$ $\varphi : D \times \mathbb{R}_+ \to \mathbb{R}_+$ is globally continuous on $E \times \mathbb{R}_+$ and satisfies $\varphi(\cdot, 0) = 0$ and there exists a constant $L_\varphi = L_\varphi(x, t, r) > 0$ such that

$$|\varphi(x, t) - \varphi(y, s)| \leq L_\varphi \left\{\|x - y\| + |t - s|\right\}$$

for every $x, y \in B_r$, and $t, s \in J$.

$(H4)$ B is a bounded linear operator from U into E, such that $\|B\| = \widetilde{M}$, for a constant $\widetilde{M} > 0$.

$(H5)$ for each $0 \leq t < b$ and $t \neq t_k, k = 1, \cdots, m$ the operators $\lambda\mathcal{R}(\lambda; \Gamma_{t_k}^b) \to 0$ and $\lambda\mathcal{R}(\lambda; \Gamma_{t_{k-1}}^{t_k}) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

$(H6)$ There exist constants $d, L_k, \ell, d_k > 0,\ k = \overline{1, m}$, such that $\|I_k(\cdot)\| < d_k,\ \sum\limits_{k=1}^{m} d_k = d$,

$\|I_k(x) - I_k(y)\| \leq L_k\|x - y\|$, for every $x, y \in E$, and $\sum\limits_{k=1}^{m} L_k = \ell$.

For brevity, let ω_1, ω_2 be the positive numbers

$$\frac{\widehat{M}^2}{\lambda}\widetilde{M}\left(\|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1 + M_2) + \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1LL_\varphi(1 + L)\right) = \omega_1,$$

$$\frac{\widehat{M}^2}{m\lambda}\widehat{M_1}\widetilde{M}\left(\|\widetilde{z}_b\| + m\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1 + M_2) + m\frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1LL_\varphi(1 + L) + m\widehat{M}d\right) = \omega_2,$$

and put

$$\widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\widetilde{M}(m\omega_2 + \omega_1) + m\widehat{M}d + (m+1)\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\Big(rM_1 + M_2 + M_1LL_\varphi(1+L)\Big) \leq r, \tag{5}$$

where $r > 0$ is a constant.

We denote $B_r = \{x \in D;\ \|x(t)\| \leq r\}$.

Theorem 3.1 *Suppose* $(H1)-(H6)$ *and* (5) *hold, moreover, let*

$$\rho = \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widetilde{M}^2+1\Big) + m\Big(\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)+\widehat{M}\ell\Big)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widehat{M}_1\widetilde{M}^2+1\Big) \tag{6}$$

be such that $\rho \in (0,1)$. *Then the problem* (1) *is approximate controllable on* J.

In this section, it will be shown that the system (1) is approximately controllable if for all $\lambda > 0$, there exists a continuous function $x(\cdot) \in D$ such that

$$\begin{aligned} x(t) &= T(t)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds \\ &+ \sum_{0<t_k<t}T(t-t_k)I_k(x(t_k)) = \widehat{z}(t)+\widetilde{z}(t). \end{aligned} \tag{7}$$

For $k = 1,\cdots,m$, we put

$$\begin{aligned} \widehat{z}(t) &= T(t)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds. \\ \widetilde{z}(t) &= \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds \\ &+ \sum_{0<t_k<t}T(t-t_k)I_k(x(t_k)). \end{aligned}$$

Proof. Transform the problem (1) into a fixed-point problem. For $\lambda > 0$, we define the operators $F_\lambda, G_\lambda : D \to D$ as $(F_\lambda x + G_\lambda x) = x(t)$, where

$$\begin{aligned} F_\lambda x(t) &= T(t)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)B\mathfrak{u}(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds, \ \text{ for } k=1,\cdots,m. \end{aligned} \tag{8}$$

$$
\begin{aligned}
G_\lambda x(t) = &\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) B\mathfrak{v}(s) ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) f(s, x(s), x(\varphi(x(s), s))) ds \\
&+ \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k)), \ \text{ for } \ k = 1, \cdots, m.
\end{aligned} \tag{9}
$$

We take the controls

$$
\mathfrak{u}(t) = B^* T^*(b-t) \mathcal{R}(\lambda, \Gamma_{t_k}^b) p(x(\cdot)), \tag{10}
$$

$$
\mathfrak{v}(t) = \frac{1}{m} B^* T^*(t_k - t) T^{-1}(b - t_k) \mathcal{R}(\lambda, \Gamma_{t_{k-1}}^{t_k}) q(x(\cdot)), \tag{11}
$$

where, for $k = 1, \cdots, m$

$$
p(x(\cdot)) = \widehat{z}_b - T(b)x_0 - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{b} (b-s)^{\alpha-1} T(b-s) f(s, x(s), x(\varphi(x(s), s))) ds, \tag{12}
$$

$$
\begin{aligned}
q(x(\cdot)) = &\widetilde{z}_b - \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(b-s) f(s, x(s), x(\varphi(x(s), s))) ds \\
&- \sum_{k=1}^{m} T(b-t_k) I_k(x(t_k)),
\end{aligned} \tag{13}
$$

for any $\lambda > 0$, we shall show that $F_\lambda + G_\lambda$ has a fixed point on D, which is a mild solution of the system (1). Clearly, $(F_\lambda x + G_\lambda x)(b) = x_b = \widehat{z}_b + \widetilde{z}_b$.

From (10) and (11), we have

$$
\|\mathfrak{u}(t)\| \le \frac{\widehat{M}^2}{\lambda} \widetilde{M} \|p(x(\cdot))\|; \ \|\mathfrak{v}(t)\| \le \frac{\widehat{M}^2}{m\lambda} \widehat{M}_1 \widetilde{M} \|q(x(\cdot))\|,
$$

using $(H1) - (H6)$ and (2), we get

$$
\begin{aligned}
\|p(x(\cdot))\| \le &\|\widehat{z}_b\| + \|T(b)\| \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{b} (b-s)^{\alpha-1} \|T(b-s)\| \|f(s, x(s), x(\varphi(x(s), s)))\| ds \\
\le &\|\widehat{z}_b\| + \widehat{M} \|x_0\| + \frac{\widehat{M}}{\Gamma(\alpha)} \int_{t_k}^{b} (b-s)^{\alpha-1} \Big\{ \|f(s, x(s), x(\varphi(x(s), s))) - f(s, 0, x(0))\| \\
&+ \|f(s, 0, x(0))\| \Big\} ds \\
\le &\|\widehat{z}_b\| + \widehat{M} \|x_0\| + \frac{\widehat{M}}{\Gamma(\alpha)} M_1 \int_{t_k}^{b} (b-s)^{\alpha-1} \Big\{ \|x(s)\| \\
&+ \|x(\varphi(x(s), s)) - x(\varphi(x(0), 0))\| \Big\} ds + \frac{b^\alpha}{\Gamma(\alpha+1)} \widehat{M} M_2
\end{aligned}
$$

$$
\begin{aligned}
\|p(x(\cdot))\| &\leq \|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}M_1 L \int_{t_k}^{b}(b-s)^{\alpha-1}|\varphi(x(s),s)-\varphi(x(0),0)|ds \\
&\leq \|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}M_1 L L_{\varphi} \int_{t_k}^{b}(b-s)^{\alpha-1}\{\|x(s)-x(0)\|+|s|\}ds \\
&\leq \|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) + \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_{\varphi}(1+L),
\end{aligned}
$$

in the same way for $t_k < b, k = 1, \cdots, m$, we get

$$
\|q(x(\cdot))\| \leq \|\widetilde{z}_b\| + m\frac{b^{\alpha}}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) + m\frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_{\varphi}(1+L) + m\widehat{M}d.
$$

Thus there exist positive numbers ω_1, ω_2 such that

$$
\begin{aligned}
\|\mathfrak{u}(t)\| &\leq \frac{\widehat{M}^2}{\lambda}\widetilde{M}\Bigg(\|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) \\
&\qquad + \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_{\varphi}(1+L)\Bigg) = \omega_1, \\
\|\mathfrak{v}(t)\| &\leq \frac{\widehat{M}^2}{m\lambda}\widehat{M}_1\widetilde{M}\Bigg(\|\widetilde{z}_b\| + m\frac{b^{\alpha}}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) \\
&\qquad + m\frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_{\varphi}(1+L) + m\widehat{M}d\Bigg) = \omega_2.
\end{aligned} \tag{14}
$$

The proof will be given in two steps.

Step 1. $F_\lambda + G_\lambda$ maps B_r into itself.

Let $x \in B_r$. By (14), we have for each $t \in J$

$$
\begin{aligned}
\|(F_\lambda x)(t) + (G_\lambda x)(t)\| &\leq \|T(t)\|\|x_0\| + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\|\|B\|\|\mathfrak{v}(s)\|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|B\|\|\mathfrak{u}(s)\|ds \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\|\|f(s,x(s),x(\varphi(x(s),s)))\|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|f(s,x(s),x(\varphi(x(s),s)))\|ds \\
&+ \sum_{k=1}^{m}\|T(t-t_k)\|\|I_k(x(t_k))\|.
\end{aligned}
$$

Using the same method to find $\|p(x(\cdot))\|, \|q(x(\cdot))\|$ and (14), we get

$$\begin{aligned}\|(F_\lambda x + G_\lambda x)(t)\| &\le \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\widetilde{M}(m\omega_2+\omega_1) + m\widehat{M}d \\ &\quad + (m+1)\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\Big(rM_1 + M_2 + M_1 L L_\varphi(1+L)\Big) \\ &\le r.\end{aligned}$$

Thus, $F_\lambda + G_\lambda$ maps B_r into itself.

Step 2. We shall show now that the operator $F_\lambda + G_\lambda$ is a contraction mapping. Let $x, y \in B_r$. By (10) and (11), for each $t \in J$, we have

$$\begin{aligned}\|F_\lambda(x)(t) - F_\lambda(y)(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|B\|\|B^*\|\|T^*(b-t)\|\|\mathcal{R}(\lambda,\Gamma_{t_k}^b)\| \\ &\quad\times\left(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-\tau)^{\alpha-1}\|T(b-\tau)\|\right. \\ &\quad\times\left.\Big\{\|f(\tau,y(\tau),y(\varphi(y(\tau),\tau))) - f(\tau,x(\tau),x(\varphi(x(\tau),\tau)))\|\Big\}d\tau\right)ds \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|f(s,x(s),x(\varphi(x(s),s))) \\ &\qquad\qquad - f(s,y(s),y(\varphi(y(s),s)))\|\,ds \\ &\le \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\| \\ &\qquad\qquad + \|y(\varphi(y(s),s)) - x(\varphi(x(s),s))\|\Big\}ds\Big) \\ &\quad+\frac{\widehat{M}}{\Gamma(\alpha)}M_1\int_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\| \\ &\qquad\qquad + \|x(\varphi(x(s),s)) - y(\varphi(y(s),s))\|\Big\}ds \\ &\le \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\| \\ &\quad+\|y(\varphi(y(s),s)) - x(\varphi(y(s),s))\| \\ &\qquad\qquad + \|x(\varphi(y(s),s)) - x(\varphi(x(s),s))\|\Big\}ds\Big) \\ &\quad+\frac{\widehat{M}}{\Gamma(\alpha)}M_1\int_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\| \\ &\quad+\|x(\varphi(x(s),s)) - y(\varphi(x(s),s))\| \\ &\qquad\qquad + \|y(\varphi(x(s),s)) - y(\varphi(y(s),s))\|\Big\}ds\end{aligned}$$

$$
\begin{aligned}
\|F_\lambda(x)(t)-F_\lambda(y)(t)\| &\le \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\|\\
&+\|y(\varphi(y(s),s))-x(\varphi(y(s),s))\|+L|\varphi(y(s),s)-\varphi(x(s),s)|\Big\}ds\Big)\\
&+\frac{\widehat{M}}{\Gamma(\alpha)}M_1\int\limits_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\|\\
&+\|x(\varphi(x(s),s))-y(\varphi(x(s),s))\|+L|\varphi(x(s),s)-\varphi(y(s),s)|\Big\}ds\\
&\le \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\|\\
&+\|y(\varphi(y(s),s))-x(\varphi(y(s),s))\|+LL_\varphi\|y(s)-x(s)\|\Big\}ds\Big)\\
&+\frac{\widehat{M}}{\Gamma(\alpha)}M_1\int\limits_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\|\\
&+\|x(\varphi(x(s),s))-y(\varphi(x(s),s))\|+LL_\varphi\|x(s)-y(s)\|\Big\}ds\\
&\le \frac{b^{2\alpha}}{\lambda(\Gamma(\alpha+1))^2}\widehat{M}^4\widetilde{M}^2 M_1(2+LL_\varphi)\|y-x\|\\
&+\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)\|x-y\|\\
&\le \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widetilde{M}^2+1\Big)\|x-y\|.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|(G_\lambda x)(t)-(G_\lambda y)(t)\| &\le \frac{1}{m\Gamma(\alpha)}\sum_{k=1}^{m}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\|\|B\|\|B^*\|\|T^*(t_k-t)\|\\
&\times\|T^{-1}(b-t_k)\|\|\mathcal{R}(\lambda,\Gamma_{t_{k-1}}^{t_k})\|\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int\limits_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}\|T(b-\tau)\|\\
&\times\|f(\tau,y(\tau),y(\varphi(y(\tau),\tau)))-f(\tau,x(\tau),x(\varphi(x(\tau),\tau)))\|d\tau\\
&+\sum_{k=1}^{m}\|T(b-t_k)\|\|I_k(y(t_k))-I_k(x(t_k))\|\Bigg)ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\|\\
&\times\|f(s,x(s),x(\varphi(x(s),s)))-f(s,y(s),y(\varphi(x(s),s)))\|ds\\
&+\sum_{k=1}^{m}\|T(t-t_k)\|\|I_k(x(t_k))-I_k(y(t_k))\|\\
&\le m\Big(\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)+\widehat{M}\ell\Big)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widehat{M}_1\widetilde{M}^2+1\Big)\|x-y\|.
\end{aligned}
$$

So, we write

$$\begin{aligned}
\|(F_\lambda x+G_\lambda x)(t)-(F_\lambda+G_\lambda)(y)(t)\| &\le \|F_\lambda(x)(t)+F_\lambda(y)(t)\|+\|G_\lambda(x)(t)-G_\lambda(y)(t)\| \\
&\le \Bigg\{\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widetilde{M}^2+1\Big) \\
&+m\Big(\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)+\widehat{M}\ell\Big) \\
&\times\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widehat{M}_1\widetilde{M}^2+1\Big)\Bigg\}\|x-y\|.
\end{aligned}$$

From (6), we have

$$\|(F_\lambda+G_\lambda)(x)-(F_\lambda+G_\lambda)(y)\|\le\rho\|x-y\|.$$

So, for $\lambda>0$, we say the operator $F_\lambda+G_\lambda$ is a contraction mapping on B_r. Hence there exists a unique fixed point $x\in B_r$ such that $(F_\lambda x+G_\lambda x)(t)=x(t)$. The unique fixed point of $F_\lambda+G_\lambda$ is a mild solution of (1) on J, which satisfies $x(b)=x_b$. Hence, by the Banach contraction principle, the semilinear fractional system (1) is approximate controllable on J.

4 An Example

Throughout this section, we provide an illustrative example to demonstrate the effectiveness of the previously proven theoretical results using the heat equation, which is a parabolic partial differential equation, to describe the physical phenomenon of thermal conduction in a metal bar. Then, we consider an initial boundary value problem with time-fractional differential equation of the following form:

$$\begin{cases}
\dfrac{\partial^\alpha\nu}{\partial t^\alpha}(t,\varepsilon)=\dfrac{\partial^2\nu}{\partial\varepsilon^2}(t,\varepsilon)+\mu(t,\varepsilon)+\sin\big(|\nu(t,\varepsilon)|\big)+\big(1+e^{(\nu(t,\varepsilon))}\big)^\beta,\ \beta\in\mathbb{R}, \\
\nu(t,0)=\nu(t,1)=0,\quad t\in[0,b], \\
\nu(0,\varepsilon)=\nu_0(\varepsilon),\quad \varepsilon\in(0,1), \\
\Delta\nu(t_k)(\varepsilon)=\varepsilon\big(|\nu(t_k)(\varepsilon)|+e^{t_k}\big),\quad k=1,\cdots,m,
\end{cases}\tag{15}$$

where $\alpha\in(0,1)$, and $\mu:J\times(0,1)\to(0,1)$ is the control function and it is continuous.

- $\nu(t,\varepsilon)$ is the temperature at any point ε and any time t.
- $Q(t,\varepsilon)=\sin\big(|\nu(t,\varepsilon)|\big)+\big(1+e^{(\nu(t,\varepsilon))}\big)^\beta$ is the heat energy generated per unit volume per unit time.

  If $Q(t,\varepsilon)>0$, then the heat energy is being added to the system at that location and time, and if $Q(t,\varepsilon)<0$, then the heat energy is being removed from the system at that location and time.
- $\nu(t,0)$ and $\nu(t,1)$ are the temperatures at the ends of the bar. These are called the boundary conditions.
  To keep things simple, we will solve the IBVP (15) for the heat equation with $\nu(t,0)=\nu(t,1)=0\,°C$. These are called the homogeneous boundary conditions.

- $\nu(0,\varepsilon)$ is the initial temperature distribution. This is called the initial condition.
- $\Delta\nu(t_k)(\varepsilon)$ is the sudden instantaneous perturbation in heat distribution. This is called the impulsive condition.

One end of the bar is assumed to be at $\varepsilon_0 = 0$ and the other is at $\varepsilon_1 = 1$ (a long metal bar of length $|\varepsilon_0 - \varepsilon_1| = 1$). The bar is much longer than it is thick, so we can treat the distribution of heat as a function of just t and ε. Assuming that the bar specific heat capacity is known, we will know how heat is distributed if we can find a function for the temperature $\nu(t,\varepsilon)$.

Now, we will satisfy the previous assumptions and theoretical results using the IBVP (15) and get the required controllability.

Set $E = L^2\big[(0,1)\big]$, and $A : D(A) \subset E \to E$ is an operator defined by

$$A\omega = \omega^{''}, \quad \omega \in D(A)$$

with the domain

$$D(A) = \{\omega \in E; \omega, \omega^{'} \text{ are absolutely continuous, } \omega^{''} \in E, \omega(0) = \omega(1) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{2}\sin(nx), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A.

It is well known that A is a generator of an analytic semigroup $\big(T(t)\big)_{t\geq 0}$ in E which is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega,\omega_n)\omega_n, \quad \omega \in E,\ t > 0.$$

Further, for each $t \in J$, we have $T^*(t)x = T(t)x$, where $x \in E$.
Therefore, for $(t,\varepsilon) \in [0,b] \times (0,1)$, we have

$$\begin{gathered} x(t)(\varepsilon) = \nu(t,\varepsilon), \\ f(t,\nu(t),\nu(a(\nu(t),t)))(\varepsilon) = Q(t,\varepsilon), \\ I(\nu(t_k))(\varepsilon) = \varepsilon\big(\big|\nu(t_k)(\varepsilon)\big| + e^{t_k}\big), \quad k = 1,2,\cdots,m, \\ Bu(t)(\varepsilon) = \mu(t,\varepsilon). \end{gathered}$$

The system (15) is the abstract form of (1).

We define an infinite dimensional control space as

$$U = \{u : u = \sum_{n=2}^{\infty} u_n\omega_n, \sum_{n=2}^{\infty}|u_n|^2 < \infty\},$$

endowed with the norm $\|u\|_U = \big(\sum\limits_{n=2}^{\infty}|u_n|^2\big)^{1/2}$.

Let $B : U \to E$ and

$$Bu = 2u_2\omega_1 + \sum_{n=2}^{\infty} u_n\omega_n.$$

Then B is a bounded linear map and the adjoint is

$$B^*v = (2v_1 + v_2)\omega_2 + \sum_{n=3}^{\infty} v_n\omega_n.$$

Moreover,

$$B^*T^*(t)y = (2y_1e^{-t} + y_2e^{-4t})\omega_2 + \sum_{n=3}^{\infty} y_ne^{-n^2t}\omega_n$$

for $v = \sum\limits_{n=1}^{\infty} v_n\omega_n$ and $y = \sum\limits_{n=1}^{\infty} y_n\omega_n$. For $t \in J$, it can be shown that

$$\|B^*T^*(t)y\| = 0 \Rightarrow \|2y_1e^{-t} + y_2e^{-4t}\|^2 + \sum_{n=3}^{\infty} \|y_ne^{-n^2t}\|^2 = 0 \Rightarrow y = 0.$$

Therefore, by Theorem 4.1.7 [6], the linear system corresponding to (15) is approximately controllable. On the other hand, we have $\lambda\mathcal{R}(\lambda, \Pi_{t_k}^b) \to 0, \lambda\mathcal{R}(\lambda, \Pi_{t_{k-1}}^{t_k}) \to 0$, as $\lambda \to 0^+$, for $k = 1, \cdots, m$ in the strong operator topology, which is a necessary and sufficient condition for the linear system to be approximately controllable. Further, the conditions $(H1) - (H6)$ are satisfied. Hence, by Theorem 3.1, the IBVP (15) is approximate controllable on J.

5 Conclusions

This paper focuses on establishing the approximate controllability of an impulsive fractional semilinear system with deviated argument in Hilbert spaces through the application of one of the most important results of the analysis and it is considered the main source of the metric fixed point theory known as the "Banach Contraction Principle" that accompanied the formulation of a certain set of sufficient conditions. These ease the proof of the existence and uniqueness of the mild solution attached to the system under study.

In the future, we aim to expand this study by adapting some techniques used to other ideas and extracting new results that show the effectiveness of this study and its effect in the midst of scientific research. The closest result we would like to prove is the establishment of the approximate controllability of an impulsive stochastic differential system with deviated argument delay of fractional order.

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Asymptotic Behavior in Product and Conjugate Dynamical Systems Using Bi-Shadowing Properties

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Abstract: In this paper, we study the persistence of asymptotic behavior of trajectories generated by the product of discrete-time dynamical systems and also generated by conjugate systems as well, using some shadowing and bishadowing properties. In particular, we establish a relationship between the asymptotic behavior of product systems and their subsystems and give some new results in this direction. We also show that bishadowing is invariant for the systems that are topologically conjugate under certain conditions.

Keywords: *bishadowing; dynamical systems; product systems; conjugate systems.*

Mathematics Subject Classification (2010): 37C50.

1 Introduction

In recent years, the theory of shadowing has become a significant part of qualitative theory of dynamical systems. It plays an important role in the investigation of stability theory and asymptotic behavior of discrete systems, see also [2, 3]. It is usually used to verify computer calculations of the system by ensuring the existence of the true trajectory of the system close to the calculated trajectory (also known as the pseudo trajectory), see [4, 15]. Shadowing firstly appeared in the work of Anosov [6], see also Bowen [7] and is now developed as part of the global theory of dynamical systems, see Palmer [20] and Pilyugin [21]. The relationship between pseudo trajectories and true trajectories became more important particularly in chaotic systems.

Nevertheless, it is natural to pose the inverse problem: given a dynamical system and a family of approximate trajectories, is it possible, for any true trajectory to find

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a close approximate trajectory? In general, the answer to this problem is yes, but in practice, only approximate trajectories with specific properties are considered. This is known as inverse shadowing. In the last three decades, shadowing and inverse shadowing properties have been studied extensively by many authors and various extensions of these concepts have also been obtained and applied for dynamical systems in different ways. For example, the Lipschitz shadowing property [17], limit shadowing property [11], orbital and weak shadowing properties [19], shadowing property for maps on Banach spaces [14], asymptotic shadowing [21], average shadowing property for diffeomorphisms [22], pseudo–orbit tracing property for flows [16], weak inverse shadowing [8], inverse shadowing for set-valued systems [18] and continuous inverse shadowing [12]. A combination of the two concepts of direct and inverse (indirect) shadowing, called bishadowing, was introduced in [10], see also [9]. Bishadowing was considered for set-valued systems with an application to iterated function systems in [1] and for infinite dimensional dynamical systems in [5].

The paper is organized as follows. Some definitions and preliminaries needed throughout the paper will be given in Section 2. The results of bishadowing for the product of systems will be established in Section 3 and those for conjugate systems will be given in Section 4.

2 Definitions and Preliminaries

Let (X, d_X) be a metric space and let $f : X \to X$ be a continuous map from X into itself. In this paper, we shall consider the discrete-time dynamical system on X generated by f along with its iterates. That is,

$$f^0 = id_X \qquad \text{and} \qquad f^{i+1} = f^i \circ f, \qquad i = 0, 1, 2, \cdots$$

We usually identify the map f with the dynamical system that generates.

A sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying $x_{i+1} = f(x_i)$, $i = 0, 1, \cdots$ is called a true trajectory of (the system) f. While a sequence $\{y_i\}_{i=0}^{\infty} \subset X$ satisfying

$$d_X(f(y_i), y_{i+1})) \leq \delta$$

for $i = 0, 1, \cdots$ and for some $\delta > 0$ is called a δ-pseudo-trajectory of f. Note that a true trajectory is also a δ-pseudo trajectory with $\delta = 0$. The totality of all continuous maps from the metric space X into itself will be denoted by $C(X)$.

For a given $\varepsilon > 0$, a δ-pseudo trajectory $\{x_i\}_{i=0}^{\infty}$ of f is said to be ε-traced by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for all $i = 0, 1, 2, \cdots$. We say that the dynamical system generated by f has the shadowing property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo trajectory is ε-traced by some point $x \in X$. We say that the dynamical system has the inverse shadowing property if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any continuous map $\phi \in C(X)$ satisfying $\sup_{x \in X} d(f(x), \phi(x)) < \delta$ and for any $x \in X$ there exists a point $x' \in X$ for which $d(f^i(x'), \phi^i(x)) < \varepsilon$, $i = 0, 1, 2, \cdots$.

We now give the definition of bishadowing of [10] in the context of a metric space.

Definition 2.1 A continuous map $f : X \to X$ is said to be bishadowing on a subset K of X with positive parameters α and β, (or (α, β)-bishadowing), if for any given δ-pseudo trajectory $\mathbf{y} = \{y_i\}_{i=0}^{\infty}$ of f in the set K with $0 \leq \delta \leq \beta$ and any continuous comparison map $\Phi \in C(X)$ satisfying

$$\delta + \sup_{x \in X} d_X(\Phi(x), f(x)) \leq \beta, \tag{1}$$

there exists a trajectory $\mathbf{x} = \{x_i\}_{i=0}^{\infty}$ of Φ in K such that

$$d_X(x_i, y_i) \leq \alpha \left(\delta + \sup_{x \in X} d_X(\Phi(x), f(x)) \right) \tag{2}$$

for all i for which x_i and y_i are defined.

Note that the definition of bishadowing include both the direct shadowing by taking as a special case $\Phi = f$ and the inverse shadowing by taking $\delta = 0$ in Definition 2.1.

3 The Product System

Throughout this section, we consider two compact metric spaces (X, d_X) and (Y, d_Y) and their product metric space $X \times Y$ with metric defined by

$$D((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

for $(x_1, y_1), (x_2, y_2) \in X \times Y$. We also consider continuous maps $f : X \to X$, $g : Y \to Y$, and the product $f \times g : X \times Y \to X \times Y$, where the product map is defined by $(f \times g)(x, y) = (f(x), g(y))$.

The proof of the following lemma is straightforward, so will be omitted.

Lemma 3.1 *Let $\psi : X \to X$ and $\varphi : Y \to Y$ be two continuous maps and let $a = \sup_{x \in X} d_X(f(x), \psi(x))$ and $b = \sup_{y \in Y} d_Y(g(y), \varphi(y))$, then for $x \in X$ and $y \in Y$ we have*

$$\sup_{x \in X, y \in Y} \{\max\{d_X(f(x), \psi(x)), d_Y(g(y), \varphi(y))\}\} = \max\{a, b\}.$$

Theorem 3.1 *Assume that both f and g are (α, β)- bishadowing with respect to the comparison classes $C(X)$ and $C(Y)$, respectively. Then the product system $f \times g$ is (α, β)-bishadowing with respect to the class $C(X) \times C(Y)$.*

Proof. Let $\{(x_i, y_i)\}_{i=0}^{\infty}$ be a δ-pseudo trajectory of the map $f \times g$, with $0 \leq \delta \leq \beta$, and let $\psi \times \varphi \in C(X) \times C(Y)$ satisfying

$$\delta + \sup_{(x,y) \in X \times Y} D((f \times g)(x, y), (\psi \times \varphi)(x, y)) \leq \beta. \tag{3}$$

Since $\{(x_i, y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of the map $f \times g$ we have, for each $i = 0, 1, \cdots$, that

$$\begin{aligned} \max\{d_X(f(x_i), x_{i+1}), d_Y(g(y_i), y_{i+1})\} &= D((f(x_i), g(y_i)), (x_{i+1}, y_{i+1})) \\ &= D((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})) < \delta. \end{aligned}$$

Thus, $d_X(f(x_i), x_{i+1}) < \delta$ and $d_Y(g(y_i), y_{i+1}) < \delta$, for $i = 0, 1, \cdots$, which implies that both $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are δ-pseudo trajectories of f and g, respectively. Since both f and g are bishadowing, then for any $\psi \in C(X)$ and $\varphi \in C(Y)$ satisfying

$$\delta + \sup_{x \in X} d_X\left(f(x), \psi(x)\right) \leq \beta \tag{4}$$

and

$$\delta + \sup_{y \in Y} d_Y\left(g(y), \varphi(y)\right) \leq \beta, \tag{5}$$

there exist true trajectories $\{w_i\}_{i=0}^{\infty}$ of ψ and $\{z_i\}_{i=1}^{\infty}$ of φ such that

$$d_X(x_i, w_i) \le \alpha \left(\delta + \sup_{x\in X} d_X(f(x), \psi(x)\right) \tag{6}$$

and

$$d_Y(y_i, z_i) \le \alpha \left(\delta + \sup_{y\in Y} d_Y(g(y), \varphi(y)\right), \text{ for } i = 0, 1, \cdots . \tag{7}$$

We may assume $\sup_{x\in X} d_X(f(x), \psi(x)) > \sup_{y\in Y} d_Y(g(y), \varphi(y))$, as the other direction can be treated similarly. We consider the following three cases.

Case 1: For the values of i, for which $d_X(x_i, w_i) > d_Y(y_i, z_i)$, and using the relation (6), we have

$$\begin{aligned} D((x_i, y_i), (w_i, z_i)) &= \max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} = d_X(x_i, w_i) \\ &\le \alpha \left(\delta + \sup_{x\in X} d_X(f(x), \psi(x))\right). \end{aligned}$$

So, for every $x \in X$, $y \in Y$, and using Lemma 3.1, we have

$$\begin{aligned} D((x_i, y_i), (w_i, z_i)) &\le \alpha \left(\delta + \sup_{x\in X} d_X(f(x), \psi(x))\right) \\ &= \alpha \left(\delta + \max\left\{\sup_{x\in X} d_X(f(x), \psi(x)), \sup_{y\in Y} d_Y(g(y), \varphi(y))\right\}\right) \\ &= \alpha \left(\delta + \sup_{x\in X, y\in Y} \max\{d_X(f(x), \psi(x)), d_Y(g(y), \varphi(y))\}\right) \\ &= \alpha \left(\delta + \sup_{(x,y)\in X\times Y} D((f\times g)(x, y), (\psi \times \varphi)(x \times y))\right). \end{aligned}$$

Case 2: For the values of i, for which $d_X(x_i, w_i) < d_Y(y_i, z_i)$, and using the relation (7), we have

$$\begin{aligned} D((x_i, y_i), (w_i, z_i)) &= \max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} = d_Y(y_i, z_i) \\ &\le \alpha \left(\delta + \sup_{y\in Y} d_Y(g(y), \varphi(y))\right) \\ &\le \alpha \left(\delta + \sup_{x\in X} d_X(f(x), \psi(x))\right). \end{aligned}$$

From the argument of **Case 1** above we obtain

$$D((x_i, y_i), (w_i, z_i)) \le \alpha \left(\delta + \sup_{(x,y)\in X\times Y} D((f\times g)(x, y), (\psi \times \varphi)(x \times y))\right).$$

Case 3: For the values of i, for which $d_X(x_i, w_i) = d_Y(y_i, z_i)$, we have the same result as in **Case 1** and **Case 2**.

By combining the three cases, we have

$$D((x_i, y_i), (w_i, z_i)) \le \alpha \left(\delta + \sup_{(x,y)\in X\times Y} D((f\times g)(x, y), (\psi \times \varphi)(x \times y))\right), \ i = 0, 1, \cdots .$$

Finally, the sequence $\{(w_i, z_i)\}_{i=0}^{\infty}$ is a true trajectory of $\psi \times \varphi$ since

$$(\psi \times \varphi)(w_i, z_i) = (\psi(w_i), \varphi(z_i)) = (w_{i+1}, z_{i+1}), \quad i = 0, 1, \cdots .$$

This means that $f \times g$ is (α, β)- bishadowing with respect to the class $C(X) \times C(Y)$. This ends the proof of Theorem 3.1. $\Box$

For the converse direction of Theorem 3.1, we have the following partial result.

Theorem 3.2 *Assume that f and g are both continuous and that the product system $f \times g$ is (α, β)- bishadowing with respect to the class $C(X) \times C(Y)$. Then at least one of the maps f and g is (α, β)-bishadowing with respect to the corresponding class of comparison maps.*

Proof. Let $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ be δ-pseudo trajectories of f and g, respectively, with $0 \leq \delta \leq \beta$, and let $\psi \in C(X)$ and $\varphi \in C(Y)$ satisfy the relations (4) and (5) respectively. Then for $i = 0, 1, \cdots$ we have the following estimates:

$$\begin{aligned} D((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})) &= D((f(x_i), g(y_i)), (x_{i+1}, y_{i+1})) \\ &= \max\{d_X(f(x_i), x_{i+1}), d_Y(g(y_i), y_{i+1})\} < \delta. \end{aligned}$$

Thus, the sequence $\{(x_i, y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of $f \times g$ and since $f \times g$ is bishadowing, for any map $\psi \times \varphi \in C(X) \times C(Y)$ satisfying the relation (3) there exists a true trajectory $\{(w_i, z_i)\}_{i=0}^{\infty}$ of $\psi \times \varphi$ such that

$$D((x_i, y_i), (w_i, z_i)) \leq \alpha \left(\delta + \sup_{(x,y) \in X \times Y} D\left((f \times g)(x, y), (\psi \times \varphi)(x, y)\right) \right), \; i = 0, 1, \cdots .$$

So, for every $x \in X$, $y \in Y$, and using Lemma 3.1, we have

$$\begin{aligned} &\max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} \\ &\quad \leq \alpha \left(\delta + \sup_{(x,y) \in X \times Y} D((f \times g)(x, y), (\psi \times \varphi)(x, y)) \right) \\ &\quad = \alpha \left(\delta + \sup_{(x,y) \in X \times Y} \max\{d_X(f(x), \psi(x)), d_Y(g(y), \varphi(y))\} \right) \\ &\quad = \alpha \left(\delta + \max\left\{ \sup_{x \in X} d_X(f(x), \psi(x)), \sup_{y \in Y} d_Y(g(y), \varphi(y)) \right\} \right). \end{aligned}$$

These estimates imply the following three cases.

Case 1: If $\sup_{x \in X} d_X(f(x), \psi(x)) > \sup_{y \in Y} d_Y(g(y), \varphi(y))$, then we have

$$d_X(x_i, w_i) \leq \alpha \left(\delta + \sup_{x \in X} d_X(f(x), \psi(x)) \right), \quad i = 0, 1, \cdots , \tag{8}$$

and hence f is (α, β)- bishadowing with respect to $C(X)$.

Case 2: If $\sup_{x \in X} d_X(f(x), \psi(x)) < \sup_{y \in Y} d_Y(g(y), \varphi(y))$, then we have

$$d_X(y_i, z_i) \leq \alpha \left(\delta + \sup_{y \in Y} d_Y(g(y), \varphi(y)) \right), \quad i = 0, 1, \cdots , \tag{9}$$

and hence g is (α, β)-bishadowing with respect to $C(Y)$.

Case 3: If $\sup_{x\in X} d_X(f(x), \psi(x)) = \sup_{y\in Y} d_Y(g(y), \varphi(y))$, then the relations in (8) and (9) are both satisfied, and consequently, both f and g are (α, β)-bishadowing with respect to $C(X)$ and $C(Y)$, respectively.

Finally, it should be mentioned that both $\{w_i\}_{i=0}^{\infty}$ and $\{z_i\}_{i=0}^{\infty}$ are true trajectories of ψ and φ, respectively, since

$$(\psi(w_i), \varphi(z_i)) = (\psi \times \varphi)(w_i, z_i) = (w_{i+1}, z_{i+1}), \quad i = 0, 1, \cdots .$$

This completes the proof of Theorem 3.2. $\square$

Now, we introduce the following definition of *mutual* bishadowing for a pair of systems generated by the continuous maps $f : X \to X$ and $g : Y \to Y$ and then establish a result that is related to the converse of Theorem 3.1.

Definition 3.1 The pair of systems (f, g) is called mutually bishadowing with respect to the comparison classes $C(X)$ and $C(Y)$ of f and g, respectively, with positive parameters α and β (or mutually (α, β)-bishadowing), if for any given δ-pseudo trajectories $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ for f and g, respectively, and for any $\psi \in C(X)$ and $\varphi \in C(Y)$ satisfying the relations (4) and (5), respectively, there exist true trajectories $\{w_i\}_{i=1}^{\infty}$ of ψ and $\{z_i\}_{i=1}^{\infty}$ of φ such that

$$d_X(x_i, w_i) \leq \alpha \left(\delta + \max \left\{ \sup_{x\in X} d_X(f(x), \psi(x)), \sup_{y\in Y} d_Y(g(y), \varphi(y)) \right\} \right) \tag{10}$$

and

$$d_X(y_i, z_i) \leq \alpha \left(\delta + \max \left\{ \sup_{x\in X} d_X(f(x), \psi(x)), \sup_{y\in Y} d_Y(g(y), \varphi(y)) \right\} \right) \tag{11}$$

for $i = 0, 1, \cdots$ and $x \in X$, $y \in Y$.

In the context of the preceding definition of mutually bishadowing for a pair of systems generated by the continuous maps $f : X \to X$ and $g : Y \to Y$, we have the following result, which is an improved version of Theorem 3.2.

Theorem 3.3 *The pair of systems (f, g) is mutually (α, β)-bishadowing with respect to the comparison classes $C(X)$ and $C(Y)$ of f and g, respectively, and with positive parameters α and β if and only if $f \times g$ is bishadowing with respect to the comparison class $C(X) \times C(Y)$.*

Proof. Let $\{(x_i, y_i)\}_{i=0}^{\infty}$ be a δ-pseudo trajectory of the map $f \times g$, with $0 \leq \delta \leq \beta$, and let $\psi \times \varphi \in C(X) \times C(Y)$ satisfying the relation (3). It follows from the proof of Theorem 3.1 that both $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are δ-pseudo trajectories of f and g, respectively. Since the pair of systems (f, g) is mutually bishadowing, then for any $\psi \in C(X)$ and $\varphi \in C(Y)$ satisfying the relations (4) and (5) there exist true trajectories $\{w_i\}_{i=0}^{\infty}$ of ψ and $\{z_i\}_{i=0}^{\infty}$ of φ satisfying the relations (10) and (11). Thus, using Lemma 3.1 and from the preceding estimates we obtain for $x \in X$ and $y \in Y$ and for $i = 0, 1, \cdots$

that

$$\begin{aligned}
&\max\{d_X(x_i,w_i),d_Y(y_i,z_i)\}\\
&\qquad\le \alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right)\\
&\qquad= \alpha\left(\delta+\sup_{(x,y)\in X\times Y}\{\max\{d_X(f(x),\psi(x)),d_Y(g(y),\varphi(y))\}\right)\\
&\qquad= \alpha\left(\delta+\sup_{(x,y)\in X\times Y} D((f\times g)(x,y),(\psi\times\varphi)(x,y))\right).
\end{aligned}$$

Therefore

$$D((x_i,y_i),(w_i,z_i))\le \alpha\left(\delta+\sup_{(x,y)\in X\times Y} D((f\times g)(x,y),(\psi\times\varphi)(x,y))\right),\ i=0,1,\cdots.$$

This shows that the map $f\times g$ is (α,β)- bishadowing with respect to the comparison class $C(X)\times C(Y)$.

Conversely, let $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ be δ-pseudo trajectories of f and g, respectively, with $0\le\delta\le\beta$, and let $\psi\in C(X)$ and $\varphi\in C(Y)$ satisfy the relations (4) and (5), respectively. From the proof of Theorem 3.2 it follows that the sequence $\{(x_i,y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of the map $f\times g$. Since the map $f\times g$ is (α,β)-bi- shadowing, for any map $\psi\times\varphi\in C(X)\times C(Y)$ satisfying the relation (3) there exists a true trajectory $\{(w_i,z_i)\}_{i=0}^{\infty}$ of $\psi\times\varphi$ such that

$$D((x_i,y_i),(w_i,z_i))\le \alpha\left(\delta+\sup_{(x,y)\in X\times Y} D((f\times g)(x,y),(\psi\times\varphi)(x,y))\right),\ i=0,1,\cdots.$$

Thus

$$\begin{aligned}
\max\{d_X(x_i,w_i),d_Y(y_i,z_i)\}\le \alpha\Bigg(\delta+\max\Bigg\{&\sup_{x\in X} d_X(f(x),\psi(x)),\\
&\sup_{y\in Y} d_Y(g(y),\varphi(y))\Bigg\}\Bigg),\ i=0,1,\cdots.
\end{aligned}$$

This implies that

$$d_X(x_i,w_i)\le \alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right),$$

and

$$d_X(y_i,z_i)\le \alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right),\ i=0,1,\cdots.$$

Hence, the pair of systems (f,g) is mutually (α,β)- bishadowing. This ends the proof of Theorem 3.3. □

Finally, for the continuous maps $f:X\to X$ and $g:Y\to Y$, we relate mutual bishadowing of the pair of systems (f,g) with bishadowing of f and g.

Corollary 3.1 *If both f and g are (α, β)- bishadowing with respect to the classes $C(X)$ and $C(Y)$, respectively, then the pair of systems (f, g) is mutually (α, β)-bishadowing with respect to $C(X)$ and $C(Y)$.*

Proof. The proof follows from Theorem 3.1 and Theorem 3.3. □

Corollary 3.2 *If the pair of systems (f, g) is mutually (α, β)-bishadowing with respect to $C(X)$ and $C(Y)$, respectively, then at least one of the two maps f and g is (α, β)-bishadowing with the same corresponding classes.*

Proof. The proof follows from Theorem 3.2 and Theorem 3.3. □

4 Topologically Conjugate Systems

If (X, d_X) and (Y, d_Y) are two metric spaces and $f : X \to X$ and $g : Y \to Y$ are two maps on X and Y, respectively, then we say that f and g are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. If, in addition, h is uniformly continuous, then f and g are called uniformly topologically conjugate, or simply uniformly conjugate. The two classes $C(X)$ and $C(Y)$ are called $(C(X), C(Y))$-topologically conjugate by h in the sense that individual maps in one class are topologically conjugate to a map in the other class by h.

The following two results on invariance of the shadowing and inverse shadowing properties for topologically conjugate systems are standard in the theory of shadowing, see, for example, [23].

Theorem 4.1 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be two continuous maps. If f and g are uniformly conjugate then f has the shadowing property if and only if g has the shadowing property.*

Theorem 4.2 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be two continuous maps. Assume that $C(X)$ and $C(Y)$ are $(C(X), C(Y))$-topologically conjugate. If f and g are uniformly conjugate, then f has the inverse shadowing property with respect to the class $C(X)$ if and only if g has the inverse shadowing property with respect to the class $C(Y)$.*

For the invariance of (α, β)-bishadowing for topologically conjugate systems we have the following result.

Theorem 4.3 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be two continuous maps that are topologically conjugate by $h : X \to Y$. Assume also that $C(X)$ and $C(Y)$ are $(C(X), C(Y))$-topologically conjugate. If there exists $\lambda \geq 1$ such that*

$$d_X(x', x'') \leq d_Y(h(x'), h(x'')) \leq \lambda\, d_X(x', x''), \qquad \text{for all } x', x'' \in X, \tag{12}$$

then we have

- **a)** *If f is (α, β)-bishadowing with respect to the comparison class $C(X)$, then g is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(Y)$.*
- **b)** *If g has the $(\alpha, \lambda\beta)$-bishadowing with respect to $C(Y)$, then f is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(X)$.*

Proof. a) Let $\{y_i\}_{i=1}^{\infty}$ be a δ-pseudo trajectory of g with $0 \leq \delta \leq \beta$, which implies that $d_Y(g(y_i), y_{i+1}) < \delta$, and let $\phi \in C(Y)$ satisfy

$$\delta + \sup_{y \in Y} d_Y(g(y), \phi(y)) \leq \beta. \tag{13}$$

Note that condition (12) is equivalent to the following condition:

$$d_X(h^{-1}(y'), h^{-1}(y'')) \leq d_Y(y', y'') \leq \lambda d_X(h^{-1}(y'), h^{-1}(y'')) \qquad \text{for all } y', y'' \in Y. \tag{14}$$

Now,

$$d_X(f(h^{-1}(y_i)), h^{-1}(y_{i+1})) = d_X(h^{-1}(g(y_i)), h^{-1}(y_{i+1})) \leq d_Y(g(y_i), y_{i+1}) < \delta.$$

Hence $\{x_i\}_{i=0}^{\infty} = \{h^{-1}(y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of f. Let $x \in X$ and $\psi \in C(X)$, then using (12) and the conjugacy h, we obtain

$$d_X(f(x), \psi(x)) \leq d_Y(h(f(x)), h(\psi(x))) = d_Y(g(h(x)), \phi(h(x))) = d_Y(g(y), \phi(y)),$$

where $y = h(x)$. This implies that

$$\sup_{x \in X} d_X(f(x), \psi(x)) \leq \sup_{y \in Y} d_Y(g(y), \phi(y)). \tag{15}$$

From(13) and(15) and for any $\psi \in C(X)$, where $\psi = h^{-1} \circ \phi \circ h$, we have $\delta + \sup_{x \in X} d_X(f(x), \psi(x)) \leq \beta$. Since f is (α, β)-bishadowing then there exists a true trajectory $\{w_i\}_{i=1}^{\infty}$ of ψ such that

$$d_X(x_i, w_i) \leq \alpha(\delta + \sup_{x \in X} d_X(f(x), \psi(x))) \leq \alpha(\delta + \sup_{y \in Y} d_Y(g(y), \phi(y))).$$

Thus, using the second part of (12), we obtain

$$d_Y(y_i, h(w_i)) = d_Y(h(x_i), h(w_i)) \leq \lambda\alpha(\delta + \sup_{y \in Y} d_Y(g(y), \phi(y))).$$

Note that $\{h(w_i)\}_{i=0}^{\infty} := \{a_i\}_{i=0}^{\infty}$ is a true trajectory of ϕ since

$$\phi(a_i) = \phi(h(w_i)) = h(\psi(w_i)) = h(w_{i+1}) = a_{i+1}.$$

This ends the proof that g is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(Y)$. □

Proof. b) Let $\{x_i\}_{i=1}^{\infty}$ be a δ-pseudo trajectory of f with $0 \leq \delta \leq \beta$, which implies that $d_X(f(x_i), x_{i+1}) < \delta$, and let $\phi \in C(Y)$ satisfy

$$\delta + \sup_{x \in X} d_X(f(x), \phi(x)) \leq \beta. \tag{16}$$

Now, $d_Y(g(h(x_i)), h(x_{i+1})) = d_Y(h(f(x_i)), h(x_{i+1})) \leq \lambda d_X(f(x_i), x_{i+1})$, hence

$$d_Y(g(h(x_i)), h(x_{i+1})) < \lambda\delta.$$

It follows that $\{y_i\}_{i=0}^{\infty} = \{h(x_i)\}_{i=0}^{\infty}$ is a $\lambda\delta$-pseudo trajectory of g. Now let $y \in Y$ and $\psi \in C(Y)$, then using (14) we have

$$\begin{aligned} d_Y(g(y), \varphi(y)) &\leq \lambda d_X(h^{-1}(g(y)), h^{-1}(\psi(y))) \\ &= \lambda d_X(f(h^{-1}(y)), \phi(h^{-1}(y))) = \lambda d_X(f(x), \phi(x)), \end{aligned}$$

where $x = h^{-1}(y)$. Therefore $d_Y(g(y), \psi(y)) \leq \lambda d_X(f(x), \phi(x))$ for every $y \in Y$ and $x = h^{-1}(y)$, hence

$$\sup_{y\in Y} d_Y(g(y), \psi(y)) \leq \lambda \sup_{x\in X} d_X(f(x), \phi(x)). \tag{17}$$

From (16) and (17) we get for any $\psi \in C(Y)$, where $\psi = h \circ \phi \circ h^{-1}$, that

$$\lambda\delta + \sup_{y\in Y} d_Y(g(y), \psi(y)) \leq \lambda\beta.$$

Since g has the $(\alpha, \lambda\beta)$-bishadowing, there exists a true trajectory $\{z_i\}_{i=1}^{\infty}$ of ψ such that

$$\begin{aligned} d_Y(y_i, z_i) \leq \alpha(\lambda\delta + \sup_{y\in Y} d_Y(g(y), \psi(y))) &\leq \alpha(\lambda\delta + \lambda \sup_{x\in X} d_X(f(x), \phi(x))) \\ &= \lambda\alpha(\delta + \sup_{x\in X} d_X(f(x), \phi(x))). \end{aligned}$$

Thus, using (13) we obtain

$$d_X(x_i, h^{-1}(z_i)) = d_X(h^{-1}(y_i), h^{-1}(z_i)) \leq \lambda\alpha(\delta + \sup_{x\in X} d_X(f(x), \phi(x))),$$

Note that $\{h^{-1}(z_i)\}_{i=0}^{\infty} = \{b_i\}_{i=0}^{\infty}$ is a true trajectory of ϕ since

$$\phi(b_i) = \phi(h^{-1}(z_i)) = h^{-1}(\psi(z_i)) = h^{-1}(z_{i+1}) = b_{i+1}.$$

This shows that f is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(X)$. □

Remark 4.1 In the preceding theorem, if $\lambda = 1$, then we conclude that the map $f : X \to X$ is (α, β)-bishadowing if and only if $g : Y \to Y$ is (α, β)-bishadowing.

5 Conclusion

We have obtained some results regarding the asymptotic behavior in product and conjugate dynamical systems using bishadowing properties in Theorems 3.1, 3.2, 3.3, Corollaries 3.1, 3.2 and Theorem 4.3. These results generalize some existing results, for example, Theorems 4.1 and 4.2. This is due to the fact that the concept of bishadowing relies on two ways of comparing the trajectories of the system, unlike the case of using the direct shadowing or the inverse shadowing only.

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The Qualitative Analysis of an n-Dimensional Nonlinear Dynamical System Arising From the Modeling of Multilayer Scales on Pure Metals

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Abstract: A metal oxide is a compound containing oxygen and metal. Certain pure metals can form different oxides, and oxidation of such metals produces a multilayer oxide scale on the metal. In one of their publications, F. Gesmundo and F. Viani qualitatively analyzed the parabolic growth of three-layer oxide scales on those metals which can form three oxides. They obtained a non-linear three-dimensional dynamical system as a model for the growth of such scales. In the present paper we generalize this dynamical system of Gesmundo and Viani to n-dimensions; we then qualitatively analyze this n-dimensional dynamical system.

Keywords: *differential equations; dynamical systems; nonlinear dynamical systems; cooperative dynamical systems.*

Mathematics Subject Classification (2010): 34C35, 70K05.

1 Introduction

A metal oxide is a compound containing oxygen and metal. For instance, common rust is caused by the oxidation of metal. Certain pure metals can form different oxides, and oxidation of such metals produces a multilayer oxide scale on the metal, where the oxide layer containing the highest concentration of metal is in contact with the surface of the metal, while the oxide layer containing the highest concentration of oxygen is in contact with the gas or oxygen to which the surface of the metal is exposed. In paper [4], F. Gesmundo and F. Viani analyzed the parabolic growth of three-layer oxide scales on

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those metals which can form three oxides. They obtained the following non-linear three-dimensional dynamical system as a model for the growth of such scales:

$$\begin{aligned}\dot{q}_1 &= m_1\frac{K_1}{2q_1} - \frac{m_1-1}{m_1}\frac{K_2}{2q_2}, \\ \dot{q}_2 &= -m_1\frac{K_1}{2q_1} + \left(\frac{m_1-1}{m_1} + \frac{m_2}{m_1}\right)\frac{K_2}{2q_2} - \frac{m_2-1}{m_2}\frac{K_3}{2q_3}, \\ \dot{q}_3 &= -\frac{m_2}{m_1}\frac{K_2}{2q_2} + \frac{K_3}{2q_3}.\end{aligned} \tag{1}$$

Here $K_i > 0$ $(i = 1, 2, 3)$ are rate constants, m_1, m_2 are parameters, $q_i > 0$ is the weight of oxygen contained in oxide i per unit area, and $\dot{q}_i$ $(i = 1, 2, 3)$ is the derivative of q_i with respect to time, t.

In the present paper we present the following n-dimensional generalization of the 3-dimensional system (1). This n-dimensional dynamical system models the parabolic growth of n-oxide scales on pure metals

$$\begin{aligned}\dot{q}_1 &= m_1\frac{K_1}{2q_1} - \frac{m_1-1}{m_1}\frac{K_2}{2q_2}, \\ &\dots \\ \dot{q}_i &= -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2q_{i-1}} + \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2q_i} - \frac{m_i-1}{m_i}\frac{K_{i+1}}{2q_{i+1}}, \quad 1 < i < n, \\ &\dots \\ \dot{q}_n &= -\frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2q_{n-1}} + \frac{K_n}{2q_n}.\end{aligned} \tag{2}$$

Here, for $i = 1, ..., n$, $K_i > 0$ are rate constants, m_i are parameters (with $m_0 = 1$), and q_i is the weight of oxygen contained in oxide i per unit area.

Theorem 1.1 below is the main result of the present paper; this theorem provides a qualitative analysis of the n-dimensional system (2).

Theorem 1.1 *Assume that in the dynamical system (2), we have* $n \geq 3$, *and* $m_i > 1$, $i = 1, ..., n$. *Then every solution* $\mathbf{p}[0, a] \to [0, a] \to \mathbf{R}^n_{++}$, $0 < a < +\infty$, *of (2) extends uniquely to a solution* $\mathbf{p} : [0, +\infty) \to \mathbf{R}^n_{++}$ *such that* $\lim_{t\to+\infty} p_i(t) = +\infty$, $i = 1, ..., n$, *and this solution is eventually monotone strictly increasing on* $[0 + \infty)$. *Moreover, the system (2) has a unique parabolic solution* $q_i(t) = c_i\sqrt{t}$, $c_i > 0$, $i = 1, ..., n$, $0 < t < +\infty$. *Finally, if* $\mathbf{p} : [0, +\infty) \to \mathbf{R}^n_{++}$ *is any other solution of (2), then*

$$\lim_{t\to+\infty} \|\mathbf{p}(t) - \mathbf{q}(t)\| = 0.$$

2 Preliminaries

In [1], [2], and [3], the present author (*et al.*) studied the following n-dimensional non-linear dynamical system, of which (1) and (2) are special cases:

$$\dot{q}_i = -\sum_{j=1}^{n} \frac{a_{ij}}{q_j}, \quad q_i(t) > 0, \quad i = 1, \dots, n. \tag{3}$$

In [3], we established the following key result ([3], Corollary I).

Theorem 2.1 *Assume that the $n \times n$ matrix $A = (a_{ij})$ in (3) satisfies the following four conditions:*

(i) $\det A \neq 0$ *and* $a_{ij} \geq 0$, *for* $i \neq j$;

(ii) *A is irreducible;*

(iii) *for all* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n_+$, *if* $x_i \sum_{j=1}^n a_{ij} x_j = 0$ *for* $i = 1, \ldots, n$, *then* $\mathbf{x} = 0$;

(iv) *every real eigenvalue of A is negative.*

Then every solution of (3) of the form

$$\mathbf{q} = (q_1, \ldots, q_n) : [0, a] \to \mathbf{R}^n_{++}, \quad 0 < a < +\infty,$$

extends uniquely to a solution

$$\mathbf{q} : [0, +\infty) \to \mathbf{R}^n_{++},$$

such that

$$\lim_{t \to +\infty} q_i(t) = \infty, \quad i = 1, \ldots, n.$$

Moreover, if $\mathbf{r}(t)$, $t \in [0, +\infty)$, *is any other solution of (3) in* $\mathbf{R}^n_{++}$, *then we have*

$$\lim_{t \to +\infty} \| \mathbf{q}(t) - \mathbf{r}(t) \| = 0.$$

Finally, if the matrix A is tridiagonal, then any solution solution, $\mathbf{q}(t)$, $t \in [0, +\infty)$ *of (3) in* $\mathbf{R}^n_{++}$ *is eventually monotonically increasing on* $[0, +\infty)$.

Definition 2.1 Let $A = (a_{ij})$ be the tridiagonal matrix whose entries are defined as follows, where the index i has range $1 < i < n$,

$$\begin{aligned}
a_{11} &= -\frac{m_1 K_1}{2}, & a_{12} &= \frac{m_1 - 1}{m_1} \frac{K_2}{2}; \\
a_{i,i-1} &= \frac{m_{i-1}}{m_{i-1}} \frac{K_{i-1}}{2}, & a_{i,i} &= -\left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}} \right) \frac{K_i}{2}, \quad a_{i,i+1} = \frac{m_i - 1}{m_i} \frac{K_{i+1}}{2}; \\
a_{n,n-1} &= \frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2}, & a_{n,n} &= -\frac{K_n}{2}.
\end{aligned}$$

Theorem 2.2 *Let* $(x_1, x_2, \cdots, x_n) \in \mathbf{R}_+$ *be arbitrary. Let* $1 < j < n$, *and assume that the following equations hold:*

$$\begin{aligned}
0 &= \frac{m_1 K_1}{2} x_1 - \frac{m_1 - 1}{m_1} \frac{K_2}{2} x_2, \qquad (4) \\
0 &= -\frac{m_1 K_1}{2} x_1 + \left(\frac{m_1 - 1}{m_1} + \frac{m_2}{m_1} \right) \frac{K_2}{2} x_2 - \frac{m_2 - 1}{m_2} \frac{K_3}{2} x_3, \\
&\cdots \\
0 &= -\frac{m_{j-1}}{m_{j-2}} \frac{K_{j-1}}{2} x_j + \left(\frac{m_{j-1} - 1}{m_{j-1} i} + \frac{m_j}{m_{j-1}} \right) \frac{K_j}{2} x_j - \frac{m_j - 1}{m_j} \frac{K_{j+1}}{2} x_{j+1}.
\end{aligned}$$

Then we must have

$$0 = \frac{m_j}{m_{j-1}} \frac{K_j}{2} x_j - \frac{m_j - 1}{m_j} \frac{K_{j+1}}{2} x_{j+1}. \qquad (5)$$

Note that in terms of the matrix A, the system (4) can be written as follows:

$$-\sum_{k=1}^{n} a_{ik}x_k = 0, \quad i = 1, \cdots, j. \tag{6}$$

Proof. It is easy to prove this by using mathematical induction on $2 \leq j \leq n-1$. □

Theorem 2.3 *Let $(x_1, x_2, \cdots, x_n) \in \mathbf{R}_+$ be arbitrary. Let $1 < i < j < n$, and assume that the following system of equations is satisfied:*

$$0 = -\frac{m_{i-1}}{m_{i-1}}\frac{K_{i-1}}{2}x_{i-1} + \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2}x_i - \frac{m_i - 1}{m_i}\frac{K_{i+1}}{2}x_{i+1}, \tag{7}$$
$$\cdots$$
$$0 = -\frac{m_{j-1}}{m_{j-2}}\frac{K_{j-1}}{2}x_{j-1} + \left(\frac{m_{j-1}-1}{m_{j-1}} + \frac{m_j}{m_{j-1}}\right)\frac{K_j}{2}x_j - \frac{m_j - 1}{m_j}\frac{K_{j+1}}{2}x_{j+1}.$$

Then we must have

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{m_j}{m_{j-1}}\frac{K_j}{2}x_j - \frac{m_j-1}{m_j}\frac{K_{j+1}}{2}x_{j+1}. \tag{8}$$

Note that Equation (7) is equivalent to the following system:

$$-\sum_{k=1}^{n} a_{pk}x_k = 0, \quad p = i, \cdots, j. \tag{9}$$

Proof. This theorem is easily proved using induction on $3 \leq j < n$. □

Theorem 2.4 *For all $(x_1, \cdots, x_n) \in \mathbf{R}^n_+$, if*

$$x_i \sum_{j=1}^{n} a_{ij}x_j = 0, \text{ for } i =, \cdots, n,$$

then $x_i = 0$, for $i = 1, \cdots, n$.

Proof. Assume that

$$x_i \sum_{j=1}^{n} a_{ij}x_j = 0, \text{ for } i = 1, \cdots, n. \tag{10}$$

One of the following cases must hold.—We will show that only **Case 1** does not lead to a contradiction.
Case 1: In this case, $x_i = 0$, for $i = 1, \cdots, n$.
Case 2: In this case,

$$x_i \neq 0, \text{ for } i = 1, \cdots, n.$$

Then (10) implies that

$$-\sum_{j=1}^{n} a_{ij}x_j = 0, \text{ for } i = 1, \cdots, n-1. \tag{11}$$

Hence, by taking $j = n-1$ in Theorem 2.2, we see that (9) implies

$$0 = \frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2} x_{n-1} - \frac{m_{n-1}-1}{m_{n-1}} \frac{K_n}{2} x_n.$$

But (10) also implies that

$$0 = \frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2} x_{n-1} - \frac{K_n}{2} x_n.$$

Adding together these last two equations produces

$$0 = -\frac{1}{m_{n-1}} \frac{K_n}{2} x_n.$$

This contradicts the assumption that $x_n \neq 0$.
Case 3: In this case, for some $1 \leq j < n$, we have

$$x_1, \cdots, x_j \neq 0; \quad x_{j+1}, \cdots, x_n = 0.$$

Hence (10) implies that

$$-\sum_{k=1}^{n} a_{ik} x_k = 0, \quad i = 1, \cdots, j. \tag{12}$$

If $j = 1$, then (12) is equivalent to

$$0 = \frac{m_1 K_1}{2} x_1 - \frac{m_1 - 1}{m_1} \frac{K_2}{2} x_2.$$

But if $j = 1$, then $x_2 = 0$, so we get $0 = -\frac{m_1 K_1}{2} x_1$, which contradicts $x_1 \neq 0$. Hence in (12) we may assume that $1 < j < n$. Then Theorem 2.2 implies that

$$0 = \frac{m_j}{m_{j-1}} \frac{K_j}{2} x_j - \frac{m_j - 1}{m_j} \frac{K_{j+1}}{2} x_{j+1}.$$

By assumption, $x_{j+1} = 0$, hence we have

$$0 = \frac{m_j}{m_{j-1}} \frac{K_j}{2} x_j,$$

i.e., $x_j = 0$. This contradiction shows that **Case 3** can not hold.
Case 4: In this case, for some $1 < i \leq n$, we have

$$x_1, \cdots, x_{i-1} = 0; x_i, \cdots, x_n \neq 0.$$

Then (10) implies that

$$-\sum_{k=1}^{n} a_{pk} x_k = 0, \quad p = i, \cdots, n. \tag{13}$$

First, assume that $i = n$ or $i = n-1$. If $i = n$, then $x_{n-1} = 0$, and (13) implies that

$$0 = \frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2} x_{n-1} - \frac{K_n}{2} x_n.$$

This leads to the contradiction that $-\dfrac{K_n}{2}x_n = 0$. If $i = n-1$, then (11) implies that

$$0 = -\frac{m_{n-2}}{m_{n-3}}\frac{K_{n-2}}{2}x_{n-2} + \left(\frac{m_{n-2}-1}{m_{n-2}} + \frac{m_{n-1}}{m_{n-2}}\right)\frac{K_{n-1}}{2}x_{n-1} - \frac{m_{n-1}-1}{m_{n-1}}\frac{K_n}{2}x_n,$$
$$0 = -\frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} + \frac{K_n}{2}x_n.$$

Adding together these two equations, and taking into consideration that $x_{n-2} = 0$, we see that

$$0 = \frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} + \frac{1}{m_{n-1}}\frac{K_n}{2}x_n.$$

This contradicts the assumption that $x_{n-1}, x_n \neq 0$.—Thus, we may assume that $1 < i < n-1$. Then (13) implies

$$\sum_{k=1}^{n} a_{pk}x_k = 0, \quad p = i, \cdots, n-1. \tag{14}$$

Now, (14) allows us to apply Theorem 2.3 to the case where $1 < i < j = n-1 < n$, producing

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} - \frac{m_{n-1}-1}{m_{n-1}}\frac{K_n}{2}x_n.$$

But (14) also implies that

$$0 = -\frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} + \frac{K_n}{2}x_n.$$

Adding together these last two equations, we obtain

$$0 = \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{1}{m_{n-1}}\frac{K_n}{2}x_n.$$

Hence, $x_i, x_n = 0$. This contradicts our assumption that $x_i, \cdots, x_n \neq 0$. Therefore, **Case 4** can not hold.
Case 5: In this case, there exist $1 < i < j < n$ such that

$$x_{i-1} = 0; x_i, \cdots, x_j \neq 0; x_{j+1} = 0.$$

Then (10) implies that

$$-\sum_{k=1}^{n} a_{pk}x_k = 0, \quad p = i, \cdots, j. \tag{15}$$

Because (15) implies (9), we may invoke Theorem 2.3, obtaining

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \frac{m_{i-1}-1}{m_{i-1}} + \frac{m_j}{m_{j-1}}\frac{K_j}{2}x_j - \frac{m_j-1}{m_j}\frac{K_{j+1}}{2}x_{j+1}.$$

Because $x_{i-1} = x_{j+1}$, this equation implies that

$$0 = \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{m_j}{m_{j-1}}\frac{K_j}{2}x_j.$$

But then we have the contradiction that $x_i = x_j = 0$. This shows that **Case 5** can not hold.

Case 6: In this final case, there exists $1 < i < n$ such that

$$x_{i-1} = 0; x_i \neq 0; x_{i+1} = 0.$$

It then follows from (10) that

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2}x_i + \frac{m_i - 1}{m_i}\frac{K_{i+1}}{2}x_{i+1}.$$

Because $x_{i-1} = x_{i+1} = 0$, we obtain

$$0 = \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2}x_i.$$

That is, $x_i = 0$. This contradiction shows that **Case 6** can not hold.

Because the above cases are the only possible cases consistent with assumption (10), we conclude that **Case 1** must hold. This completes the proof of the theorem. □

Definition 2.2 Let $B = (b_{ij})$ be the tridiagonal matrix whose entries are defined as follows, where the index i has range $1 < i < n$,

$$b_{11} = -m_1, \qquad b_{12} = \frac{m_1 - 1}{m_1};$$

$$b_{i,i-1} = \frac{m_{i-1}}{m_{i-1}}, \qquad b_{i,i} = -\left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right), \qquad b_{i,i+1} = \frac{m_i - 1}{m_i}\frac{K_{i+1}}{2};$$

$$b_{n,n-1} = \frac{m_{n-1}}{m_{n-2}}, \qquad b_{n,n} = -1.$$

Note that

$$A = B \text{ diag}\left(\frac{K_1}{2}, \cdots, \frac{K_n}{2}\right).$$

Define $P = (p_{ij})$ to be following matrix:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{m_1-1}{2m_1} & \frac{(m_1-1)(m_2-1)}{2m_1m_2} & \cdots & \frac{(m_1-1)(m_2-1)\cdots(m_n-1)}{2m_1m_2\cdots m_n} \\ \frac{m_1}{2} & \frac{m_1}{2} & \frac{m_1(m_2-1)}{2m_2} & \cdots & \frac{m_1(m_2-1)\cdots(m_n-1)}{2m_2m_3\cdots m_n} \\ \frac{m_2}{2} & \frac{m_2}{2} & \frac{m_2}{2} & \cdots & \frac{m_2(m_3-1)\cdots(m_n-1)}{2m_3\cdots m_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{m_{n-1}}{2} & \frac{m_{n-1}}{2} & \frac{m_{n-1}}{2} & \cdots & \frac{m_{n-1}(m_n-1)}{2m_n} \\ \frac{m_n}{2} & \frac{m_n}{2} & \frac{m_n}{2} & \cdots & \frac{m_n}{2} \end{pmatrix}$$

Observe that the entries of P are given by

$$P_{ij} = \begin{cases} \frac{m_{i-1}(m_i-1)\cdots(m_{j-1}-1)}{2m_i\cdots m_{j-1}}, & \text{if } \quad 1 \leq i < j \leq n; \\ \frac{m_{i-1}}{2}, & \text{if } \quad 1 \leq j \leq i \leq n. \end{cases}$$

Theorem 2.5 *The matrices B and P satisfy*

$$BP = -\frac{I}{2}.$$

Hence B is invertible, with inverse

$$B^{-1} = -2P.$$

Proof. We must prove that for all $1 \leq i \leq j \leq n$,

$$(BP)_{ij} = -\frac{1}{2}\delta_{ij}.$$

The following cases exhaust all possible cases for the pair $1 \leq i \leq j \leq n$.
Case 1: In this case, $i = j = 1$. We then have

$$\begin{aligned}(BP)_{11} &= b_{11}P_{11} + b_{12}P_{21}\\ &= -\frac{m_1}{2} + \frac{m_1 - 1}{m_1}\frac{m_1}{2}\\ &= -\frac{1}{2}.\end{aligned}$$

Case 2: In this case, $i = j = n$. Then

$$\begin{aligned}(BP)_{nn} &= b_{n,n-1}P_{n-1,n} + b_{nn}P_{nn}\\ &= \frac{m_{n-1}}{m_{n-2}}\frac{m_{n-2}(m_{n-1} - 1)}{m_{n-1}} - \frac{m_{n-1}}{2}\\ &= \frac{m_{n-1} - 1}{2} - \frac{m_{n-1}}{2}\\ &= -\frac{1}{2}.\end{aligned}$$

Case 3: In this case, $1 < i = j < n$. We then obtain

$$\begin{aligned}(BP)_{ii} &= b_{i,i-1}P_{i-1,i} + b_{ii}P_{ii} + b_{i,i+1}P_{i+1,i}\\ &= \frac{m_{i-1}}{m_{i-2}}\frac{m_{i-2}(m_{i-1} - 1)}{2m_{i-1}} - \left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{m_{i-1}}{2} + \frac{m_i - 1}{m_i}\frac{m_i}{2}\\ &= \frac{m_{i-1} - 1}{2} - \frac{m_{i-1} - 1 + m_i}{2} + \frac{m_i - 1}{2}\\ &= -\frac{1}{2}.\end{aligned}$$

Case 4: In this case, $1 < i + 1 < j \leq n$. We obtain

$$\begin{aligned}(PB)_{ij} &= b_{i,i-1}P_{i-1,j} + b_{ii}P_{i,j} + b_{i,i+1}P_{i+1,j}\\ &= \frac{m_{i-1}}{m_{i-2}}\frac{m_{i-2}(m_{i-1} - 1)\cdots(m_{j-1} - 1)}{2m_{i-1}\cdots m_{j-1}} - \left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\times\\ &\quad\times\frac{m_{i-1}(m_i - 1)\cdots(m_{j-1} - 1)}{2m_i\cdots m_{j-1}} + \frac{m_i(m_{i+1} - 1)\cdots(m_{j-1} - 1)}{2m_{i+1}\cdots m_{j-1}}\\ &= \frac{(m_{i-1} - 1)\cdots(m_{j-1} - 1)}{2m_i\cdots m_{j-1}} - ([m_{i-1} - 1] + m_i)\frac{(m_i - 1)\cdots(m_{j-1} - 1)}{2m_i\cdots m_{j-1}} +\\ &\quad+ \frac{(m_i - 1)(m_{i+1} - 1)\cdots(m_{j-1} - 1)}{2m_{i+1}\cdots m_{j-1}}\\ &= 0.\end{aligned}$$

Case 5: In this case, $1 < i < j = i + 1 \leq n$. We then have

$$\begin{aligned}
(PB)_{ij} &= b_{i,i-1}P_{i-1,j} + b_{ii}P_{ij} + b_{i,i+1}P_{i+1,j} \\
&= b_{i,i-1}P_{i-1,i+1} + b_{ii}P_{i,i+1} + b_{i,i+1}P_{i+1,i+1} \\
&= \frac{m_i - 1}{m_{i-2}} \frac{m_{i-2}(m_{i-1} - 1)(m_i - 1)}{2m_{i-1}m_i} - \left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right) \frac{m_{i-1}(m_i - 1)}{2m_i} \\
&\quad + \frac{m_i - 1}{m_i} \frac{m_i}{2} \\
&= \frac{(m_{i-1} - 1)(m_i - 1)}{2m_i} - \frac{(m_{i-1} - 1)(m_i - 1)}{2m_i} - m_i \frac{(m_i - 1)}{2m_i} + \frac{(m_i - 1)}{2} \\
&= 0.
\end{aligned}$$

Case 6: In this case, we have $i = 1 < 2 < j \leq n$. We see that

$$\begin{aligned}
(BP)_{ij} = (BP)_{1j} &= b_{11}P_{ij} + b_{12}P_{2j} \\
&= -m_1 \frac{(m_1 - 1) \cdots (m_{j-1} - 1)}{m_1 \cdots m_{j-1}} + \frac{m_1 - 1}{m_1} \frac{m_1(m_2 - 1) \cdots (m_{j-1} - 1)}{m_2 \cdots m_{j-1}} \\
&= 0.
\end{aligned}$$

Case 7: In this case, $1 = i < 2 = j \leq n$. We then obtain

$$\begin{aligned}
(BP)_{ij} = (BP)_{12} &= b_{11}P_{12} + b_{12}P_{22} \\
&= -m_1 \frac{(m_1 - 1)}{m_1} + \frac{m_1 - 1}{m_1} \frac{m_2}{2} \\
&= 0.
\end{aligned}$$

Case 8: In this case, $1 \leq j < i < n$. We then have

$$\begin{aligned}
(BP)_{ij} &= b_{i,i-1}P_{i-1,j} + b_{ii}P_{ij} + b_{i,i+1}P_{i+1,j} \\
&= \frac{m_{i-1}}{m_{i-2}} \frac{m_{i-2}}{2} - \left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right) \frac{m_{i-1}}{2} + \frac{m_i - 1}{m_i} \frac{m_i}{2} \\
&= \frac{m_{i-1}}{2} - \frac{(m_{i-1} - 1 + m_i)}{2} + \frac{m_i - 1}{2} \\
&= 0.
\end{aligned}$$

Case 9: In this final case, $1 \leq j < i = n$. Consequently,

$$\begin{aligned}
(BP)_{ij} = (BP)_{nj} &= b_{n,n-1}P_{n-1,j} + bnnP_{nj} \\
&= \frac{m_{n-1}}{m_{n-2}} \frac{m_{n-2}}{2} - \frac{m_{n-1}}{2} \\
&= 0.
\end{aligned}$$

The proof of the theorem is now complete. □

Theorem 2.6 *Every real eigenvalue of the matrix A is negative.*

Proof. We apply Gershgoin's Circle Theorem to the transpose of A, concluding that the eigenvalues of A are contained in the union of the following closed disks in the complex plane:

D_1: center $= -\dfrac{m_1K_1}{2}$, radius $= \dfrac{m_1K_1}{2}$;
D_i: center $= -\left(\dfrac{m_{i-1}-1}{m_{i-i}} + \dfrac{m_i}{m_{i-1}}\right)\dfrac{K_i}{2}$, radius $= \left(\dfrac{m_{i-1}-1}{m_{i-i}} + \dfrac{m_i}{m_{i-1}}\right)\dfrac{K_i}{2}$, $1 < i < n$;
D_n: center $= -\dfrac{K_n}{2}$, radius $= \dfrac{m_{n-1}-1}{m_{n-1}}\dfrac{K_n}{2}$.

Because $m_i > 1$ for $i = 1 \cdots n$, these disks are all in the closed left half-plane, hence all eigenvalues have non-positive real parts. By Theorem 2.5 the matrix B is invertible, and hence A is invertible, because $A = B \text{ diag}\left(\dfrac{K_1}{2}, \cdots, \dfrac{K_n}{2}\right)$, with $K_i > 0$, $i = 1, \cdots, n$. Therefore, all real eigenvalues of A are negative. □

Theorem 2.7 *The matrix A satisfies conditions* (i)-(iv) *of Theorem* 2.1.

Proof. By Theorem 2.5, A is invertible, hence condition (i) of Theorem 2.1 is satisfied. Because A is tridiagonal and $a_{ij} \neq 0$ whenever $|i-j| = 1$, condition (ii) of Theorem 2.1 holds. By Theorem 2.4, condition (iii) of Theorem 2.1 is satisfied. Finally, Theorem 2.6 implies that condition (iv) of Theorem 2.1 holds. □

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first prove that under the hypothesis of Theorem 1.1, there exists a unique parabolic solution of (2). We prove the remainder of Theorem 1.1 by an application of Theorem 2.1.

The next theorem is a key to proving the existence of a parabolic solution of (2). The following notation is used. We denote by $\triangle^m$ the *standard m-simplex*, i.e., the set of all points $\mathbf{x} = (x_1, \cdots, x_{m+1}) \in \mathbf{R}_+^{m+1}$ such that $\sum_{i=1}^{m+1} x_i = 1$; we denote the boundary of $\triangle^m$ by $\partial\triangle^m$. Let $e_1 = (1, 0, \cdots, 0), \cdots, e_{m+1} = (0, \cdots, 0, 1)$ be the standard basis for $\triangle^m$. For $1 \le i, j \le m+1$, we let $[e_i, e_j]$ be the boundary simplex determined by the pair e_i, e_j, that is, $[e_i, e_j]$ is the convex hull of the pair e_i, e_j. Observe that $\partial\triangle^m$ is the union of all the boundaries $[e_i, e_j]$, $i \neq j$, $1 \le i, j \le m+1$.

Theorem 3.1 *Let $f : \triangle^m \to \triangle^m$ be a continuous map which maps each vertex to itself and each edge into itself. Then $f(\triangle^m) = \triangle^m$.*

Proof. Standard theorems in algebraic topology show that any extension to the simplex, of a continuous map of the boundary of a simplex to itself having nonzero degree, must map onto the simplex. By looking at each edge it is easy to prove that the restriction of f to the boundary is a map of the boundary to itself which is homotopic to the identity; it is well known that this implies degree 1. Therefore f is onto. □

Theorem 3.2 *Define $f = (f_1, \cdots, f_n) : \triangle^{n-1} \to \triangle^{n-1}$ by*

$$f_i(\mathbf{x}) = \left(x_i \sum_{j=1}^{n} P_{ij}x_j\right) \Bigg/ \left(\sum_{k=1}^{n} x_k \sum_{j=1}^{n} P_{kj}x_j\right), \quad \mathbf{x} = (x_1, \cdots, x_n) \in \triangle^{n-1}, \ i = 1, \cdots, n.$$

Then f maps $\triangle^{n-1}$ onto itself.

Proof. First of all, observe that f sends $\triangle^{n-1}$ into itself, because the entries of P are positive. It is easy to check that f is continuous and maps each vertex of the simplex to itself and each edge of the simplex into itself. Therefore Theorem 3.1 implies f is onto. $\square$

Theorem 3.3 *There exists a unique parabolic solution of (2).*

Proof. Uniqueness: Let $\mathbf{q} = (q_1, \cdots, q_n)$ and $\mathbf{r} = (r_1, \cdots, r_n)$ be two parabolic solutions of (2), with $q_i(t) = c_i\sqrt{t}, r_i(t) = d_i\sqrt{t}, c_i > 0, d_i > 0, i = 1, \cdots, n$. By Theorem 2.7, the matrix A in (2) satisfies conditions (i)–(iv) of Theorem 2.1, hence by that theorem we have $\lim_{t\to+\infty} |q_i(t) - r_i(t)| = 0$, for $i = 1, \cdots, n$. But $|q_i(t) - r_i(t)| = \sqrt{t}|c_i - d_i|$, $i = 1, \cdots, n$. We conclude that $c_i = d_i$, for $i = 1, \cdots, n$. Existence: Let K_i be as in (2) and define $\mathbf{y} \in \triangle^{n-1}$ by

$$y_i = K_i \Bigg/ \left(\sum_{j=1}^{n} K_j\right), \quad i = 1, \cdots, n.$$

Let $f : \triangle^{n-1} \to \triangle^{n-1}$ be defined as in Theorem 3.2; then by that theorem there exists a point $\mathbf{u} \in \triangle^{n-1}$, such that $\mathbf{y} = f(\mathbf{u})$. Define ζ, η by

$$\zeta = \left(\sum_{j=1}^{n} K_j\right)^{\frac{1}{2}}, \quad \eta = \left(\sum_{i=1}^{n} u_i \sum_{j=1}^{n} P_{ij} u_j\right)^{\frac{1}{2}}.$$

Let $\mathbf{c} = (\zeta/\eta)\mathbf{x}$. Then $\mathbf{y} = f(\mathbf{u})$ implies $K_i = c_i \sum_{j=1}^{n} P_{ij} c_j$, for $i = 1, \cdots, n$. This last set of equations is equivalent to:

$$\frac{1}{2} c_i = -\sum_{j=1}^{n} a_{ij} \left(\frac{1}{c_j}\right), \quad i = 1, \cdots, n.$$

Define $\mathbf{q}(t), t \in (0, +\infty)$, by $q_i(t) = c_i\sqrt{t}, i = 1, \cdots, n$. The preceding equations imply that $\mathbf{q}(t)$ is a parabolic solution of (2). This proves the theorem. $\square$

3.1 Proof of Theorem 1.1

To prove Theorem 1.1, let $\mathbf{p}(t) = (p_1(t), \cdots, p_n(t))$, $t \in [0, a]$, $0 < a < +\infty$, be a solution of (2) in $\mathbf{R}^3_{++}$. By Theorem 2.7, the matrix A in (2) satisfies conditions (i)-(iv) of Theorem 2.1, hence, by that theorem, there exists a unique extension of $\mathbf{p}(t)$, $t \in [0, a]$, to a solution $\mathbf{p}(t) = (p_1(t), \cdots, p_n(t))$, $t \in [0, +\infty)$, of (2) in $\mathbf{R}^3_{++}$, such that $\lim_{t\to+\infty} p_i(t) = +\infty$, $i = 1, \cdots, n$. Moreover, if $\mathbf{r}(t)$, $t \in [0, \infty)$ is any other solution of (2), then by Theorem 2.1, we have

$$\lim_{t\to\infty} ||\mathbf{p}(t) - \mathbf{r}(t)|| = 0.$$

Because the matrix A of the system (2) is tridiagonal, Theorem 2.1 implies that the extended solution $\mathbf{p}(t)$ is eventually monotone increasing on $[0, +\infty)$. By Theorem 3.3, there exists a unique parabolic solution $\mathbf{q}(t) = (c_1\sqrt{t}, \cdots, c_n\sqrt{t})$, $(c_1, \cdots, c_n) > 0$, $t \in (0, +\infty)$, of (2). This completes the proof of Theorem 1.1.

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Conform Fractional Semi-Dynamical Systems

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Abstract: The aim of this work is to present the notion of a conform semi-dynamical system, unlike the concept of a dynamical system, here we can work with the continuous functions. Some examples are presented to illustrate the result of the autonomous case.

Keywords: *conform dynamical systems; orbit; omega-set limit; autonomous system.*

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1 Introduction

Fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, starting from some speculations of G.W. Leibeniz (1967) and L. Euler (1730) and since then, it has continued to be developed up to nowadays. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true for problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering. We recall that the fractional partial derivatives are difficult to handle analytically, especially those describing real world processes, and the researchers sometimes have to rely on the numerical methods to solve these equations. One of the well-known fractional derivatives is the Riemann-Liouville fractional order derivative, which is not always appropriate for modeling real world problems. The second one is the so-called Caputo derivative, this one is opposite with relation to displaying physical field complications and has been intensively used for this purpose.

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However, new derivatives should be proposed in order to deal better with the dynamics of the complex systems [9, 11]. In [8] the authors give a new definition of fractional derivative and fractional integral. The form of the definition shows that it is the most natural definition, and the most fruitful one. The definition for $0 < \alpha < 1$ coincides with the classical definitions on polynomials (up to a constant). Further, if $\alpha = 1$, the definition coincides with the classical definition of the first derivative. They presented some applications to fractional differential equations. A. Atanagana in [3] investigated in more detail some new properties of the conform derivative and has proved some useful related theorems. Dynamical systems as a generalization of solutions of ordinary differential equations are already a classical subject in the mathematical literature. Its systematic generalization to systems with nonunique solutions was developed by Barbashin [7]. We note that this study depends on the nature of derivative, our objective is to give the analogue with the conform derivative, in order to weaken the hypothesis of class C functions in continuous functions.

The paper is organized as follows. After this introductory section, we will present and demonstrate some properties concerning the conform derivative in Section 2. The definitions of α-semi-dynamical system, orbit and omega-set and their properties are given in Section 3. The last Section 4 contains qualitative studies of autonomous system in dimension 2.

2 Conform Fractional Derivative

In this section, we will give some definitions and properties concerning the new derivative important in the following.

Definition 2.1 (see [8]) Let $\alpha \in (n, n+1]$ and $f : [0, \infty) \to \mathbb{R}$ be n-differentiable at $t > 0$, then the conformable fractional derivative of f of order α is defined by

$$\left\{ f^{(\alpha)}(t) = \lim_{\epsilon \to 0} \frac{f^{(n)}(t+\epsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\epsilon}, f^{(\alpha)}(0) = \lim_{t \to 0} f^{(\alpha)}(t). \right.$$

Remark 2.1 (see [8]) As consequence of the previous definition, one can easily show that

$$f^{(\alpha)}(t) = t^{n+1-\alpha} f^{(n+1)}(t),$$

where $\alpha \in (n, n+1]$, and f is $(n+1)$-differentiable at $t > 0$.

In [3] we find the following proposition.

Proposition 2.1 *[3] We have the following properties:*

1. $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$,
2. $(fg)^{(\alpha)} = f^{(\alpha)}g + fg^{(\alpha)}$,
3. $(t^p)^{(\alpha)} = pt^{p-\alpha}$,
4. $\left(\frac{f}{g}\right)^{(\alpha)} = \frac{f^{(\alpha)}g - fg^{(\alpha)}}{g^2}$,
5. *If* $c \in \mathbb{R}$, $c^{(\alpha)} = 0$.

Proposition 2.2 *If x is a continuous map, then $t \to x^{(\alpha)}(t)$ is a continuous map.*

Proof. Since x is a continuous map, $t \in \mathbb{R}_+^* \to x\left(t + \epsilon t^{1-\alpha}\right)$ is continuous, thus $\forall \beta > 0$, $\exists \alpha > 0$,

$$|\frac{x\left(t + \epsilon t^{1-\alpha}\right) - x\left(t_0 + \epsilon t_0^{1-\alpha}\right)}{\epsilon}| \leq \beta,$$

whenever $|t - t_0| \leq \alpha$, by passing to the limit $\epsilon \to 0$ we get $|x^{(\alpha)}(t) - x^{(\alpha)}(t_0)| \leq \beta$ as desired.

Proposition 2.3 *Let $f : X \to X$ be a Lipschitziane map, i.e., $|f(x) - f(y)| \leq k|x - y|$, $\forall x, y \in X$ and $k \in]0, 1[$. The Cauchy problem*

$$\begin{cases} x^{(\alpha)}(t) = f\left(x(t)\right), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

has a unique solution.

Proof. By Proposition 2.2 x is continuous, the sequence $x_{x+1} = f(x_n)$ is a Cauchy sequence, since $\mathbb{R}$ is a complete space, then x_n converges to the unique solution of (1).

Definition 2.2 (see [8]) Let $\alpha \in (1, 2]$, $(I^\alpha f)(t) = \int_0^t s^{\alpha-2} f(s)ds$.

Theorem 2.1 *(see [8]) $(I^\alpha f)^{(\alpha)}(t) = f(t)$ for $t \geq 0$.*

Example 2.1

$$I^\alpha(\sin(t)) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\alpha}}{(2n+\alpha)(2n+1)!}$$

where $\alpha \in (1, 2)$.

Definition 2.3 (see [6]) Let $\alpha > 0$. For a Banach space X, a family $\{T(t)\}_{t \geq 0} \subset \mathcal{L}(X, X)$ is called a fractional α-semigroup if

1. $T(0) = I$.
2. $T\left((s+t)^{\frac{1}{\alpha}}\right) = T\left(s^{\frac{1}{\alpha}}\right) T\left(t^{\frac{1}{\alpha}}\right)$, for all $s, t \in [0, \infty)$.

Example 2.2 Let A be a bounded linear operator on X. Define $T(t) = e^{2\sqrt{t}A}$. Then $T(t)_{t \geq 0}$ is a $\frac{1}{2}$ semigroup. Indeed,

1. $T(0) = e^{0A} = I$.
2. $\forall s, t \in [0, \infty)$, $T\left((s+t)^2\right) = e^{2(t+s)A} = e^{2tA} e^{2sA} = T(s)T(t)$.

Definition 2.4 (see [6]) An α-semigroup $T(t)$ is called a c_0-semigroup if, for each fixed $x \in X$, $T(t)x \to x$ as $t \to 0^+$.

The conformable α-derivative of $T(t)$ at $t = 0$ is called the α-infinitesimal generator of the fractional α-semigroup $T(t)$, with the domain equal to $\left\{x \in X : \lim_{t \to 0} T(t)x \text{ exists} \right\}$.

3 Conform Fractional Dynamical Systems

3.1 Definition and examples

In this subsection we will introduce the notion of α-dynamical system.

Definition 3.1 Let X be a complete metric space. An α-semi-dynamical system is a couple (X, π_α), where X state space of the system. Each point of X is a state of the system, and $\pi_\alpha : \mathbb{R}_+ \times X \longrightarrow X$ satisfies

$$\begin{cases} \pi_\alpha(0, x) = x, \quad \forall x \in X, \\ \pi_\alpha\big(t, \pi_\alpha(s, x)\big) = \pi_\alpha(t + s, x), \quad \forall x \in X, t, s > 0. \end{cases}$$

Example 3.1

1. Here X is a Banach space. Let $f : X \to X$, be a Lipschitzian map, i.e., $|f(x) - f(y)| \leq k|x - y|$, $\forall x, y \in X$ and $k \in]0, 1[$. The Cauchy problem

$$\begin{cases} x^{(\alpha)}(t) = f\left(x(t)\right), \quad t > 0, \\ x(0) = x_0. \end{cases}$$

has unique solution. The mapping $\pi_\alpha(t, x_0) = x\left(t^{\frac{1}{\alpha}}\right)$ defines an α-semi-dynamical system.

2. Here $X = \mathcal{C}\big([0, 1], \mathbb{R}\big)$, X is a Banach space. For all $f \in X$, we define $\pi_\alpha(t, f)$ by

$$\pi_\alpha(t, f)(s) = f\Big(\min\Big((t^{\frac{1}{\alpha}} + s), 1\Big)\Big), \quad 0 \leq s \leq 1.$$

We will demonstrate that this corresponds well to the α-semi-dynamical system.

3.2 Orbit of α-semi-dynamical system

The term orbit generally refers to the image (in the state space) of a solution. It is defined as

$$\mathcal{O}(x) = \big\{\pi_\alpha\left(t, x\right),\ t \geq 0\big\}.$$

Definition 3.2 A point $x \in X$ is said to be a critical point if $\pi_\alpha\left(t, x\right) = x$.

Example 3.2 We take $x^{(\alpha)}(t) = x(t) - x^2(t)$, $x = 0$ and $x = 1$ are two critical points.

Definition 3.3 A point $x \in X$ is said to be a periodic point if there is $\tau > 0$ such that $\pi_\alpha\left(t + \tau, x\right) = \pi_\alpha\left(t, x\right)$.

Remark 3.1 $x \in X$ is a periodic point if and only if $\pi_\alpha\left(\tau, x\right) = x$.

Proposition 3.1 *Let x be a periodical point of π_α, of period τ. It goes through one and only one periodic solution, of period τ, defined on $\mathbb{R}$.*

Proof. The mapping defined by

$$u(t) = \begin{cases} \pi_\alpha(t, x), \forall t \geq 0, \\ \pi_\alpha(t + n\tau, x), \forall t \in [-n\tau, (-n + 1)\tau[,\ n \in \mathbb{N}, \end{cases}$$

is a periodic extension of $\pi_\alpha(t, x)$ on $\mathbb{R}$. We get $u(t + s) = \pi_\alpha(s, u(t))$, $\forall t, s \in \mathbb{R}$.

3.3 Omega-limit set

We will discuss some properties of the Omega-limit set related to the orbit. We will start by the following definition.

Definition 3.4 Let $x \in X$. The Omega-limit set, denoted $\omega(x)$, is defined as

$$\omega(x) = \big\{ y = \lim_{n\to\infty} \pi_\alpha(t_n, x) : \ t_n \to \infty, \text{ such that } \pi_\alpha(t_n, x) converge \big\}.$$

Remark 3.2 We can write

$$\omega(x) = \bigcap_{t>0} \overline{\pi_\alpha([t_n, \infty[, X)},$$

where $\overline{\pi_\alpha([t_n, \infty[, X)}$ is the closure of $\pi_\alpha([t_n, \infty[, X)$.

We will etablish several properties of the Omega-limit set.

Lemma 3.1 *$\omega(x)$ is closed, and satisfies*

$$y \in \omega(x) \Rightarrow \pi_\alpha(t, y) \in \omega(x).$$

Proof. It is closed because there is an intersection of closed parts.
For all $s \geq 0$, we have

$$\begin{aligned} \pi_\alpha\left(s, \omega(x)\right) &\subset \bigcap \pi_\alpha\Big(s, \overline{\pi_\alpha([t, \infty[, X)}\Big) \\ &\subset \bigcap \overline{\pi_\alpha([t, \infty[, X)} \\ &\subset \omega(x). \end{aligned}$$

Lemma 3.2 *If $\mathcal{O}(x)$ is precompact, then $\omega(x) \neq \emptyset$.*

Proof. The sets $\pi_\alpha([t, \infty[, X)$ are compacts, whose finite intersection can not be avoided, thus $\omega(x) \neq \emptyset$.

Proposition 3.2 *If $\mathcal{O}(x)$ is precompact, then $\omega(x)$ is a compact susbset, connected.*

Proof. By Proposition 3.2 then $\omega(x)$ is an intersection of compacts, thus it is a compact. It remains to demonstrate that it is connected.

Theorem 3.1 *If $\mathcal{O}(x)$ is precompact without double point (i.e., $\forall t_1 < t_2$, $\pi_\alpha(t_1, x) \neq \pi_\alpha(t_2, x)$), then $\omega(x) \setminus \mathcal{O}(x)$ is dense in $\omega(x)$.*

Proof. We have the following alternative: $\mathcal{O}(x) \cap \omega(x) = \emptyset$, in this case the result is clear, or $\mathcal{O}(x) \cap \omega(x) \neq \emptyset$, in this case we get $\pi_\alpha([\tau, \infty[, X) \subset \omega(x)$, for some $\tau > 0$. On the other hand we can write

$$\omega(x) \setminus \mathcal{O}(x) = \bigcap_n \omega(x) \setminus \pi_\alpha([0, n], x),$$

each $\omega(x) \setminus \pi_\alpha([0, n], x)$ is open in $\omega(x)$ which is compact, then it is complete. Using the Baire theorem we conclude that each $\omega(x) \setminus \pi_\alpha([0, n], x)$ is dense for all $n \in \mathbb{N}$.
Since for all $y \in \omega(x)$, and $\epsilon > 0$, there is $t > \max n, \tau$, such that $d\left(\pi_\alpha(t, x), y\right) \leq \epsilon$, it becomes that $\pi_\alpha(t, x) \in \omega(x) \setminus \prime(x)$, which proves the density of $\pi_\alpha([0, n], x)$ in $\omega(x)$.

4 Fractional Autonomous Differential Systems

The purpose of this section is to study the following system in dimension 2, first we begin with some notion

$$x^{(\alpha)}(t) = f\left(x(t)\right), \tag{2}$$

where $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, Ω is open and f is a Lipschitzian map on Ω and differentiable at 0.

4.1 Definitions and notations

Definition 4.1 An equilibrium point of (2) is a point x_0 such that $f(x_0) = 0$.

Remark 4.1 If x_0 is an equilibrum point of (2) then $t \to x_0$ is a solution of (2).

Definition 4.2 Let x_0 be an equilibrum point of (2). We say that:

1. x_0 is stable if for all $\epsilon > 0$ there is $\eta > 0$ such that, if x is a solution of (2) which for t_0 satisfies $|x(t_0) - x_0| < \eta$, we have
   - x is defined for all $t \geq t_0$,
   - $|x(t) - x_0| < \epsilon$ for all $t \geq t_0$.
2. x is not stable if x_0 is stable,
3. x is asymptotically stable if
   - x_0 is stable,
   - $\lim_{t\to\infty} x(t) = x_0$.

4.2 Qualitative study of linear systems in dimension 2

In this subsection we consider the following differential system:

$$x^{(\alpha)}(t) = Ax(t), \tag{3}$$

where $x : \mathbb{R} \to \mathbb{R}^2$ and A is a constant matrix in $\mathcal{M}_2(\mathbb{R})$.

Remark 4.2 $x_0 = 0$ is a stable point of (3).

Before we study the above mentioned system, let us first solve the following equation:

$$x^{(\alpha)}(t) = ax(t), \tag{4}$$

where $a \in \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}$.

We put $\phi_0(t) = e^{\frac{a}{\alpha}t^\alpha}$, it is clear that ϕ_0 is a solution of (4).
Let ϕ be another solution, we get

$$\Big(\frac{\phi}{\phi_0}\Big)^{(\alpha)}(t) = \frac{\phi^{(\alpha)}(t)\phi_0(t) - \phi(t)\phi_0^{(\alpha)}(t)}{\left(\phi_0(t)\right)^2} = \frac{a\phi(t)\phi_0(t) - a\phi(t)\phi_0(t)}{\left(\phi_0(t)\right)^2} = 0.$$

Then $\frac{\phi}{\phi_0}$ is constant, which implies that $\phi(t) = ce^{\frac{a}{\alpha}t^\alpha}$, where $c \in \mathbb{R}$. The trajectories of the system depend on the nature of the eigenvalues of the matrix.

Case 1. Let v_1 and v_2 be two eigenvectors of A associated, respectively, with two eigenvalues λ_1 and λ_2, note that (v_1, v_2) is a basis of $\mathbb{R}^2$. Let P be the transit matrix from the canonical basis to (v_1, v_2) and we put $x = Py$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$, we get $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. The system (4) is written as

$$y^{(\alpha)} = Dy,$$

where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. It becomes $\begin{cases} y_1(t) = y_1^0 e^{\frac{\lambda_1}{\alpha} t^\alpha}, \\ y_1(t) = y_2^0 e^{\frac{\lambda_2}{\alpha} t^\alpha}. \end{cases}$

Example 4.1

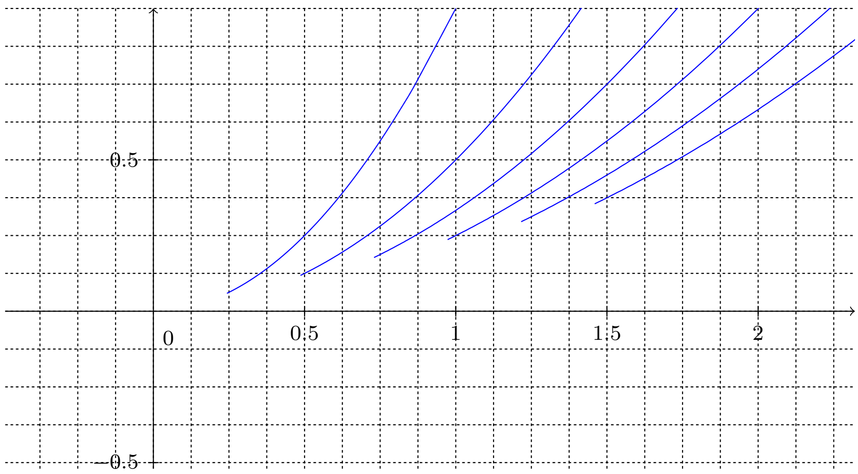

Figure 1. $\lambda_2 < \lambda_1 < 0$. The equilibrium point is asymptotically stable.

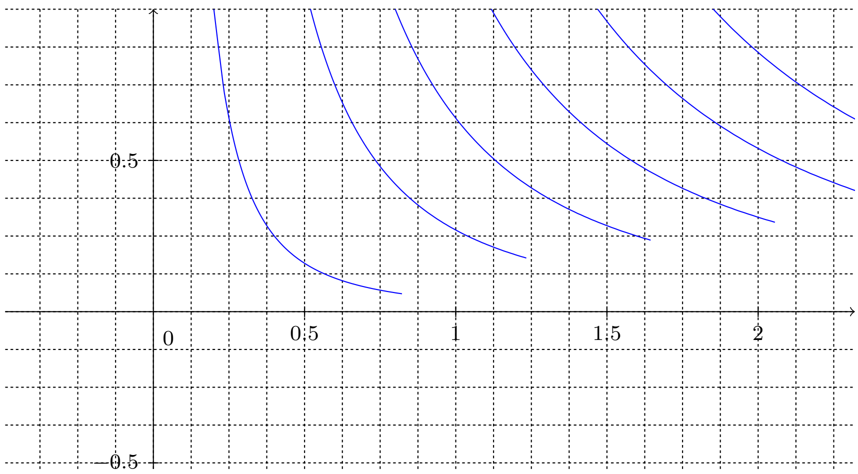

Figure 2. $\lambda_2 > \lambda_1 > 0$. The equilibrium point is not stable.

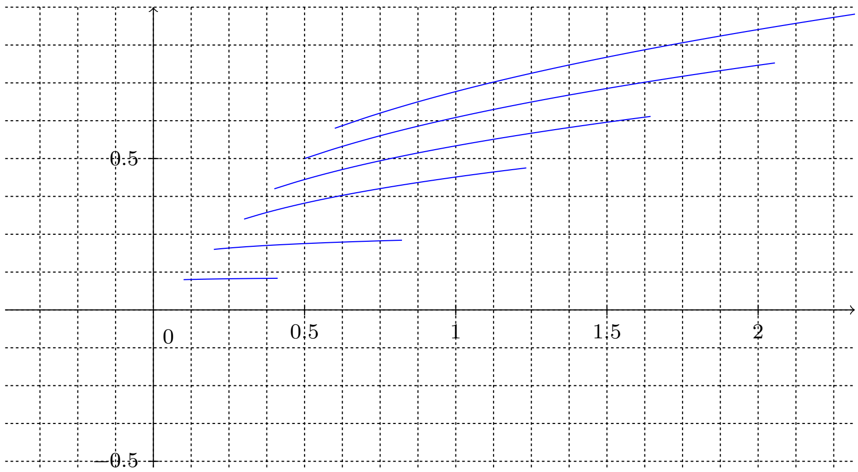

Figure 3. $\lambda_2 > 0 > \lambda_1$. Saddle point.

The trajectories described in Figures 1,2,3 are the curves given in parametric coordinates $y_1(t) = y_1^0 e^{\frac{\lambda_1}{\alpha} t^\alpha}$ and $y_1(t) = y_2^0 e^{\frac{\lambda_2}{\alpha} t^\alpha}$.

Case 2. Let $Z = u + iv$ be a complex eigenvector of A associated with the eigenvalue $\lambda = \eta - i\delta$. We have $Au = \eta u + \delta v$ and $Av = -\delta u + \eta v$. Thus (u, v) is a basis in which the matrix is writen as follows: $\begin{pmatrix} \alpha & -\delta \\ \delta & \alpha \end{pmatrix}$. If we denote by P the transit matrix from the canonical basis to (u, v), and we put $x = Py$, we get

$$\begin{cases} y_1(t) = Re^{\frac{\eta}{\alpha} t^\alpha} \cos\left(\frac{\delta}{\alpha} t^\alpha - \varphi\right), \\ y_2(t) = Re^{\frac{\eta}{\alpha} t^\alpha} \sin\left(\frac{\delta}{\alpha} t^\alpha - \varphi\right). \end{cases}$$

Example 4.2 In this example we take $\alpha = \frac{1}{2}$, $\delta = \frac{1}{2}$, $\varphi = 0$.

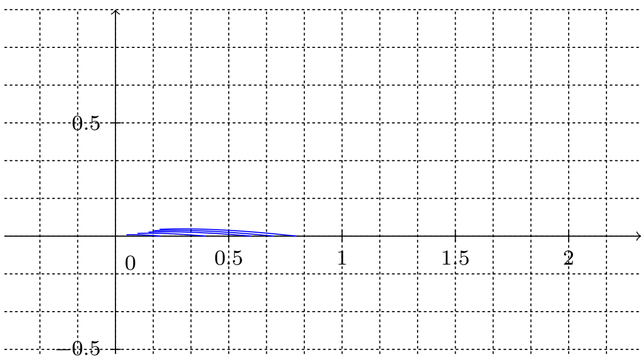

Figure 4: $\eta < 0$. The equilibrium point is stable.

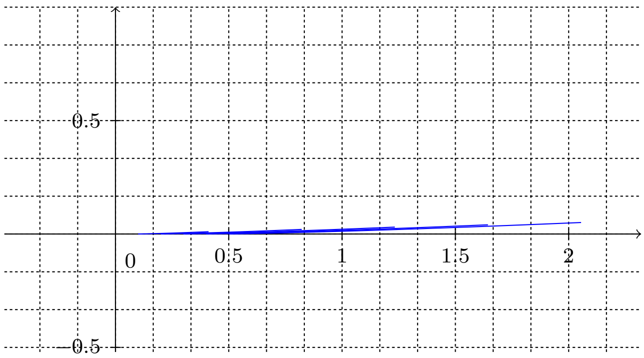

Figure 5: $\eta > 0$. The equilibrium point is not unstable.

Remark 4.3 If $\eta = 0$, then $y_1^2 + y_2^2 = R^2$.

5 Conclusion

This study is a basic idea for beginning the study of dynamical system in the conform frame.

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Almost Periodic Solutions for a Class of Nonlinear Duffing System with Time-Varying Coefficients and Stepanov-Almost Periodic Forcing Terms

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Abstract: In this paper, we study the existence of almost periodic solutions for a class of nonlinear Duffing system with time-varying coefficients and Stepanov-almost periodic forcing terms. Some sufficient conditions for the existence and uniqueness of an almost periodic solution of the system are established. We provide an example to illustrate the main result.

Keywords: *nonlinear Duffing system; almost periodic; Stepanov-almost periodic; contraction mapping principle.*

Mathematics Subject Classification (2010): 34C27, 34K14, 34A34.

1 Introduction

In recent years, various kinds of dynamic behaviors of nonlinear Duffing equations have been investigated by many authors due to its applications in many fields such as physics, mechanics, engineering and other scientific fields, for example, see [4, 5, 14]. In such applications, the existence of almost periodic solutions for nonlinear Duffing equations is an important topic. Many authors have studied the existence of periodic and almost periodic solutions of nonlinear differential equations, for more details we refer, [1–3, 6, 9–11, 13, 15–18] and the references cited therein.

Peng and Wang [13] considered the following model for a nonlinear Duffing equation with deviating argument

$$u''(t) + cu'(t) - au(t) + bu^m(t - \phi(t)) = \psi(t), \tag{1}$$

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where $\phi(t)$ and $\psi(t)$ are almost periodic functions on $\mathbb{R}$, $m > 1$ is an integer, and a, b, c are constants. By considering

$$v = u' + \xi u - Q_1(t),\ Q_2(t) = \psi(t) + (\xi - c)Q_1(t) - Q_1'(t), \tag{2}$$

where $Q_1(t)$ is a continuous and differentiable function on $\mathbb{R}$ and $\xi > 1$ is a constant, Peng and Wang [13] transformed (1) into the following system of differential equations

$$\begin{cases} u'(t) = -\xi u(t) + v(t) + Q_1(t), \\ v'(t) = -(c - \xi)v(t) + (a + \xi(c - \xi))u(t) - bu^m(t - \phi(t)) + Q_2(t), \end{cases} \tag{3}$$

and then proved the existence of positive almost periodic solutions of (1), and (3). Xu [15] extended the system (3) to the following nonlinear Duffing system with time-varying coefficients and delay

$$\begin{cases} u'(t) = -\delta_1(t)u(t) + v(t) + Q_1(t), \\ v'(t) = \delta_2(t)v(t) + [\mu(t) - \delta_2^2(t)]u(t) - \nu(t)u^m(t - \phi(t)) + Q_2(t), \end{cases} \tag{4}$$

where $\mu(t), \nu(t), \phi(t), \delta_1(t), \delta_2(t), Q_1(t), Q_2(t)$ are all almost periodic functions on $\mathbb{R}$, m is an integer with $m > 1$, $\mu(t) > 0, \nu(t) \neq 0$, and established some sufficient conditions for the existence of almost periodic solutions of (4).

In this paper, we extend the systems (3) and (4) to the following Duffing system

$$\begin{cases} u'(t) = -f_1(t)u(t) + v(t) + F_1(t), \\ v'(t) = -f_2(t)v(t) + [\alpha(t) + f_2^2(t)]u(t) - \beta(t)u^m(t - \phi(t)) + F_2(t), \end{cases} \tag{5}$$

where $f_1(t)$ is a bounded continuous function on $\mathbb{R}$ with $\inf_{t\in\mathbb{R}} f_1(t) > 0$; $f_2(t)$, $\alpha(t)$, $\beta(t)$, $\phi(t)$ are almost periodic functions on $\mathbb{R}$; $F_1(t)$, $F_2(t)$ are Stepanov-almost periodic continuous functions on $\mathbb{R}$, and $\inf_{t\in\mathbb{R}} f_2(t) > 0$, $m > 1$ is an integer.

In [15, 16, 18], the authors considered the following almost periodic system

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}, \tag{6}$$

where $f : \mathbb{R} \to \mathbb{R}^n$ is an almost periodic function and $A(t)$ is an $n \times n$ almost periodic matrix defined on $\mathbb{R}$, to prove the existence of almost periodic solutions for a class of nonlinear Duffing systems. In this paper, we first study the existence of almost periodic solutions of (6) when $f : \mathbb{R} \to \mathbb{R}^n$ is a Stepanov-almost periodic continuous function and $A(t)$ is an $n \times n$ bounded continuous matrix defined on $\mathbb{R}$ satisfying some suitable conditions, and using these results we find sufficient conditions for the existence and uniqueness of almost periodic solution of (5). Finally, we provide an example to illustrate the results.

2 Preliminaries

In this section we give some basic definitions, notations, and results. In the rest of this paper $\mathbb{R}$ stands for a set of real numbers. We let $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ to denote a column vector, in which the symbol $()^T$ denotes the transpose of a vector, and if $x \in \mathbb{R}^n$, then we define $\|x\| = \max_{1\leq i\leq n} |x_i|$.

Definition 2.1 A continuous function $u : \mathbb{R} \to \mathbb{R}^n$ is said to be almost periodic if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\|u(t+\tau) - u(t)\| < \epsilon \quad \forall t \in \mathbb{R}.$$

Lemma 2.1 *If $u : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ are almost periodic functions, then $u(\cdot - g(\cdot)) : \mathbb{R} \to \mathbb{R}^n$ is also an almost periodic function.*

For a detailed proof of the above lemma see [7, Lemma 2.4].

Throughout the rest of the paper we fix $p, 1 \le p < \infty$. Denote by $L^p_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ the space of all functions from $\mathbb{R}$ into $\mathbb{R}^n$ which are locally p-integrable in the Bochner-Lebesgue sense. We say that a function, $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ is p-Stepanov bounded (S^p-bounded) if

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s)\|^p ds \Big)^{1/p} < \infty.$$

We indicate by $L^p_s(\mathbb{R};\mathbb{R}^n)$ the set of all S^p-bounded functions $\mathbb{R}$ into $\mathbb{R}^n$.

Definition 2.2 A function $f \in L^p_s(\mathbb{R};\mathbb{R}^n)$ is said to be almost periodic in the sense of Stepanov (S^p-almost periodic) if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s+\tau) - f(s)\|^p ds \Big)^{1/p} < \epsilon.$$

Lemma 2.2 ([12]) *Let $A(t) = (a_{ij})$ be an $n \times n$ continuous matrix defined on $\mathbb{R}$. If*

(i) *$A(t)$ is bounded,*

(ii) *$|\det A(t)| \ge \kappa$ on $\mathbb{R}$ for some $\kappa > 0$,*

(iii) *$a_{ii}(t) \le 0$ for $i = 1, 2, \cdots, n$ and for all $t \in \mathbb{R}$,*

(iv) *$|a_{ii}| \ge \sum_{j=1, j\neq i}^{n} |a_{ji}|$ for all $i = 1, 2, \cdots, n$ and for all $t \in \mathbb{R}$,*

then there exist positive constants M, γ, and the fundamental solution matrix $X(t)$ of the linear system

$$x'(t) = A(t)x(t), \quad t \in \mathbb{R}, \tag{7}$$

satisfying

$$\|X(t)X^{-1}(s)\| \le Me^{-\gamma(t-s)} \quad \textit{for} \quad t \ge s. \tag{8}$$

In the rest of the paper, we assume that $A(t)$ satisfies all conditions given in Lemma 2.2.

Lemma 2.3 ([8]) *Let $f : \mathbb{R} \to \mathbb{R}^n$ be a continuous function. Then the solution $x : \mathbb{R} \to \mathbb{R}^n$ of (6) is given by*

$$x(t) = X(t)X^{-1}(a)x(a) + \int_a^t X(t)X^{-1}(s)f(s)ds, \quad t \ge a, \ a \in \mathbb{R}. \tag{9}$$

Lemma 2.4 *If $f : \mathbb{R} \to \mathbb{R}^n$ is an S^p-almost periodic continuous function, then the function $\Lambda : \mathbb{R} \to \mathbb{R}^n$ defined by*

$$\Lambda(t) = \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds, \quad t \in \mathbb{R},$$

is an almost periodic function.

Proof. We consider

$$\Lambda_k(t) := \int_{t-k}^{t-k+1} X(t)X^{-1}(s)f(s)ds, \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

Then

$$\begin{aligned}
\|\Lambda_k(t)\| &\le \int_{t-k}^{t-k+1} \|X(t)X^{-1}(s)f(s)\|ds \\
&\le \int_{t-k}^{t-k+1} \|X(t)X^{-1}(s)\|\|f(s)\|ds \\
&\le M \int_{t-k}^{t-k+1} e^{-\gamma(t-s)}\|f(s)\|ds \\
&\le M\Big(\int_{t-k}^{t-k+1} e^{-q\gamma(t-s)}ds\Big)^{1/q}\Big(\int_{t-k}^{t-k+1} \|f(s)\|^p ds\Big)^{1/p} \\
&= M\frac{e^{-\gamma k}\sqrt[q]{e^{\gamma q}-1}}{\sqrt[q]{q\gamma}}\|f\|_{S^p}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} e^{-\gamma k}$ is convergent, from the Weierstrass test it follows that the sequence of functions $\sum_{k=1}^{n} \Lambda_k(t)$ is uniformly convergent on $\mathbb{R}$. Thus we have

$$\Lambda(t) = \sum_{k=1}^{\infty} \Lambda_k(t).$$

Let $\epsilon > 0$. Then there exists a number $l > 0$ such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_{t}^{t+1} \|f(s+\tau) - f(s)\|^p ds\Big)^{1/p} \le \epsilon_1,$$

where

$$0 < \epsilon_1 < \frac{\epsilon\sqrt[q]{q\gamma}}{M(e^{\gamma}-1)(\sqrt[q]{e^{q\gamma}-1})}.$$

Now, we consider

$$\begin{aligned}
&\|\Lambda_k(s+\tau)-\Lambda_k(s)\| \\
&= \Big\|\int_{s+\tau-k}^{s+\tau-k+1} X(s+\tau)X^{-1}(z)f(z)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz\Big\| \\
&= \Big\|\int_{s-k}^{s-k+1} X(s+\tau)X^{-1}(z+\tau)f(z+\tau)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz\Big\| \\
&= \Big\|\int_{s-k}^{s-k+1} X(s)X(\tau)X^{-1}(\tau)X^{-1}(z)f(z+\tau)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz\Big\| \\
&= \Big\|\int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z+\tau)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz\Big\| \\
&\le \int_{s-k}^{s-k+1} \|X(s)X^{-1}(z)\|\|f(\tau+z)-f(z)\|dz \\
&\le M\int_{s-k}^{s-k+1} e^{-\gamma(s-z)}\|f(\tau+z)-f(z)\|dz \\
&\le M\Big(\int_{s-k}^{s-k+1} e^{-q\gamma(s-z)}dz\Big)^{1/q}\Big(\int_{s-k}^{s-k+1}\|f(z+\tau)-f(z)\|^p dz\Big)^{1/p} \\
&\le \epsilon_1 M\Big(\int_{s-k}^{s-k+1} e^{-q\gamma(s-z)}dz\Big)^{1/q} = \frac{\epsilon_1 M e^{-\gamma k}(\sqrt[q]{e^{q\gamma}-1})}{\sqrt[q]{q\gamma}} < \epsilon.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty}\|\Lambda_k(s+\tau)-\Lambda_k(s)\| \le \frac{\epsilon_1 M(\sqrt[q]{e^{q\gamma}-1})}{\sqrt[q]{q\gamma}}\sum_{k=1}^{\infty} e^{-\gamma k} < \epsilon.$$

Hence, $\Lambda(t)$ is an almost periodic function.

Lemma 2.5 *If $f:\mathbb{R}\to\mathbb{R}^n$ is an S^p-almost periodic continuous function, then the system* (6) *has an almost periodic solution $x:\mathbb{R}\to\mathbb{R}^n$ if and only if*

$$x(t) = \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds, \quad t\in\mathbb{R}. \tag{10}$$

Moreover, the system (6) *has a unique almost periodic solution.*

Proof. Let $x:\mathbb{R}\to\mathbb{R}^n$ be an almost periodic solution of (6). Then

$$x(t) = X(t)X^{-1}(a)x(a) + \int_a^t X(t)X^{-1}(s)f(s)ds, \quad t\ge a,\ a\in\mathbb{R}.$$

For $t\ge a$, we have

$$\begin{aligned}
\|X(t)X^{-1}(a)x(a)\| &\le Me^{-\gamma(t-a)}\|x(a)\| \\
&\le Me^{-\gamma(t-a)}\sup_{t\in\mathbb{R}}\|x(t)\|.
\end{aligned}$$

Therefore,

$$\lim_{a\to-\infty}\|X(t)X^{-1}(a)x(a)\| = 0.$$

Hence,

$$x(t) = \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds, \quad t \in \mathbb{R}.$$

Conversely, let $x : \mathbb{R} \to \mathbb{R}^n$ be a function satisfying the integral representation (10). Then, by Lemma 2.4 , x is almost periodic. For $t \geq a$, we have

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds \\ &= \int_{-\infty}^{a} X(t)X^{-1}(s)f(s)ds + \int_{a}^{t} X(t)X^{-1}(s)f(s)ds \\ &= X(t)X^{-1}(a)\int_{-\infty}^{a} X(a)X^{-1}(s)f(s)ds + \int_{a}^{t} X(t)X^{-1}(s)f(s)ds \\ &= X(t)X^{-1}(a)x(a) + \int_{a}^{t} X(t)X^{-1}(s)f(s)ds. \end{aligned}$$

Hence x is an almost periodic solution of (6). Suppose x and y are two almost periodic solutions of (6), then $u = x - y$ is an almost periodic solution of (7), hence $u = 0$. Thus (6) has a unique almost periodic solution.

3 Main Result

In this section, we prove the existence of almost periodic solution of (5).
Consider the following assumptions:

(H1) $m > 1$ is an integer, $1 \leq p < \infty$ and q is the conjugate index of p.

(H2) $f_1(t)$ is a bounded continuous function defined from $\mathbb{R}$ into $\mathbb{R}$, and $\inf_{t\in\mathbb{R}} f_1(t) > 0$.

(H3) $f_2(t),\ \alpha(t),\ \beta(t),\ \phi(t)$ are all almost periodic functions defined from $\mathbb{R}$ into $\mathbb{R}$, and $\inf_{t\in\mathbb{R}} f_2(t) > 0$.

(H4) $F_1(t),\ F_2(t)$ are S^p-almost periodic continuous functions defined from $\mathbb{R}$ into $\mathbb{R}$.

Consider the following notations:

$$\begin{aligned} &\delta_1 = \inf_{t\in\mathbb{R}} f_1(t),\ \delta_2 = \inf_{t\in\mathbb{R}} f_2(t),\ \delta = \min\{\delta_1, \delta_2\}, \\ &\theta = \max\{\frac{1}{\delta},\ \frac{\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|)}{\delta}\},\ \sigma = \sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + m|\beta(t)|), \\ &\lambda = \frac{\sqrt[q]{(e^{q\delta} - 1)}}{\sqrt[q]{q\delta}(e^{\delta} - 1)} \max\{\|F_1\|_{S^p}, \|F_2\|_{S^p}\}. \end{aligned}$$

We indicate by E the set of all functions of the form $\varphi(t) = (\varphi_1(t), \varphi_2(t))^T$, where $\varphi_1(t), \varphi_2(t)$ are almost periodic functions defined from $\mathbb{R}$ into $\mathbb{R}$. Then E forms a Banach space with respect to the norm $\|\cdot\|_E$ given by

$$\|\varphi\|_E = \max\Big\{\sup_{t\in\mathbb{R}} |\varphi_1(t)|, \sup_{t\in\mathbb{R}} |\varphi_2(t)|\Big\}.$$

Theorem 3.1 *Suppose the assumptions (H1)-(H4) hold, and the positive constants λ, θ and σ satisfy*

$$\max\{1, \sigma\} < \delta, \ \frac{\lambda}{1-\theta} < 1, \tag{11}$$

then there exists a unique almost periodic solution of system (5) *in the region*

$$E^* = \{\varphi \in E : \|\varphi - \varphi_0\|_E \le \frac{\theta\lambda}{1-\theta}\},$$

where

$$\varphi_0(t) = \left(\int_{-\infty}^t e^{-\int_s^t f_1(z)dz} F_1(s)ds, \int_{-\infty}^t e^{-\int_s^t f_2(z)dz} F_2(s)ds\right)^T.$$

Proof. Since $m > 1$, we have

$$\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|) \le \sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + m|\beta(t)|),$$

hence $\theta \le \max\{\frac{1}{\delta}, \frac{\sigma}{\delta}\} < 1$. Clearly, E^* is a closed convex subset of E. We consider

$$\begin{aligned}
\|\varphi_0\|_E &= \max\{\sup_{t\in\mathbb{R}} |\int_{-\infty}^t e^{-\int_s^t f_1(z)dz} F_1(s)ds|, \sup_{t\in\mathbb{R}} |\int_{-\infty}^t e^{-\int_s^t f_2(z)dz} F_2(s)ds|\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\int_s^t f_1(z)dz} |F_1(s)|ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\int_s^t f_2(z)dz} |F_2(s)|ds\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\int_s^t \delta dz} |F_1(s)|ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\int_s^t \delta dz} |F_2(s)|ds\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\delta(t-s)} |F_1(s)|ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\delta(t-s)} |F_2(s)|ds\} \\
&= \max\{\sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} e^{-\delta(t-s)} |F_1(s)|ds, \\
&\quad \sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} e^{-\delta(t-s)} |F_2(s)|ds\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} (\int_{t-k}^{t-k+1} e^{-q\delta(t-s)} ds)^{1/q} \|F_1\|_{S^p}, \\
&\quad \sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} (e^{-q\delta(t-s)} ds)^{1/q} \|F_2\|_{S^p}\} \\
&= \max\{\frac{\sqrt[q]{(e^{q\delta} - 1)}}{\sqrt[q]{q\delta}(e^{\delta} - 1)} \|F_1\|_{S^p}, \frac{\sqrt[q]{(e^{q\delta} - 1)}}{\sqrt[q]{q\delta}(e^{\delta} - 1)} \|F_2\|_{S^p}\} \\
&= \frac{\sqrt[q]{(e^{q\delta} - 1)}}{\sqrt[q]{q\delta}(e^{\delta} - 1)} \max\{\|F_1\|_{S^p}, \|F_2\|_{S^p}\} = \lambda.
\end{aligned}$$

Therefore, for any $\varphi \in E^*$, we get

$$\|\varphi\|_E \le \|\varphi - \varphi_0\|_E + \|\varphi_0\|_E \le \frac{\theta\lambda}{1-\theta} + \lambda = \frac{\lambda}{1-\theta} < 1. \tag{12}$$

Now, let $\varphi \in E$, and consider the following nonlinear system

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} -f_1(t) & 0 \\ 0 & -f_2(t) \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} \varphi_2(t) + F_1(t) \\ \tilde{\varphi}_1(t) \end{pmatrix}, \tag{13}$$

where $\tilde{\varphi}_1(t) = (\alpha(t) + f_2^2(t))\varphi_1(t) - \beta(t)\varphi_1^m(t - \phi(t)) + F_2(t)$. By Lemma 2.1, $\varphi_1(t - \phi(t))$ is almost periodic, and hence $\tilde{\varphi}_1(t)$ is a Stepanov-almost periodic function. Since the matrix $\begin{pmatrix} -f_1(t) & 0 \\ 0 & -f_2(t) \end{pmatrix}$ satisfies all conditions given in Lemma 2.2, by Lemma 2.5 the system (13) has a unique almost periodic solution and is given by

$$\begin{pmatrix} u^\varphi(t) \\ v^\varphi(t) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{-\infty}^{t} e^{-\int_s^t f_1(z)dz}[\varphi_2(s) + F_1(s)]ds \\ \displaystyle\int_{-\infty}^{t} e^{-\int_s^t f_2(z)dz}\tilde{\varphi}_1(s)ds \end{pmatrix}.$$

Therefore, for each $\varphi \in E$, (13) has a unique almost periodic solution $\begin{pmatrix} u^\varphi(t) \\ v^\varphi(t) \end{pmatrix}$.

Define a map $T : E \to E$ by

$$T(\varphi)(t) = \begin{pmatrix} u^\varphi(t) \\ v^\varphi(t) \end{pmatrix}.$$

To prove T is a self-mapping from E^* into E^*, we consider, for any $\varphi \in E^*$,

$$\begin{aligned}
&\|T\varphi - \varphi_0\|_E \\
&= \max\{\sup_{t\in\mathbb{R}} |\int_{-\infty}^{t} e^{-\int_s^t f_1(z)dz}\varphi_2(s)ds|, \\
&\quad \sup_{t\in\mathbb{R}} |\int_{-\infty}^{t} e^{-\int_s^t f_2(z)dz}(\alpha(s) + f_2^2(s))\varphi_1(s) - \beta(s)\varphi_1^m(t - \phi(s))ds|\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta_1(t-s)}\|\varphi\|_E ds, \\
&\quad \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta_2(t-s)}[(|\alpha(s) + f_2^2(s)|)\|\varphi\|_E + |\beta(s)|\|\varphi\|_E^m]ds\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta_1(t-s)}ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta_2(t-s)}(|\alpha(s) + f_2^2(s)| + |\beta(s)|)ds\}\|\varphi\|_E \\
&\le \max\{\frac{1}{\delta_1}, \frac{\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|)}{\delta_2}\}\|\varphi\|_E \\
&\le \max\{\frac{1}{\delta}, \frac{\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|)}{\delta}\}\|\varphi\|_E \\
&= \theta\|\varphi\|_E \le \frac{\theta\lambda}{1-\theta}.
\end{aligned}$$

Therefore, T maps E^* into itself. Next, we prove that T is a contraction mapping from E^* into itself. For $\varphi, \psi \in E^*$, we have

$$\begin{aligned}
&\|T(\varphi)-T(\psi)\|_E\\
&=\max\{\sup_{t\in\mathbb{R}}|\int_{-\infty}^{t}e^{-\int_s^t f_1(z)dz}[\varphi_2(s)-\psi_2(s)]ds|,\sup_{t\in\mathbb{R}}|\int_{-\infty}^{t}e^{-\int_s^t f_2(z)dz}[(\alpha(s)\\
&\qquad+f_2^2(s))(\varphi_1(s)-\psi_1(s))-\beta(s)(\varphi_1^m(s-\phi(s))-\psi_1^m(s-\phi(s)))]ds|\}\\
&\le\max\{\sup_{t\in\mathbb{R}}|\int_{-\infty}^{t}e^{-\int_s^t \delta_1 dz}\|\varphi-\psi\|_E ds,\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\int_s^t \delta_2 dz}[|\alpha(s)+f_2^2(s)|\times\\
&\qquad|\varphi_1(s)-\psi_1(s)|+|\beta(s)||\varphi_1^m(s-\phi(s))-\psi_1^m(s-\phi(s))|]ds\}\\
&\le\max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E,\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\delta_2(t-s)}[|\alpha(s)+f_2^2(s)|\|\varphi-\psi\|_E+|\beta(s)|\times\\
&\quad(|\varphi_1(s-\phi(s))-\psi_1(s-\phi(s))|(|\varphi_1^{m-1}(s-\phi(s))|+|\varphi_1^{m-2}(s-\phi(s))|\times\\
&\quad|\psi_1(s-\phi(s))|+\cdots+|\varphi_1(s-\phi(s))||\psi_1^{m-2}(s-\phi(s))|+|\psi_1^{m-1}(s-\phi(s))|))]ds\}\\
&\le\max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E,\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\delta_2(t-s)}[|\alpha(s)+f_2^2(s)|\|\varphi-\psi\|_E+|\beta(s)|\times\\
&\quad\|\varphi-\psi\|_E(\|\varphi\|_E^{m-1}+\|\varphi\|_E^{m-2}\|\psi\|_E+\cdots+\|\varphi\|_E\|\psi\|_E^{m-2}+\|\psi\|_E^{m-1})]ds\}\\
&\le\max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E,\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\delta_2(t-s)}[|\alpha(s)+f_2^2(s)|\|\varphi-\psi\|_E\\
&\qquad\qquad+m|\beta(s)|\|\varphi-\psi\|_E]ds\}\\
&\le\max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E,\frac{1}{\delta_2}\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)|+m|\beta(t)|)\|\varphi-\psi\|_E\}\\
&\le\max\{\frac{1}{\delta}\|\varphi-\psi\|_E,\frac{1}{\delta}\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)|+m|\beta(t)|)\|\varphi-\psi\|_E\}\\
&\le\max\{\frac{1}{\delta},\frac{1}{\delta}\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)|+m|\beta(t)|)\}\|\varphi-\psi\|_E\\
&=\max\{\frac{1}{\delta},\frac{\sigma}{\delta}\}\|\varphi-\psi\|_E.
\end{aligned}$$

Since $\max\{\frac{1}{\delta},\frac{\sigma}{\delta}\}<1$, T is a contraction map on E^*. Therefore, T has a unique fixed point $\varphi^*(t)=(u^*(t),v^*(t))^T\in E^*$, i.e., $T\varphi^*=\varphi^*$. By (13), φ^* satisfies (5), hence φ^* is an almost periodic solution of the system (5) in E^*.

4 Application

Example 4.1 The following nonlinear Duffing equation with time-varying coefficients

$$\begin{aligned}
&u''(t)+(20+\sin t+\sin t^2)u'(t)+(100.5+10\sin t-\sin^2 t+\sin t\sin t^2+10\sin t^2\\
&\qquad+2t\cos t^2)u(t)+0.5\cos t(u^3(t-\sin t)-\sin t-10)-0.5\sin\sqrt{2}t=0 \qquad (14)
\end{aligned}$$

has at least one almost periodic solution.

Proof. Consider

$$v(t)=u'(t)+(10+\sin t^2)u(t)-0.5\cos t, \qquad (15)$$

then we can transform (14) into the following system of differential equations

$$\begin{cases} u'(t) = -(10+\sin t^2)u(t) + v(t) + 0.5\cos t, \\ v'(t) = -(10+\sin t)v(t) + (\sin^2 t - 0.5)u(t) - 0.5\cos t u^3(t-\sin t) \\ \qquad\qquad +0.5\sin t + 0.5\sin\sqrt{2}t. \end{cases} \tag{16}$$

Since $f_1(t) = 10+\sin t^2$, $f_2(t) = 10+\sin t$, $\alpha(t) = -100.5 - 20\sin t$, $\beta(t) = 0.5\cos t$, $F_1(t) = 0.5\cos t$, $F_2(t) = 0.5\sin t + 0.5\sin\sqrt{2}t$, $\phi(t) = \sin t$, $p = 2$, $m = 3$, we have $\delta_1 = \delta_2 = \delta = 9$, $q = 2$,

$$\begin{aligned} \theta &= \max\{\frac{1}{\delta}, \frac{\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)| + |\beta(t)|)}{\delta}\} \\ &= \max\{\frac{1}{9}, \frac{\sup_{t\in\mathbb{R}}(|-0.5+\sin^2 t| + |0.5\cos t|)}{9}\} \\ &= \max\{\frac{1}{9}, \frac{1}{9}\} = \frac{1}{9}. \end{aligned}$$

Since

$$0 < \|F_1\|_{S^2} \le \sup_{t\in\mathbb{R}} |0.5\cos t| = 0.5 \quad \text{and}$$

$$0 < \|F_2\|_{S^2} \le \sup_{t\in\mathbb{R}} (|0.5\sin t + 0.5\sin\sqrt{2}t|) = 1,$$

$$\begin{aligned} 0 < \lambda &= \frac{\sqrt[q]{(e^{q\delta}-1)}}{\sqrt[q]{q\delta}(e^{\delta}-1)} \max\{\|F_1\|_{S^p}, \|F_2\|_{S^p}\} \\ &= \frac{\sqrt{(e^{18}-1)}}{\sqrt{18}(e^9-1)} \max\{\|F_1\|_{S^2}, \|F_2\|_{S^2}\} \le \frac{\sqrt{(e^{18}-1)}}{\sqrt{18}(e^9-1)} \\ \Longrightarrow \lambda^2 &\le \frac{(e^{18}-1)}{18(e^9-1)^2} = \frac{1}{18} + \frac{1}{9(e^9-1)} < \frac{3}{18} \\ \Longrightarrow \lambda &< \sqrt{\frac{3}{18}} < \frac{8}{9} = 1-\theta \Longrightarrow \frac{\lambda}{1-\theta} < 1. \end{aligned}$$

$$\sigma = \sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + m|\beta(t)|) = \sup_{t\in\mathbb{R}}(|-0.5+\sin^2 t| + 3|0.5\cos t|) = 2 < 9 = \delta.$$

Therefore, all the assumptions given in Theorem 3.1 are satisfied, hence (16) has at least one almost periodic solution. Thus, the nonlinear Duffing equation (14) has at least one almost periodic solution.

Remark 4.1 Notice that the function $f_1(t) = 10+\sin t^2$ is not almost periodic and the coefficient of $u(t)$ in (14) is unbounded. Thus, the results of this paper are substantially extended and improved the main results of [13, 15, 16, 18].

5 Conclusion

In this paper, we considered a class of nonlinear Duffing system (5) with time-varying coefficients and Stepanov-almost periodic forcing terms. We first considered the almost periodic system (6) with a Stepanov-almost periodic continuous forcing function and then studied the existence of almost periodic solutions of (6). Using these results, we established some sufficient conditions for the existence and uniqueness of an almost periodic solution of the system (5). Finally, we provided an example to illustrate the main result.

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Performance Evaluation of Synchronization of Chua's System Under Different Memductance

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Abstract: Practical implementation of synchronization schemes is important for secure communication. With many systems available, simple systems with varying differences will prove pertinent in user identification and inter-operability of communicating units. The implementation of Chua's circuit with different memristors is a potential candidate for the realization of such units. In this paper, a general control function for the synchronization of two Chua's circuits with similar or dissimilar memristors was developed. Three different memristor circuits were considered in this paper. Numerical simulation of the proposed control function was carried out and the performance of different memristors in the similar and dissimilar configuration was considered.

Keywords: *memristor; active control; Chua's circuit; synchronization.*

Mathematics Subject Classification (2010): 34H10, 93C10.

1 Introduction

The application of chaotic systems in secure communications has led to the development of several synchronization schemes. Initially, synchronization of chaotic systems was between two identical systems [1] before it was extended from two different chaotic systems to the increased and reduced order synchronization between two systems [2], increased and reduced order between three or more systems [3, 4], synchronization of fractional order systems [5], delay differential equations, discrete chaotic systems, and electronic realization [6].

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Different types of synchronization such as complete synchronization, lag synchronization, generalized synchronization, projective synchronization, and function projective synchronization have been developed. There exist several synchronization techniques including the active control, backstepping, Open-Plus-Closed-Loop (OPCL), recursive active control. Studies have shown that the active control method has better performance for integer order [7] and fractional order systems [5].

Implementation of synchronization is required for real time applications in secure communication. Electronic circuit design and implementation of chaotic systems are important in the understanding of dynamic systems and practical implementation. The discovery of chaos in electronic circuits by Chua [8] initiated a new line of research [9]. Chua's chaotic circuit has been extensively studied with several modifications. Combination synchronization of memristive circuit was realized through the diffusive and negative feedback coupling [6], time delayed sliding mode synchronization in a novel chaotic memcapacitor [10], design of low dimensional fractional order nonautonomous system based on the Chua system [11], experimental realization of synchronization in a network using Chua's circuit [12].

The main goal of this paper is to investigate the behaviour of different memristors under synchronization using an active control method. The performance of different memristors is important in real life implementation of synchronization for secure communication. Hence, the speed of synchronization and fluctuations before synchronization are considered in this paper. In Section 2, the system and different memristors to be considered are discovered while the synchronization of the systems is discussed in Section 3. Results are presented in Section 4 and conclusions are given in Section 5.

2 System Description

Chua's circuit is given by the expression [13]

$$\begin{aligned} \dot{x} &= \alpha(y - f(x)), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned} \tag{1}$$

where x, y, z are state space of the system, and the piecewise linear function $f(x)$ is defined as

$$f(x) = \begin{cases} \alpha(y - bx - (b - a), & \text{if } x < -1; \\ \alpha(y - ax), & \text{if } -1 \leq x \leq 1; \\ \alpha(y - bx - (a - b)), & \text{if } x > 1, \end{cases}$$

a, b, c are constants.

By replacing Chua's diode with a flux controlled memristor, Itoh and Chua [14] transformed the canonical Chua circuit into a 4 D system of the form

$$\begin{aligned} \dot{y}_1 &= \frac{1}{C_1 R}(y_2 - y_1) - \frac{1}{C_1} y_1 W_i, \\ \dot{y}_2 &= \frac{1}{C_2 R}(y_1 - y_2) - y_3, \\ \dot{y}_3 &= \frac{1}{L} y_2 - \frac{r}{L} y_3, \\ \dot{y}_4 &= y_1, \end{aligned} \tag{2}$$

where C_1, C_2, L are the circuit elements and W_i is the memductance. In this paper, the synchronization of system (2) will be investigated under three different flux controlled memristors.

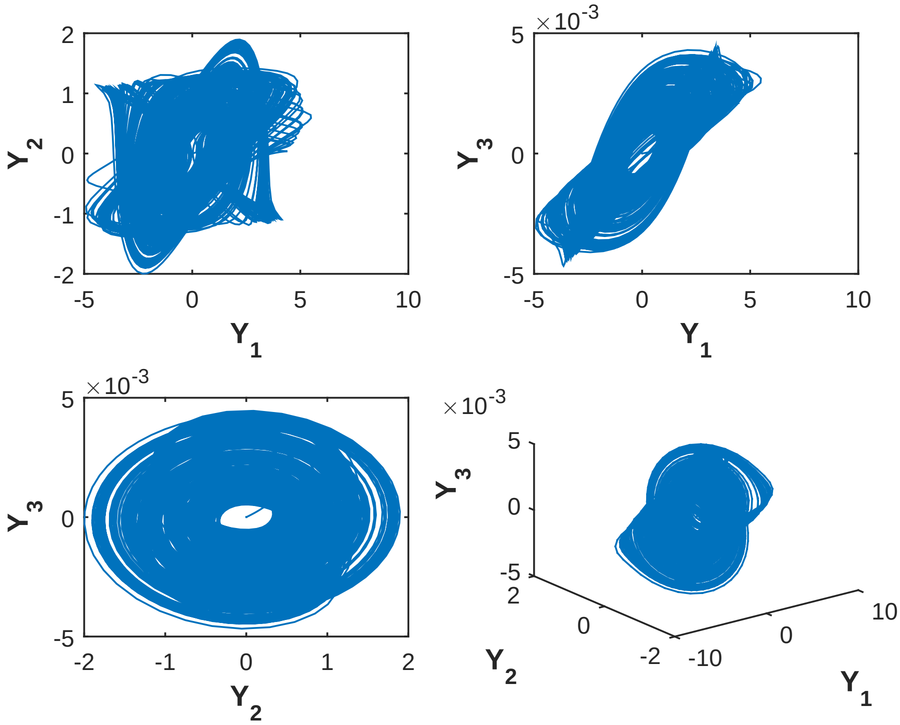

Figure 1: Phase space realization of Chua's circuit with a memristor of the form $W_i = -a + b|y_4|$, where the values of a and b are taken as 0.6667×10^{-3} and 1.4828, respectively.

2.1 Memristor of type I

By replacing Chua's diode in Chua's chaotic circuit with an active flux memristor, a new memristor based chaotic circuit was obtained by [15] as

$$\begin{aligned} \dot{x} &= \alpha(y - x - W(w)x), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y - \gamma z, \\ \dot{w} &= x, \end{aligned} \tag{3}$$

where $W(w) = -a + b|w|$. The system was found chaotic for a wide range of values of $|w|$. The circuit form was given and implemented by [16] as

$$\begin{aligned}
\frac{dv_1}{dt} &= \frac{1}{C_1}\left(\frac{v_2 - v_1}{R} - W(\phi)v_1\right), \\
\frac{dv_2}{dt} &= \frac{1}{C_2}\left(\frac{v_1 - v_2}{R} - i_3\right), \\
\frac{di_3}{dt} &= \frac{1}{L}(v_2 - v_1), \\
\frac{d\phi}{dt} &= v_1.
\end{aligned} \tag{4}$$

The phase space representation of this system is shown in Figure 1.

2.2 Memristor of type II

A memductance function of the form

$$W(\phi) = \frac{dq(\phi(t)}{d\phi(t)} = -a + 3b\phi^2(t) \tag{5}$$

was introduced to extend 4D chaotic Chua's circuit proposed in [17] to a 5D system by [18]. The proposed systems thus became

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= \frac{1}{C_1}(x_3(t) - W(x_5(t))x_1(t), \\
\frac{dx_2(t)}{dt} &= \frac{1}{C_2}(-x_3(t) + x_4(t)), \\
\frac{dx_3(t)}{dt} &= \frac{1}{L}(x_2(t) - x_1(t) - Rx_3(t)), \\
\frac{dx_4(t)}{dt} &= \frac{-x_2(t)}{L_2}, \\
\frac{dx_5(t)}{dt} &= x_1(t).
\end{aligned} \tag{6}$$

The system was reported to exhibit chaos for certain system parameters. The phase space and dynamics of equation (2) with memristor of the form (5) is presented in Figure 2.

2.3 Memristor of type III

A dimensionless flux controlled memristor model with fifth order flux polynomial was proposed by [19] as

$$\begin{aligned}
\dot{x} &= \alpha(y + x - W(w)x), \\
\dot{y} &= \beta x + \gamma y - z, \\
\dot{z} &= \delta y - z, \\
\dot{w} &= x,
\end{aligned} \tag{7}$$

where the memductance $W(\phi)$ is defined as

$$W(\phi) = a\phi^4 - b\phi^2 - c, \tag{8}$$

The phase space and dynamics of equation (2) with memristor of the form (8) is presented in Figure 3.

3 Synchronization

Theorem 1 *If the drive and response systems of the form (2) have the memristive elements given by W_i^j and W_i^k, respectively (where $i = 1, 2, 3$ are the different types of memristors being considered), complete synchronization will be achieved by the method of active control if the control function is chosen as*

$$\begin{aligned} u_1(t) &= \frac{1}{C_1} y_1 W_i^j - \frac{1}{C_1} x_1 W_i^k + \left(\lambda_1 + \frac{1}{C_1 R}\right) e_1 - \frac{1}{C_1 R} e_2, \\ u_2(t) &= \frac{1}{C_2 R} e_1 + \left(\lambda_2 + \frac{1}{C_2 R}\right) e_2 + \frac{1}{C_2} e_3, \\ u_3(t) &= -\frac{1}{L} e_2 + \left(\lambda_3 + \frac{r}{L}\right) e_3, \\ u_4(t) &= e_1 + \lambda_4 e_4, \end{aligned} \tag{9}$$

where λ_i are chosen to be negative and $e_i = y_i - x_i$, then the drive system (2) will achieve multi-switching synchronization with the response system.

Proof. Take equation (2) as the drive system and the following ones as the response system

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C_1 R}(x_2 - x_1) - \frac{1}{C_1} x_1 W_i + u_1(t), \\ \dot{x}_2 &= \frac{1}{C_2 R}(x_1 - x_2) - x_3 + u_2(t), \\ \dot{x}_3 &= \frac{1}{L} x_2 - \frac{r}{L} x_3 + u_3(t), \\ \dot{x}_4 &= x_1 + u_4(t), \end{aligned} \tag{10}$$

where u_i are the controllers to be determined. Substituting equations (2) and (10) into the error dynamics $e_i = y_i - x_i$, where $i = 1, 2, 3$, we obtain

$$\begin{aligned} \dot{e}_1 &= \frac{1}{C_1 R}(e_2 - e_1) - \frac{1}{C_1} y_1 W_i^j + \frac{1}{C_1} x_1 W_i^k + u_1(t), \\ \dot{e}_2 &= \frac{1}{C_2 R}(e_1 - e_2) - e_3 + u_2(t), \\ \dot{e}_3 &= \frac{1}{L} e_2 - \frac{r}{L} e_3 + u_3, \\ \dot{e}_4 &= e_1 + u_4, \end{aligned} \tag{11}$$

To achieve asymptotic stability of system (11), the terms, which are nonlinear in e_i, are eliminated as follows:

$$\begin{aligned} u_1 &= \frac{1}{C_1} y_1 W_i^j - \frac{1}{C_1} x_1 W_i^k + v_1(t), \\ u_2 &= v_2(t), \\ u_3 &= v_3, \\ u_4 &= v_4, \end{aligned} \tag{12}$$

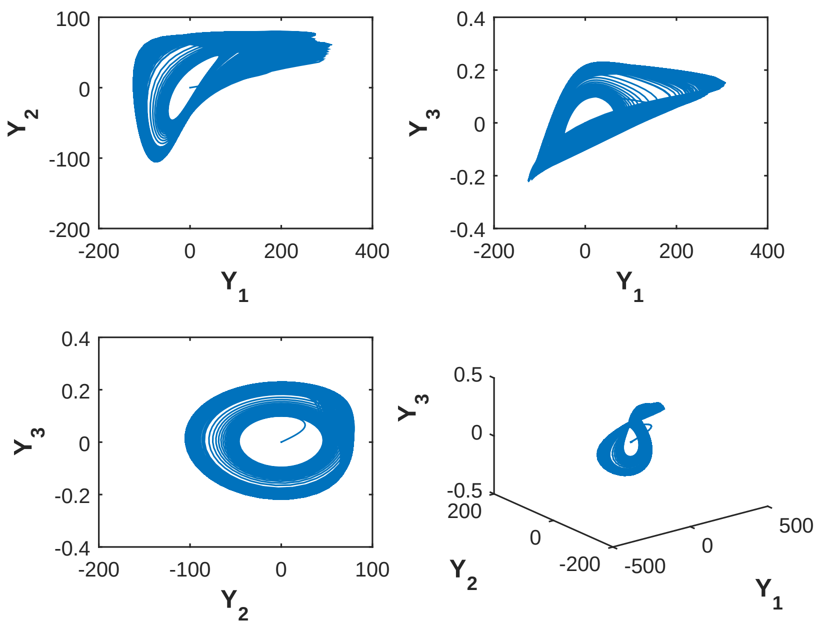

Figure 2: Phase space realization of Chua's circuit with the memristor of the form $W_i = -a + 3by_4^2$, where the values of a and b are taken as 0.6667×10^{-3} and 1.4828, respectively.

substituting (12) into (11) gives

$$\begin{aligned}
\dot{e}_1 &= \frac{1}{C_1 R}(e_2 - e_1) + v_1(t), \\
\dot{e}_2 &= \frac{1}{C_2 R}(e_1 - e_2) - e_3 + v_2(t), \\
\dot{e}_3 &= \frac{1}{L}e_2 - \frac{r}{L}e_3 + v_3, \\
\dot{e}_4 &= e_1 + v_4.
\end{aligned} \tag{13}$$

Using the active control method, a constant matrix $\mathbb{D}$ is chosen which will control the error dynamics (13) such that the feedback matrix is $V_i = \mathbb{D}e_i$. There are various choices of the feedback $\mathbb{D}$ which can be chosen to control the error dynamics [20]. We chose $\mathbb{D}$ to be of the form

$$\mathbb{D} = \begin{pmatrix} \left(\lambda_1 + \frac{1}{C_1 R}\right) & -\frac{1}{C_1 R} & 0 & 0 \\ \frac{1}{C_2 R} & \left(\lambda_2 + \frac{1}{C_2 R}\right) & 1 & 0 \\ 0 & -\frac{1}{L} & \left(\lambda_3 + \frac{r}{L}\right) & 0 \\ 1 & 0 & 0 & \lambda_4 \end{pmatrix}. \tag{14}$$

If the eigenvalues λ_i are chosen to be negative, a stable synchronization between the drive and response system will be achieved.

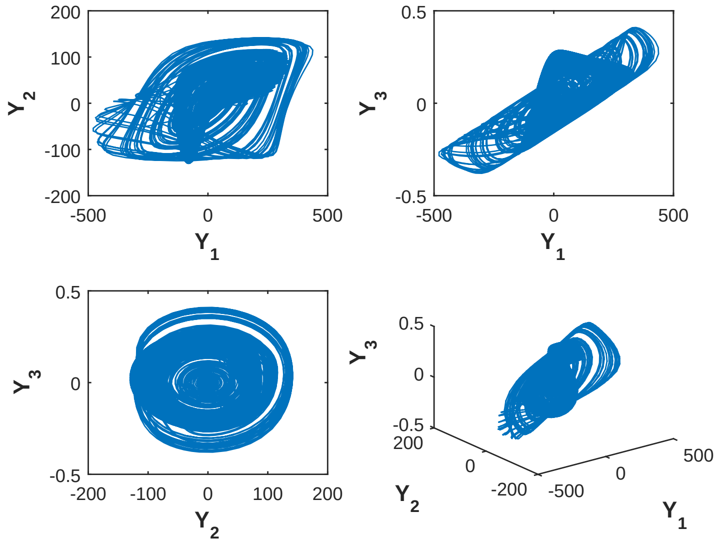

Figure 3: Phase space realization of Chua's circuit with memristor of the form $W_i = ay_4^4 - by_4^2 - c$, where the values of a b and c are taken as 1000, 1.087 and $0.33e^{-3}$, respectively.

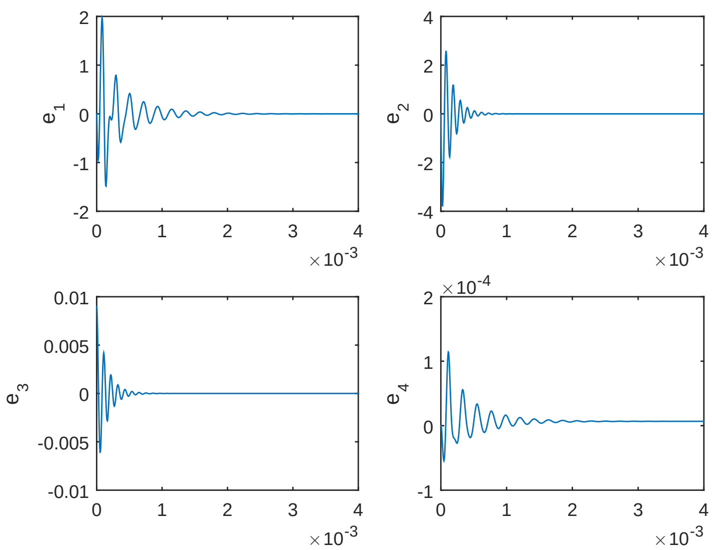

Figure 4: The synchronization error functions for two systems with the memristor of type I using control functions as described in Corollary 3.1.

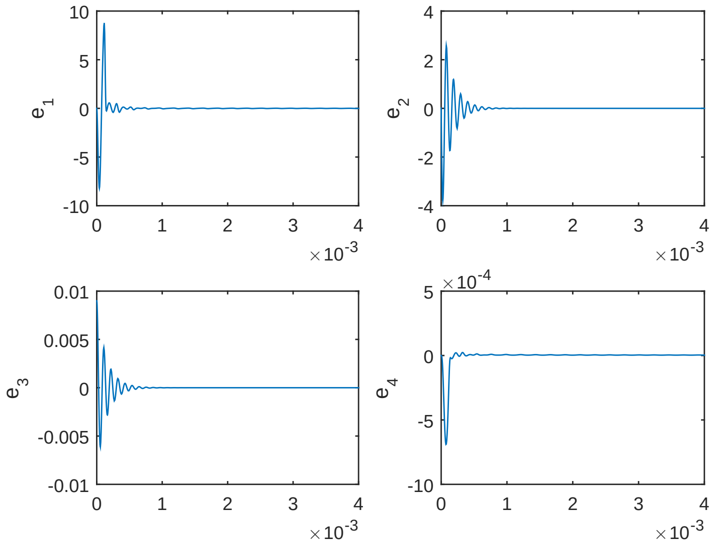

Figure 5: The synchronization error functions for two systems with the memristor of type II using control functions as described in Corollary 3.2.

Corollary 3.1 *If* $W_1^1 = -a_1^1 + b_1^1|y_4|$ *and* $W_1^2 = -a_1^2 + b_1^2|x_4|$, *then the control function (14) can be written as*

$$\begin{aligned} u_1(t) &= \frac{1}{C_1}y_1W_i^j - \frac{1}{C_1}x_1W_i^k + \left(\lambda_1 + \frac{1}{C_1R}\right)e_1 - \frac{1}{C_1R}e_2, \\ u_2(t) &= \frac{1}{C_2R}e_1 + \left(\lambda_2 + \frac{1}{C_2R}\right)e_2 + \frac{1}{C_2}e_3, \\ u_3(t) &= -\frac{1}{L}e_2 + \left(\lambda_3 + \frac{r}{L}\right)e_3, \\ u_4(t) &= e_1 + \lambda_4 e_4. \end{aligned} \tag{15}$$

Corollary 3.2 *If* $W_2^1 = -a_2^1 + 3b_2^1y_4^2$ *and* $W_2^2 = -a_2^2 + 3b_2^2x_4^2$, *then the control function (14) can be written as*

$$\begin{aligned} u_1(t) &= \frac{1}{C_1}y_1W_i^j - \frac{1}{C_1}x_1W_i^k + \left(\lambda_1 + \frac{1}{C_1R}\right)e_1 - \frac{1}{C_1R}e_2, \\ u_2(t) &= \frac{1}{C_2R}e_1 + \left(\lambda_2 + \frac{1}{C_2R}\right)e_2 + \frac{1}{C_2}e_3, \\ u_3(t) &= -\frac{1}{L}e_2 + \left(\lambda_3 + \frac{r}{L}\right)e_3, \\ u_4(t) &= e_1 + \lambda_4 e_4. \end{aligned} \tag{16}$$

Corollary 3.3 *If* $W_1^1 = a_3^1y_4^4 - b_3^1y_4^2 - c_3^1$ *and* $W_1^2 = a_3^2y_4^4 - b_3^2y_4^2 - c_3^2$, *then the control*

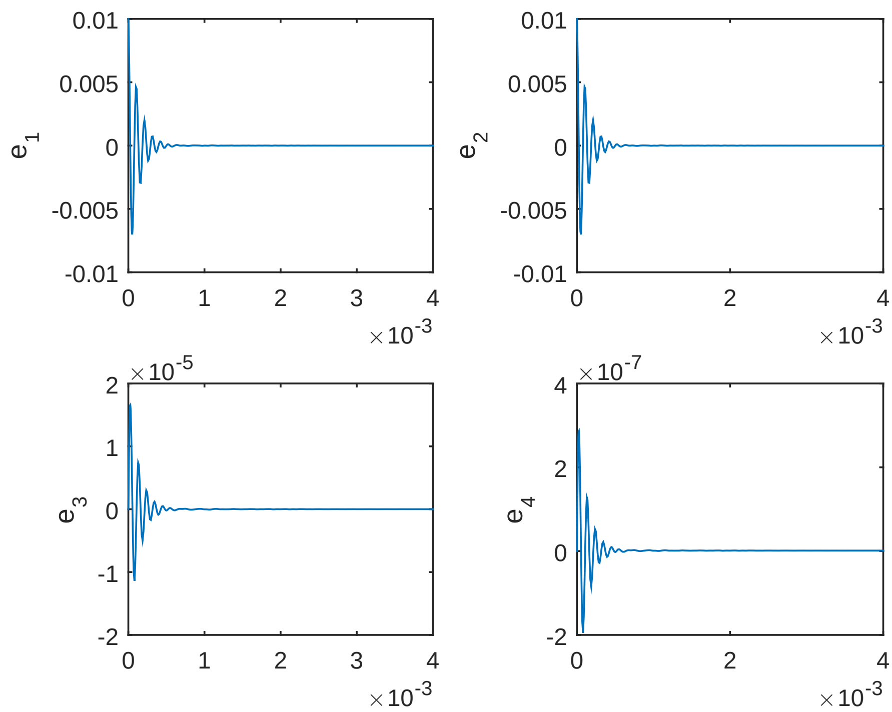

Figure 6: The synchronization error functions for two systems with the memristor of type III using control functions as described in Corollary 3.3.

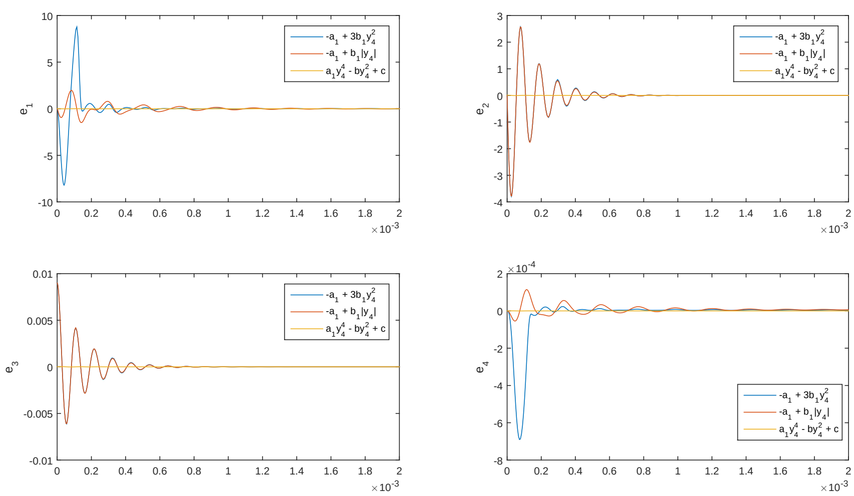

Figure 7: Error functions of different memristors when synchronized with the memristor W_{ii} (where $i = 1, 2, 3$).

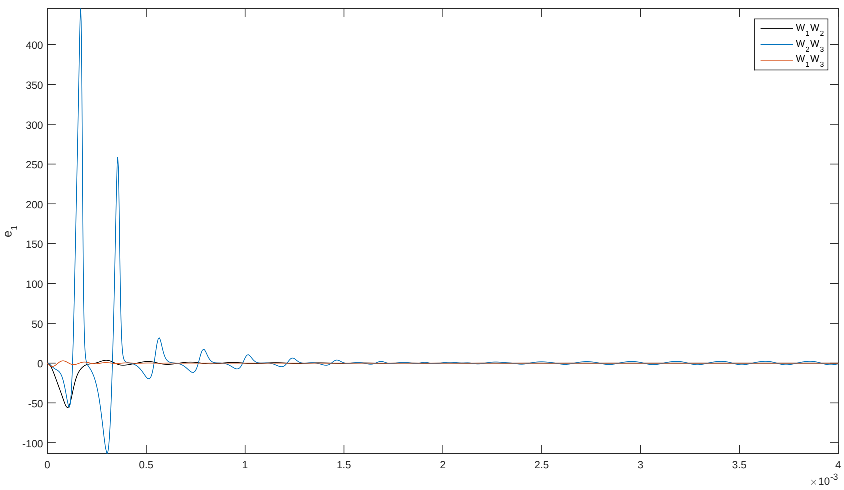

Figure 8: Comparison of the synchronization error for $W_{ij}(i \neq j)$.

function (14) can be written as

$$
\begin{aligned}
u_1(t) &= \frac{1}{C_1} y_1 W_i^j - \frac{1}{C_1} x_1 W_i^k + \left(\lambda_1 + \frac{1}{C_1 R}\right) e_1 - \frac{1}{C_1 R} e_2, \\
u_2(t) &= \frac{1}{C_2 R} e_1 + \left(\lambda_2 + \frac{1}{C_2 R}\right) e_2 + \frac{1}{C_2} e_3, \\
u_3(t) &= -\frac{1}{L} e_2 + \left(\lambda_3 + \frac{r}{L}\right) e_3, \\
u_4(t) &= e_1 + \lambda_4 e_4.
\end{aligned} \tag{17}
$$

4 Results

In order to verify the effectiveness of the proposed controllers, numerical simulations were carried out. The system of equations with the proposed controllers was solved using the fourth-order Runge-Kutta method with step size of 0.0001. Using the initial conditions $(y_1, y_2, y_3, y_4) = (10 \times 10^{-3}, 0.02, 0.01, 1 \times 10^{-3})$ and $(x_1, x_2, x_3, x_4) = (9 \times 10^{-3}, 10 \times 10^{-3} 0.0011 \times 10^{-3})$, the controllers described in Corollary 3.1 were implemented. The results are shown in Figure 4. The synchronization errors between the two systems when $W_1 = -a_1 + b_2|y_4|$ and $W_2 = -a_2 + b_2|x_4|$ were simulated. Similarly, the synchronization error between system (2) and (10) when $W_1 = -a_1 + 3b_2 y_4^2$ and $W_2 = -a_2 + b3_2 x_4^2$ using the initial conditions $(y_1, y_2, y_3, y_4) = (10 \times 10^{-3}, 0.02, 0.01, 1 \times 10^{-3})$ and (x_1, x_2, x_3, x_4) $= (9 \times 10^{-3}, 10 \times 10^{-3} 0.0011 \times 10^{-3})$ are simulated and the results are shown in Figure 5. In the same vein, the control functions as described in Corollary 3.3 were simulated using two memristor of type III. The result is shown in Figure 6. The performance of each memristor, when synchronized in the form W_{ii}, can be investigated from Figure 7. The memristor of type II showed the worst performance while the memristor of type III exhibited the best performance. The error fluctuation were the greatest for the memristor of type II and the lowest for the memristor of type III. In Figure 8, the performance of synchronization of the system under two different memristors $W_{ij}(i \neq j)$ is reported using only the first error component e_1. The combination of W_1W_3 has the lowest fluctuations before synchronization and the fastest synchronization of the three different combinations. The worst performance was the combination W_2W_3 which has the highest fluctuations amongst the three and the slowest convergence.

5 Conclusion

In this paper, we have investigated the synchronization of different memristors in the same circuit with identical and non-identical memristors. Suitable controllers were designed using the method of active control. Numerical simulation results showed that the controllers were effective. The performance of the three memristors was investigated using fluctuations before synchronization and time to synchronization. In the synchronization between two similar memristors, the memristor of type III was found to have the best performance. However, the synchronization of memristors of type I and III exhibited the fastest synchronization and least fluctuation, making it the best performing. This scheme will find application in a multiuser secure communication environment. Further investigation may be carried out on the multi-switching, time delayed and practical implementation of this scheme.

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Robust Stability of Markovian Jumping Neural Networks with Time-Varying Delays

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Abstract: In this paper, global stability of recurrent neural networks with time-varying delays is considered. The uncertainity is considered in all the parameters of the concerned neural networks. A novel LMI-based stability criterion is obtained by using the Lyapunov functional theory to guarantee the asymptotic stability of recurrent neural networks with time-varying delays. Finally, a numerical example is given to demonstrate the correctness of the theoretical results.

Keywords: *Lyapunov functional; linear matrix inequality; recurrent neural networks; time-varying delays.*

Mathematics Subject Classification (2010): 34K20, 34K50, 92B20, 94D05.

1 Introduction

A recurrent neural network naturally involves dynamic elements in the form of feedback connections used as internal memories. Unlike the feedforward neural network whose output is a function of its current inputs only and is limited to static mapping, the recurrent neural network performs dynamic mapping. Recurrent networks are needed for the problems where there exists at least one system state variable which cannot be observed. Most of the existing recurrent neural networks are obtained by adding trainable temporal elements to the feedforward neural networks (such as multilayer perceptron networks [5] and radial basis function networks [2]) to make the output history sensitive. Like feedforward neural networks, this network function as block boxes and the meaning

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of each weight in these nodes is not known. They play an important role in applications such as classification of patterns, associate memories and optimization etc. (see [2], [5] and the references therein). Thus, research on the properties of an especial stability problem and relaxed stability problem of recurrent neural networks, has become a very active area in the past few years (see for example [3], [8], [9]).

It is well known that time delays are inevitably encountered in neural networks which are usually a main source of oscillation and instability, which brings to the neural network divergence and instability and needs much attention to be payed. According to the finite switching speed of amplifiers in electronic networks, time delay is either constant or time-varying. The stability criteria of neural networks with time-varying delays are classified into two categories, i.e., delay-independent [11] and delay-dependent [14]. The delay-independent stability conditions are usually more conservative than delay-dependent conditions due to the fact that they include less information concerning the time delays, especially for the time delays which are relatively small. Recently, many important results on the delay-dependent stability analysis have been reported for neural networks with time-varying delays [10], [12], [13].

In this paper, we study stability of recurrent neural networks with time-varying delays. By using the Lyapunov functional technique, global robust stability conditions for the recurrent neural networks are given in terms of LMIs, which can be easily calculated by MATLAB LMI toolbox [4]. The main advantage of the LMI based approaches is that the LMI stability conditions can be solved numerically using the effective interior-point algorithms [1]. Numerical examples are provided to demonstrate the effectiveness and applicability of the proposed stability results.

Notations: Throughout the manuscript we will use the notation $A > 0$ (or $A < 0$) to denote that the matrix A is a symmetric and positive definite (or negative definite) matrix. The shorthand $diag\ \{\cdots\}$ denotes the block diagonal matrix. $\|\cdot\|$ stands for the Euclidean norm. Moreover, the notation * always denotes the symmetric block in one symmetric matrix. Let $r(t), t \geq 0$ be a right-continuous Markov chain on a complete probability space $(\Upsilon, \mathfrak{F}, \mathcal{P})$ taking values in a finite space $S = 1, 2, ...N$ with operator $\Lambda = \Pi_{ij}(n \times n)$ given by $P\{r(t+\Delta(t)) = j|r(t) = i\} = \begin{cases} \Pi_{ij}\Delta(t) + o(\Delta(t)), i \neq j, \\ 1 + \Pi_{ij}\Delta(t) + o(\Delta(t)), i = j, \end{cases}$ where $\Delta(t) > 0$ and $\lim_{\Delta(t)\to 0} \frac{o(\Delta(t))}{\Delta(t)} = 0$, $\Pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\Pi_{ii}{=}-\sum_{j=1, j\neq 1}^{N} \Pi_{ij}.$, $i, j \in S$.

2 System Description and Preliminaries

Consider the following uncertain recurrent neural network with time-varying delays described by

$$\dot{v}_i(t) = -a_i(r(t))v_i(t) + \sum_{j=1}^{n} b_{ij}(r(t)) + G_j(v_j(t)) + \sum_{j=1}^{n} c_{ij}(r(t))G_j(v_j(t-\tau_j(t))) + I_i, \quad (1)$$

in which $v_i(t)$ is the activation of the i^{th} neuron. Positive constant a_i denotes the rates with which the cell i resets their potential to the resting state when isolated from the other cells and inputs. b_{ij} and c_{ij} are the connection weights at the time t, I_i denotes the external input and $G_j(\cdot)$ is the neuron activation function of j^{th} neuron. $\tau_j(t)$ is the bounded time varying delay in the state and satisfies

$0 \leq \tau_j(t) \leq \bar{\tau},\ 0 \leq \dot{\tau}_j(t) \leq d < 1,\ i,j = 1,2,..,n.$

The following assumption is made on the activation function.

(A) The neuron activation function $G_j(\cdot)$ in (1) is bounded and satisfies the following Lipschitz condition:

$$|G_j(x) - G_j(y)| \leq |L_j(x-y)|$$

for all $x, y \in R, i, j = 1, 2, ..., n$, where $L_j \in R^{n\times n}$ are known constant matrices.

Assume that $v^* = (v_1^*, v_2^*, ..., v_n^*)^T$ is the equilibrium point of the system, then we shift the equilibrium points to the origin by the transformation $x_i(t) = v_i(t) - v_i^*,\ f_j(x_j(t)) = G_j(u_j(t)) - G_j(u_j^*)$. Then the transformed system is given by

$$\dot{x}_i(t) = -a_i(r(t))x_i(t) + \sum_{j=1}^{n} w_{ij}(r(t))f_j(x_j(t)) + \sum_{j=1}^{n} h_{ij}(r(t))f_j(x_j(t-\tau_j(t))). \tag{2}$$

Conveniently, we can write (2) in the form

$$\dot{x}(t) = -A(r(t))x(t) + B(r(t))f(x(t)) + C(r(t))f(x(t-\tau(t))),$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $A = diag\{a_1, a_2, \ldots, a_n\}$, $B = [(b_{ij})_{n\times n}]^T$, $C = [(c_{ij})_{n\times n}]^T$, $f(x(t)) = [f_1(x_1(t)),\ f_2(x_2(t)), \ldots, f_n(x_n(t))]^T$ and $\tau(t) = (\tau_1(t), \tau_2(t), \ldots, \tau_n(t))^T$.

Then we have

$$f^T(x(t))f(x(t)) \leq x^T(t)L^T Lx(t),$$

where $L = diag\{L_1, L_2, \ldots, L_n\}$. For convinience we denote, $r(t) = i$.

Lemma 2.1 *(Schur complement [1]). Let M, P, Q be given matrices such that $Q > 0$, then*

$$\begin{bmatrix} P & M^T \\ M & -Q \end{bmatrix} < 0 \iff P + M^T Q^{-1} M < 0.$$

The following Lemmas will be essential for the proofs in the next section.

Lemma 2.2 *Let $x \in R^n,\ y \in R^n$ and $\epsilon > 0$. Then we have $x^T y + y^T x \leq \epsilon x^T x + \epsilon^{-1} y^T y$.*

Proof. The proof follows immediately from the inequality $(\epsilon^{1/2}x - \epsilon^{-1/2}y)^T(\epsilon^{1/2}x - \epsilon^{-1/2}y) \geq 0$.

Lemma 2.3 *[6] For any constant matrix $M \in R^{n\times n}, M = M^T > 0$, scalar $\eta > 0$, vector function $\Gamma : [0,\eta] \to R^n$ such that the integrations are well defined, the following inequality holds:*

$$\left[\int_0^\eta \Gamma(s)ds\right]^T M \left[\int_0^\eta \Gamma(s)ds\right] \leq \eta \int_0^\eta \Gamma^T(s)M\Gamma(s)ds.$$

3 Stability Results

In this section, some sufficient conditions of stability for system (2) are obtained.

Theorem 3.1 *Under the assumption* (A) *the system* (2) *is robustly asymptotically stable in the mean square if there exist symmetric positive definite matrices* $P_i > 0, Q > 0,\ R > 0,\ S > 0,$ *positive scalars* $\gamma_j, (j = 0, 1, 2, 3, 4)$ *and positive diagonal matrix* $M = diag\{m_1, m_2, \ldots, m_n\} > 0$ *such that feasible solutions exist for*

$$\Psi = \begin{bmatrix} \Sigma & 0 & \gamma_2 P_i B_i & \gamma_4 P_i C_i & L^T C_i & \gamma_3 L^T M & \gamma_4 L^T M & \gamma_5 L^T M \\ * & L^T C_i & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\gamma_5 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_1 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma_4 I & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma_3 I & 0 & 0 \\ * & * & * & * & * & * & -\gamma_4 I & 0 \\ * & * & * & * & * & * & * & -\gamma_5 I \end{bmatrix} < 0, \quad (3)$$

where

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ * & \Sigma_2 & 0 \\ * & * & -\bar{\tau}^{-1} S \end{bmatrix},$$

$\Sigma_1 = -A_i^T P_i - P_i^T A_i + L^T R L + \bar{\tau} L^T S L + \gamma_1^{-1} L^T L + Q + \gamma_3^{-1} A_i^T A_i + \sum_{j=1}^N \Pi_{ij} P_j,$
$\Sigma_2 = -(1-d)Q - (1-d)L^T R L + \gamma_2^{-1} L^T L.$

Proof: We consider the following Lyapunov functional to derive the stability result:

$$V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t)) + V_4(t, x(t)),$$

where

$$V_1(t, x(t)) = x^T(t)\ P_i\ x(t),$$
$$V_2(t, x(t)) = 2\sum_{i=1}^n m_i \int_0^{x_i} f_i(s) ds,$$
$$V_3(t, x(t)) = \int_{t-\tau(t)}^t [x^T(s) Q x(s) ds + f^T(x(s)) R f(x(s))] ds,$$
$$V_4(t, x(t)) = \int_{t-\bar{\tau}}^t (s - t + \bar{\tau}) f^T(x(\theta)) S f(x(\theta)) d\theta ds.$$

We can calculate the derivative of V along the trajectories of the system (2), then we have

$$\dot{V}(t, x(t)) = \dot{V}_1(t, x(t)) + \dot{V}_2(t, x(t)) + \dot{V}_3(t, x(t)) + \dot{V}_4(t, x(t)),$$

where

$\dot{V}_1(t, x(t)) = 2x^T(t) P \dot{x}(t) = 2x^T(t) P_i[-A_i x(t) + B_i f(x(t)) + C_i f(x(t - \tau(t)))]$

$+ \sum_{j=1}^N \Pi_{ij} x^T(t) P_j x(t),$

$\dot{V}_2(t, x(t)) = 2\sum_{i=1}^n m_i f_i(x_i(t)) \dot{x}_i(t)$

$= f^T(x(t))[-2MA_i x(t) + 2f^T(x(t))MB_i f(x(t)) + 2f^T(x(t))MC_i f(x(t-\tau(t))),$

$\dot{V}_3(t,x(t)) = x^T(t)Qx(t) - (1-d)x^T(t-\tau(t))Qx(t-\tau(t)) + f^T(x(t))Rf(x(t))$

$-(1-d)f^T(x(t-\tau(t)))Rf(x(t-\tau(t))),$

and using Lemma 2.3, we have

$\dot{V}_4(t,x(t)) = \bar{\tau} f^T(x(t))Sf(x(t)) - \int_{t-\bar{\tau}}^{t} f(x(s))Sf(x(s))ds$

$\leq \bar{\tau} f^T(x(t))Sf(x(t)) - \big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\big)^T \ \bar{\tau}^{-1}S \ \big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\big).$

It follows from Lemma 2.2 that

$$2x^T(t)P_iBf(x(t)) \leq \gamma_1 x^T(t)P_iB_iB_i^TP_ix(t) + \gamma_1^{-1}x^T(t)L^TLx(t),$$
$$2x^T(t)P_iC_if(x(t-\tau(t))) \leq \gamma_2 x^T(t)P_iC_iC_i^TP_ix(t) + \gamma_2^{-1}x^T(t-\tau(t))L^TLx(t-\tau(t)),$$
$$-2f^T(x(t))MA_ix(t) \leq \gamma_3 x^T(t)L^TMM^TLx(t) + \gamma_3^{-1}x^T(t)A_i^TA_ix(t),$$
$$2f^T(x(t))MW_if(x(t)) \leq \gamma_4 x^T(t)L^TMM^TLx(t) + \gamma_4^{-1}x^T(t)L^TB_iB_i^TLx(t),$$
$$2f^T(x(t))MC_if(x(t-\tau(t))) \leq \gamma_5 x^T(t)L^TMM^TLx(t) + \gamma_5^{-1}x^T(t-\tau(t))L^TC_iC_i^TLx(t-\tau(t)).$$

We obtain

$$\dot{V} \leq x^T(t)\Big[-A_i^TP_i - P_iA_i + \gamma_1P_iB_iB_i^TP_i + \gamma_1^{-1}L^TL + \gamma_3^{-1}A_i^TA_i + \gamma_4^{-1}L^TB_iB_i^TL \quad (4)$$
$$+\gamma_4P_iC_iC_i^TP_i + (\gamma_3+\gamma_4+\gamma_5)L^TMM^TL + Q + L^TRL + \bar{\tau}L^TSL\Big]x(t)$$
$$-\Big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\Big)^T \ \bar{\tau}^{-1}S \ \Big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\Big) + x^T(t-\tau(t))[\gamma_2^{-1}L^TL$$
$$+\gamma_5^{-1}L^TC_iC_i^TL - (1-d)Q - (1-d)L^TRL]x(t-\tau(t))\Big\}$$
$$\leq \xi^T(t)\Gamma\xi(t), \quad (5)$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & 0 & 0 \\ * & \Gamma_{22} & 0 \\ * & * & -\bar{\tau}^{-1}S \end{bmatrix}, \qquad \Gamma_{11} = -A_i^TP_i - P_iA_i + \gamma_1P_iB_iB_i^TP_i$$

$+\gamma_1^{-1}L^TL+\gamma_3^{-1}A_i^TA_i+\gamma_4^{-1}L^TB_iB_i^TL+\gamma_2P_iC_iC_i^TP_i+(\gamma_3+\gamma_4+\gamma_5)L^TMM^TL+Q+L^TRL$
$+\bar{\tau}L^TSL + \sum_{j=1}^{N}\Pi_{ij}P_j, \Gamma_{22} = \gamma_2^{-1}L^TLL - (1-d)Q - (1-d)L^TRL + \gamma_5^{-1}L^TC_iC_i^TL,$

$$\xi^T(t) = [x^T(t) \ x^T(t-\tau(t)) \ \Big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\Big)^T]^T.$$

By using the Schur complement (Lemma 2.1), Σ can be written as $\Omega < 0$. We get

$$\dot{V}(t,x(t)) \leq [\xi^T(t)\Psi\xi(t)],$$

which indicates, from the Lyapunov stability theory [7], that the dynamics of the neural network (2) is asymptotically stable, which completes the proof.

4 Numerical Example

Consider the recurrent neural network (2) with (s=2) of the following form:

$$\dot{x}(t) = -A_i \ x(t) + B_if(x(t)) + C_i \ f(x(t-\tau(t))),$$

where $f(x) = tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Obviously, $tanh(\cdot)$ satisfies $tanh(x) < 1$ for every $x \in R$, further

$$|tanh(x) - tanh(y)| = \left|\frac{d(tanh(z))}{dt}\right|_{z=\zeta}|x-y| = \left|\frac{4}{(e^{\zeta}+e^{-\zeta})^2}\right||x-y| \leq |x-y|,$$

for every $x, y \in R$, and $\zeta \in (x, y)$ or $\zeta \in (y, x)$. Thus, $L = diag(1, 1)$. The membership functions for Rule 1 and Rule 2 are $\eta^1 = \frac{1}{e^{-2u_1(t)}}$, $\eta^2 = 1 - \eta^1$,

$$A_1 = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.01 & -0.02 \\ -0.10 & 0.01 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.01 & -0.02 \\ -0.10 & 0.01 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.02 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.02 \end{bmatrix}.$$

By using the Matlab LMI toolbox [4], we solve the LMI (3) for $\gamma_i > 0$, $(i = 1, 2, ..., 11)$, $\bar{\tau} = 0.5$ and $d = 0.5$, the feasible solutions are

$$P_1 = \begin{bmatrix} 3.8210 & -0.3112 \\ -0.1312 & 0.6531 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.8210 & -0.3112 \\ -0.1312 & 0.6531 \end{bmatrix},$$

$$Q = R = 10^3 \begin{bmatrix} 1.7364 & 0.0024 \\ 0.0024 & 1.7364 \end{bmatrix},$$

$$M = 10^3 \begin{bmatrix} 1.3264 & 0 \\ 0 & 3.3264 \end{bmatrix}, \qquad S = 10^3 \begin{bmatrix} 2.0110 & -0.0002 \\ -0.0002 & 2.0110 \end{bmatrix}.$$

Therefore, the concerned neural network with time-varying delays is asymptotically stable.

5 Conclusion

In this paper, we have performed the robust stability analysis for a class of uncertain recurrent neural networks with time varying delays and uncertainties. Some new stability criteria have been presented to guarantee the recurrent neural network to be robustly asymptotically stable. The linear matrix inequality (LMI) approach has been used to solve the underlying problem. The applicability of the derived results has been demonstrated through the numerical examples for the effectiveness of less conservative numerical solutions.

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Reduced Order Multiswitching Synchronization between Two Hyperchaotic Systems of Different Order

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Abstract: In this paper, we have investigated the problem of reduced order multiswitching synchronization using the active control method. Reduced order multiswitching synchronization can be considered as a combination of multi-switching with reduced order synchronization. Apt controllers have been constructed to establish the asymptotically stable synchronized state by using different laws of switching and Lyapunov stability theory. To analyze the proposed methodology, a six-dimensional Lorenz model and four-dimensional hyper-chaotic coupled dynamos system have been considered as a drive and response system, respectively. Theoretical results are validated by numerical simulations performed in MATLAB.

Keywords: *multiswitching synchronization; reduced order synchronization; Lyapunov stability theory; Lorenz model; dynamos system.*

Mathematics Subject Classification (2010): 34D06.

1 Introduction

Nonlinear dynamical systems manifest extreme sensitive dependence on initial conditions [1]. Different aspects of nonlinear dynamical systems such as chaos, stability, bifurcation, Poincare surface and synchronization have many useful applications in the modelling of brain activity [8], secure communication [11], information processing [10], medicine[8, 9], signal processing [10] and chemical networks. This has led to the discovery of various kinds of synchronization such as projective synchronization [6], reduced order synchronization [3], generalized synchronization, lag synchronization [5], phase synchronization [4], complete synchronization [7], anticipated synchronization and increased order synchronization [2].

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Ucar et al. [12] first introduced the concept of multiswitching synchronization. It is an important and interesting extension of the existing synchronization schemes because in this scheme, a greater number of synchronization directions exist as the different states of the drive system are synchronized with the desired states of the response system. Various synchronization schemes have been investigated to achieve multiswitching synchronization such as multiswitching combination synchronization (MSCS), multiswitching combination combination synchronization (MSCCS) and a dual combination multiswitching scheme. Most of the work in multiswitching synchronization, till now, has been restricted to the multiswitching synchronization between the drive system and the response system of same order, here the order means the number of state variables. The multiswitching synchronization problem between chaotic systems of different order is still a relatively unexplored area of research. In recent years, problems related to the reduced-order synchronization of chaotic systems have fascinated researchers because of its occcurence in biological and social sciences. The main feature of the reduced-order synchronization is the synchronization of state variables of the response system with the projections of state variables of the drive system where the order of the response system is less than the order of the drive system. Here all states of the response system will be synchronized during synchronization. This kind of synchronization is required between heart and lungs or between neurons and in ecological systems as these dynamical systems are of different orders thus it makes them a very relevant topic to be investigated. Motivated by the above discussion, in this paper, we have made an effort to study the multiswitching reduced order synchronization between chaotic systems. We have designed appropriate controllers to achieve the reduced-order multi-switching synchronization between a six-dimensional Lorenz model and a four-dimensional hyper-chaotic coupled dynamos system.

2 Problem Formulation

In this section, we explain the reduced order multiswitching synchronization between chaotic systems via the active control method. Consider the hyperchaotic system (the drive system) described as

$$\dot{\xi}(t) = f(\xi(t)), \tag{1}$$

where $\xi = (\xi_1, \xi_2, ...\xi_m) \in \Re^m$ denotes the state variable of the master system and $f(\xi(t)) \in \Re^m$ represents the nonlinear functional vector. Now we consider the following chaotic system as our response system:

$$\dot{\zeta}(t) = g(\zeta(t)) + U(t), \tag{2}$$

where $\zeta = (\zeta_1, \zeta_2, ...\zeta_n) \in \Re^n$ denotes the state variable of the slave system and $g(\zeta(t)) \in \Re^n$ represents the nonlinear functional vector and $U(t) = (u_1, u_2, ...u_n) \in \Re^n$ is the control input to be evaluated which will synchronize the state of the drive and the response system.

Since the order of response system is less than the order of drive system, we have $n < m$, and thus we may select any n variables out of m variables of drive system for the projection because the reduced order synchronization is the problem of synchronizing a response system with the projection of the drive system. Thus, we can divide the drive system into two parts, the projection part and the remaining part, given by

$$\dot{\xi}_p(t) = f_p(\xi(t)), \tag{3}$$

where $\xi_p = (\xi_{p1}, \xi_{p2}, ..., \xi_{pn}) \in \Re^n$ and $f_p : \Re^m \to \Re^n$, and

$$\dot{\xi}_r(t) = f_r(\xi(t)), \tag{4}$$

where $\xi_r = (\xi_{p(n+1)}, \xi_{p(n+2)}, ..., \xi_{pm}) \in \Re^{m-n}$ and $f_r : \Re^m \to \Re^{m-n}$. Clearly, ξ_{pi}'s are the rearrangement of ξ_i's.

To obtain the multiswitching reduced order synchronization between the projection of master system (2) and slave system (3) we need to calculate the controller $U = (u_1, u_2, ..., u_n)$ such that

$$\lim_{t\to\infty} e_{ij} = \lim_{t\to\infty} ||\zeta_j - \xi_{pi}|| = 0. \tag{5}$$

3 System Description

Recently, Carolini C. Felicio and Paulo C. Rech [14] generalized the Lorentz model of $3rd$ order to $6th$ order by adding three more state variables to it. The $6th$ order hyperchaotic Lorentz model (the drive system) is given by

$$\left.\begin{aligned}
\dot{\xi}_1 &= a_1(\xi_2 - \xi_1), \\
\dot{\xi}_2 &= a_2\xi_1 - \xi_2 - \xi_1\xi_3 + \xi_3\xi_4 - 2\xi_4\xi_6, \\
\dot{\xi}_3 &= \xi_1\xi_2 - a_3\xi_3 - \xi_1\xi_5 - \xi_2\xi_6, \\
\dot{\xi}_4 &= -(1 + 2a_3)a_1\xi_4 + \frac{a_1}{(1 + 2a_3)}\xi_5, \\
\dot{\xi}_5 &= \xi_1\xi_3 - 2\xi_1\xi_6 + a_2\xi_4 - (1 + 2a_3)\xi_5, \\
\dot{\xi}_6 &= 2\xi_1\xi_5 + 2\xi_2\xi_4 - 4a_3\xi_6,
\end{aligned}\right\}, \tag{6}$$

where a_1, a_2 and a_3 are constant parameters.

Figures below show the chaotic attractor of drive system for particular values of parameters given by $a_1 = 10, a_2 = 100$ and $a_3 = \frac{8}{3}$.

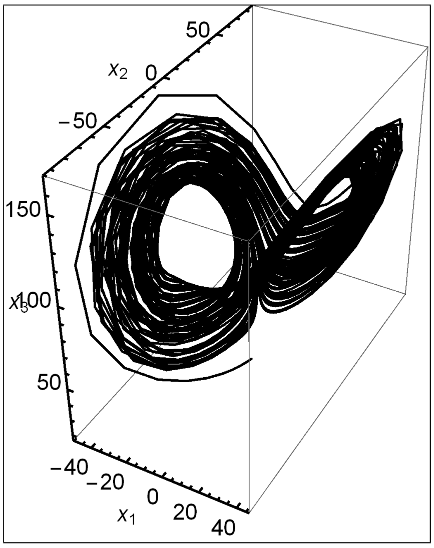

Fig.1: Chaotic behavior of system (6) in ξ_1, ξ_2, ξ_3 plane.

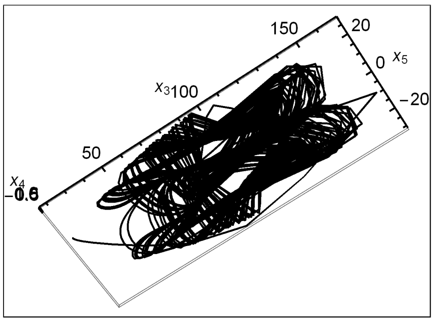

Fig.2: Chaotic behavior of system (6)in ξ_3, ξ_4, ξ_5 plane.

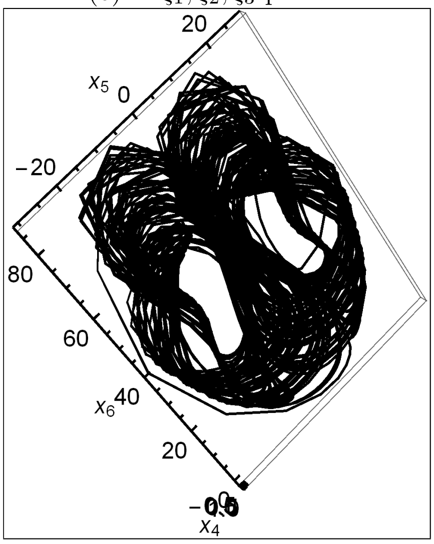

Fig.3: Chaotic behavior of system (6) in ξ_4, ξ_5, ξ_6 plane.

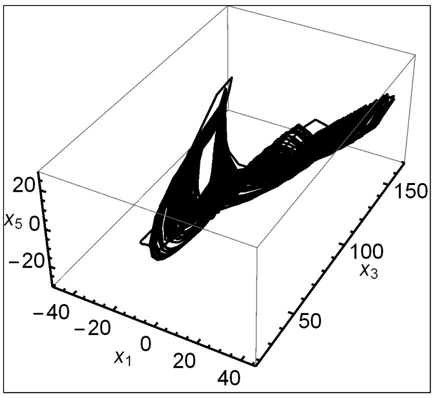

Fig.4: Chaotic behavior of system (6)in ξ_1, ξ_3, ξ_5 plane.

Yanyun Xie and Wenliang Cai[15], in 2017, modified $3rd$ order coupled dynamos system to a new $4th$ order hyperchaotic coupled dynamos (response) system given by

$$\left.\begin{aligned}
\dot{\zeta}_1 &= -2\zeta_1 + \zeta_2(\zeta_3 + 3) + \zeta_4 + u_1, \\
\dot{\zeta}_2 &= -2\zeta_2 + \zeta_1(\zeta_3 - 3) + u_2, \\
\dot{\zeta}_3 &= \zeta_3 - \zeta_1\zeta_2 + u_3, \\
\dot{\zeta}_4 &= -m\zeta_2 + u_4,
\end{aligned}\right\}, \tag{7}$$

where m is a constant parameter. Figures below show the chaotic attractor of response system for $m = 100$

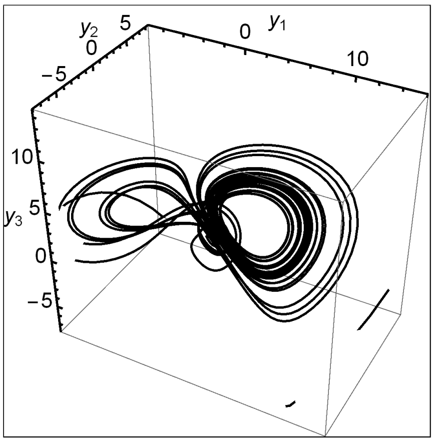

Fig.5: Chaotic behavior of system (7) in $\zeta_1, \zeta_2, \zeta_3$ plane.

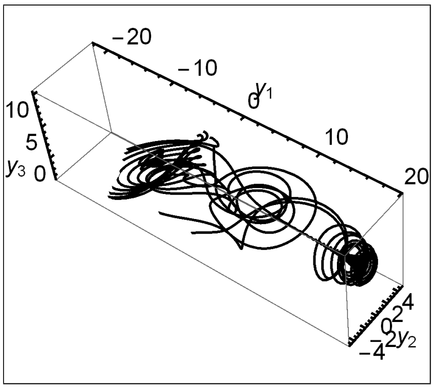

Fig.6: Chaotic behavior of system (7)in $\zeta_2, \zeta_3, \zeta_4$ plane.

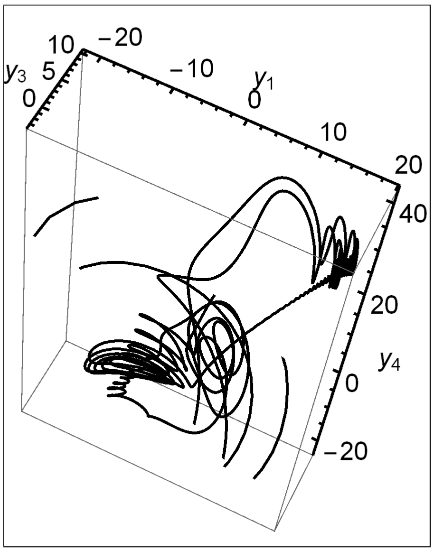

Fig.7: Chaotic behavior of system (8) in $\zeta_1, \zeta_3, \zeta_4$ plane.

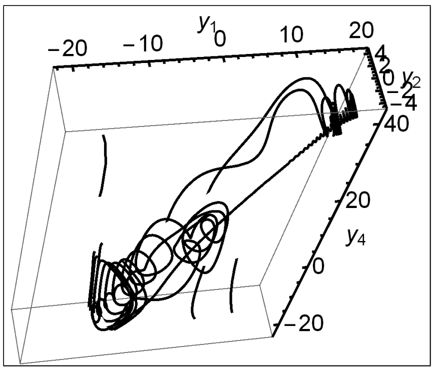

Fig.8: Chaotic behavior of system (9)in $\zeta_1, \zeta_2, \zeta_4$ plane.

4 Illustration

In this section, we present all the possible ways of projection of state variables in the drive system with respect to the order of the response system that is $\binom{6}{4}$ ways. Thus, we have 15 cases for the reduced order multiswitching synchronization and 24 ways of multiswitching for each case. Likewise, we can take a projection to the hyperplane occupied by $\xi_1 - \xi_2 - \xi_3 - \xi_4$, $\xi_2 - \xi_3 - \xi_4 - \xi_5$, $\xi_3 - \xi_4 - \xi_5 - \xi_6$ and so on, which implies we can synchronize these two systems of different order in 360 ways.

First we arbitrarily considered the case of projection of system variables to the hyperplane occupied by $\xi_1 - \xi_3 - \xi_4 - \xi_6$. Again, for this projection we have 24 possible switches given by Switching 1.

The errors are given by

$$\left.\begin{aligned} e_{11} &= \zeta_1(t) - \xi_1(t), \\ e_{12} &= \zeta_2(t) - \xi_3(t), \\ e_{13} &= \zeta_3(t) - \xi_4(t), \\ e_{14} &= \zeta_4(t) - \xi_6(t), \end{aligned}\right\} \tag{8}$$

Switching 2

$$\left.\begin{aligned} e_{21} &= \zeta_1(t) - \xi_1(t), \\ e_{22} &= \zeta_2(t) - \xi_3(t), \\ e_{23} &= \zeta_3(t) - \xi_6(t), \\ e_{24} &= \zeta_4(t) - \xi_4(t), \end{aligned}\right\} \tag{9}$$

Switching 3

$$\left.\begin{aligned} e_{31} &= \zeta_1(t) - \xi_1(t), \\ e_{32} &= \zeta_2(t) - \xi_4(t), \\ e_{33} &= \zeta_3(t) - \xi_3(t), \\ e_{34} &= \zeta_4(t) - \xi_6(t), \end{aligned}\right\} \tag{10}$$

and so on. We now calculate suitable controllers for Switching 1 to achieve the reduced order multiswitching synchronization. Let us define the error dynamics between the drive system (6) and the response system (7) as

$$\left.\begin{aligned} \dot{e}_{11} &= -(1+a_1)e_{11} + a_1\zeta_3 + 2\zeta_3 - \zeta_1\zeta_2 + a_1\xi_2 + u_3, \\ \dot{e}_{12} &= -(2+a_3)e_{12} + \zeta_2\zeta_3 + 3\zeta_2 + \zeta_4 - \xi_1\xi_2 + \xi_1\xi_5 \\ &\quad +\xi_2\xi_4 - 2\xi_3 + a_3\zeta_1 + u_1, \\ \dot{e}_{13} &= -(\frac{a_1}{1+2a_3} + a_2)e_{13} - m\zeta_2 - (\frac{a_1}{1+2a_3})\xi_5 + a_2\zeta_4 \\ &\quad -a_2\xi_4 + (\tfrac{a_1}{1+2a_3})\zeta_4 + u_4, \\ \dot{e}_{14} &= -(2+a_2)e_{14} + \zeta_1\zeta_3 - 3\zeta_1 - 2\xi_1\xi_5 - 2\xi_2\xi_4 + 4a_3\xi_6 \\ &\quad -2\xi_6 + a_2\zeta_2 - a_2\xi_6 + u_2. \end{aligned}\right\} \tag{11}$$

Next, we give the following proposition for the control parameters based on the error dynamic system (11)

Proposition 4.1 *Considering the error dynamics (8) the reduced order multiswitching synchronization between the drive system (6) and the response system (7) will be achieved if the control functions* u_{11}, u_{12}, u_{13} *and* u_{14} *are chosen as*

$$\left.\begin{aligned}
u_{11} &= \xi_1\xi_2 + 2\xi_3 - \xi_1\xi_5 - \xi_2\xi_4 - a_3\zeta_1 - \zeta_2\zeta_3 - 3\zeta_2 - \zeta_4 + A_1,\\
u_{12} &= 2\xi_1\xi_5 + 3\zeta_1 + 2\xi_2\xi_4 + 2\xi_6 + a_2\xi_6 - \zeta_1\zeta_3 - 4a_3\xi_6 - a_2\zeta_2 + A_2,\\
u_{13} &= -a_1\zeta_3 - 2\zeta_3 + \zeta_1\zeta_2 - a_1\xi_2 + A_3,\\
u_{14} &= (\tfrac{a_1}{1+2a_3})\xi_5 + m\zeta_2 + a_2\xi_4 - (\tfrac{a_1}{1+2a_3})\zeta_4 - a_2\zeta_4 + A_4,
\end{aligned}\right\} \tag{12}$$

where A_1, A_2, A_3 *and* A_4 *are the functions of* e_{11}, e_{12}, e_{13} *and* e_{14}.

Proof: The error dynamics (11), after using the controllers given by (12), can be written as

$$\left.\begin{aligned}
\dot{e}_{11} &= -e_{11}(1 + a_1) + A_3,\\
\dot{e}_{12} &= -e_{12}(2 + a_3) + A_1,\\
\dot{e}_{13} &= -e_{13}(\frac{a_1}{1+2a_3} + a_2) + A_4,\\
\dot{e}_{14} &= -e_{14}(2 + a_2) + A_2.
\end{aligned}\right\} \tag{13}$$

We select A_1, A_2, A_3 and A_4 in such a way that the error dynamical system given by (13) gets stabilized, that means the errors will asymptotically tend to zero. Let us consider

$$\begin{bmatrix} A_1\\ A_2\\ A_3\\ A_4 \end{bmatrix} = P \begin{bmatrix} e_{11}\\ e_{12}\\ e_{13}\\ e_{14} \end{bmatrix}, \tag{14}$$

where P is a 4×4 matrix whose enteries are selected such that the values of A_1, A_2, A_3 and A_4 will make (13) stable. Let us consider

$$P = \begin{bmatrix} 0 & a_3 & 0 & 0\\ 0 & 0 & 0 & a_2\\ a_1 & 0 & 0 & 0\\ 0 & 0 & \frac{a_1}{1+2a_3} & 0 \end{bmatrix}. \tag{15}$$

Therefore,

$$\begin{bmatrix} A_1\\ A_2\\ A_3\\ A_4 \end{bmatrix} = \begin{bmatrix} a_3e_{12}\\ a_2e14\\ a_1e_{11}\\ \frac{a_1}{1+2a_3}e_{13} \end{bmatrix}. \tag{16}$$

The error dynamical system (13), after using the values of A_1, A_2, A_3 and A_4, reduced to

$$\left.\begin{aligned} \dot{e}_{11} &= -e_{11}, \\ \dot{e}_{12} &= -2e_{12}, \\ \dot{e}_{13} &= -a_2 e_{13}, \\ \dot{e}_{14} &= -2e_{14}. \end{aligned}\right\} \tag{17}$$

Now, we choose the following Lyapunov function:

$$V_1 = e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2. \tag{18}$$

Clearly, V_1 is positive definite in $\mathbb{R}^4$. After differentiating V_1 with respect to time, we get

$$\frac{dV_1}{dt} = 2e_{11}\dot{e}_{11} + 2e_{12}\dot{e}_{12} + 2e_{13}\dot{e}_{13} + 2e_{14}\dot{e}_{14}. \tag{19}$$

Using (17), we get

$$\frac{dV_1}{dt} = -2e_{11}^2 - 4e_{12}^2 - 2a_2 e_{13}^2 - 4e_{14}^2. \tag{20}$$

Since V_1 is a positive definite function and $\frac{dV_1}{dt}$ is a negative definite function, the Lyapunov stability theory proves that the state of the drive and response systems synchronize asymptotically. Hence the result.

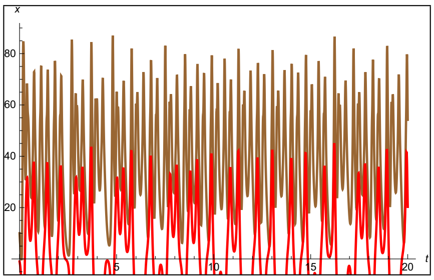

Fig.9: Multiswitching synchronization state between ξ_1, ζ_1.

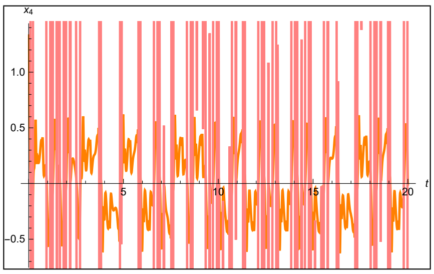

Fig.10: Multiswitching synchronization state between ξ_3, ζ_2.

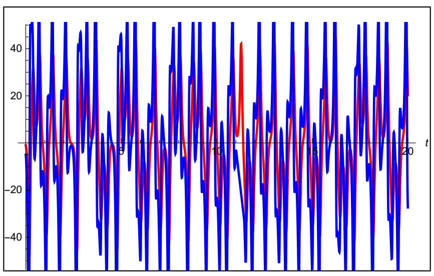

Fig.11: Multiswitching synchronization state between ξ_4, ζ_3.

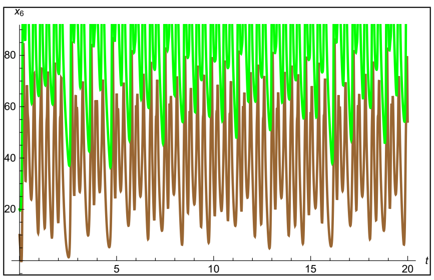

Fig.12: Multiswitching synchronization state between ξ_6, ζ_4.

5 Numerical Simulations

This section presents the synchronization of numerical simulations of the sixth order hyperchaotic Lorentz system and the fourth order hyperchaotic coupled dynamos system. The reduced order multiswitching synchronization is used as an approach to synchronize the systems. The simulations are done in MATLAB. We set the initial values and the parameters as follows: $\xi_1(0) = -1, \xi_2(0) = -5, \xi_3(0) = 20, \xi_4(0) = 5, \xi_5(0) = 3, \xi_6(0) = 10, \zeta_1(0) = -1, \zeta_2(0) = -5, \zeta_3(0) = 20, \zeta_4 = 5, a_1 = 10, a_2 = \frac{8}{3}, a_3 = 100$ and $m = 100$. These figures represent the simulation errors e_1, e_2, e_3, e_4 converging to zero asymptotically, which prove that the hyperchaotic system is synchronized.

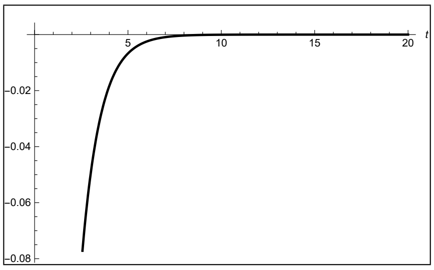

Fig.13: Convergence of error-e_1.

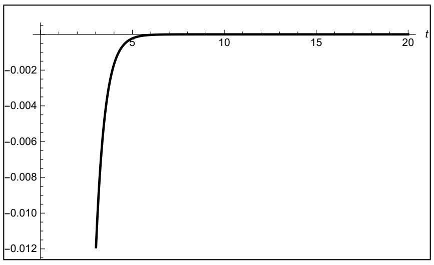

Fig.14: Convergence of error-e_2.

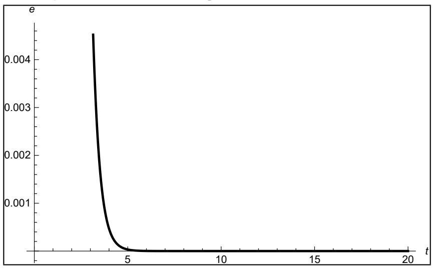

Fig.15: Convergence of error-e_3.

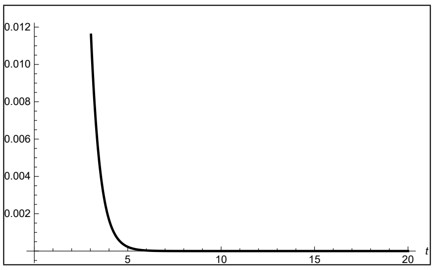

Fig.16: Convergence of error-e_4.

6 Conclusion

In this paper, we have studied the reduced order multiswitching synchronization for two hyperchaotic systems with different order. We have proposed controllers and updating laws based on active control theories. Thus we achieved the error system asymptotically stable. Further, the simulation the results demonstrate the effectiveness and feasibility of results which are performed in MATLAB.

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Bio-Economics of a Renewable Resource in the Presence of Pollution: The Problem of Optimal Effort Allocation

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Abstract: This paper deals with the bio-economics of a renewable resource in a polluted environment. A decline in the revenue due to pollution drives the harvester to allocate a part of the total effort capacity towards pollutant inflow reduction. Hence, the interest is to find an optimal allocation of the available effort capacity between harvesting and pollutant inflow reduction so that the revenue is as large as possible. Therefore, we formulate the optimal harvest problem on an infinite horizon, and it is solved using the standard techniques of optimization. We verify the applicability of the results by considering some practical examples.

Keywords: *bio-economics; renewable resource; pollution; effort allocation; inflow reduction.*

Mathematics Subject Classification (2010): 34H05, 49J15, 92D25.

1 Introduction

The link between renewable resource harvesting and pollution has occupied the attention of researchers and scientists from various disciplines such as economics, biology, engineering, and mathematics. Pollution of water bodies (such as rivers and lakes from the discharge of municipal sewage, septage, industrial chemicals, agricultural run off containing pesticides, etc.) affects the livestock surviving in that environment. Consequently, the economy dependent on the exploitation of such stock suffers as the presence of pollution in the environment affects the health, longevity, and reproductivity of biomass.

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Economic damages to livestock (fisheries), caused by the presence of pollution in the environment have been the main focus in several studies, some of which were undertaken in response to particular pollution incidents [7,8]. The effect of pollutants on the survival of biological species is studied by different authors [10, 11, 16, 20]. Dubey and Hussain [11] studied the survival of species dependent on a resource in a polluted environment. Shukla et al. [16] investigated the effects of pollutants emitted from external sources as well as from their precursors on the survival of a resource-dependent species. Hallam [10] studied the crucial role played by pollutant uptake by the resource. Thomas et al. [20] considered the control problem in a polluted environment to investigate the effect of environmental pollution on a single species population.

Considerable investigations have been made to study the connection between renewable resource harvesting and environmental pollution [12, 17, 19]. Siebert [17] presented the dynamic optimization model that combines renewable resource harvesting and pollution. It underlines the effect of pollution on the regeneration of the resource. Tahvonen [19] studied the dynamics of a harvested renewable resource and pollution with the control strategy, where pollutants are assumed to affect both intrinsic per capita growth rate and saturation level of the resources. A study presented in [12] underlines the effectiveness of pollution reduction expenditures under the assumption that there are constant returns to de-pollution efforts. In particular, the pollutant inflow reduction reduces the pollution of water bodies.

In the literature, we come across studies dealing with optimal exploitation of renewable resource and pollution control wherein the rate of emissions (generated from the production sector) and the harvest rate are considered as controllable variables. These models mostly consider benefits from both the natural resource sector and the pollution generating-production sector. In this study, we focus on the optimal allocation problem in a polluted environment, where there is no benefit from other sectors (such as pollution generating-production sectors).

Pollution affects both the resource growth rate and the quality of the catch, resulting in a decline in the revenue. Since environmental pollution is inevitable, the harvester wishes to allocate a part of the available effort capacity towards pollutant inflow reduction with a hope that it would improve the productivity of the resource resulting in enhanced revenues. To address this problem, it is essential to gain an understanding of the interplay between resource dynamics in the presence of harvesting and the dynamics of pollution under pollutant inflow reduction. Hence, we formulate a dynamic optimization model on an infinite horizon, and it is solved using the standard techniques of dynamic optimization [4, 5, 9, 13, 15].

We organize the paper as follows. In Section 2, we consider the resource-pollution dynamics (with no harvesting and inflow reduction) to investigate the effect of pollutants on the survival of the resource. In Section 3, we investigate the influence of investing in pollutant inflow reduction on the harvest. Here, we evaluate the optimal effort allocation that maximizes the sustainable yield. By including the stock benefit as another source of revenue, we present an optimal harvest problem on an infinite horizon in Section 4. Practical examples that demonstrate the main results and the concluding remarks, respectively, are given in Sections 5 and 6.

2 Resource Dynamics in a Polluted Environment

Consider a renewable resource (fish) that is surviving in a polluted lake environment. Let $x(t)$ and $P(t)$ represent the resource stock and the stock of pollution in the environment at each time t, respectively. Given the inflow rate of pollutants v at each time t, the dynamics of a renewable resource and pollution (in the environment) can be given by (ref. [10, 20])

$$\frac{dx}{dt} = r(P)x\big(1 - \frac{x}{K(P)}\big),\ x(0) = x_0 > 0, \tag{1a}$$

$$\frac{dP}{dt} = v - \gamma Px - \eta P,\ P(0) = P_0 > 0. \tag{1b}$$

The resource x in Eqn.(1a) grows as per Logistic equation where the intrinsic per capita growth rate ($r(P)$) and environmental carrying capacity ($r(P)$) are pollution dependent. These parameters are given by $r(P) = a - \phi P > 0$ and $K(P) = \frac{a-\phi P}{b} > 0$, which are both decreasing functions of P. a and $\frac{a}{b}$ represent the intrinsic per capita growth rate and the environmental carrying capacity, respectively, in the absence of pollution. The expressions γPx and ηP in Eqn.(1b) stand for pollutant uptake by the resource and decay of pollution, respectively. Next, we study the steady-state equilibria and its stability for the system (1).

We know that the resource and pollution stocks are nonnegative, and hence the discussion to come is meaningful only if both the components are nonnegative. Throughout the paper, we use the terms "axial and interior equilibria" to represent the steady-state solutions of (1) that belong to the boundary and interior regions, respectively, of the positive quadrant of xP-space. It can be easily seen that the system (1) admits an axial equilibrium $\big(0, \frac{v}{\eta}\big)$ and the interior equilibrium $(\overline{x}, \overline{P})$ (provided it exists), whose components are given by

$$\overline{x} = \frac{1}{b}(a - \phi\overline{P}), \tag{2a}$$

$$\overline{P} = \frac{v}{\gamma\overline{x} + \eta}. \tag{2b}$$

To express the components ($\overline{x}$ and $\overline{P}$) explicitly, the following equations can be used (which are derived from (2)):

$$\gamma b\overline{x}^2 - (a\gamma - b\eta)\overline{x} + \phi v - a\eta = 0, \tag{3a}$$

$$\gamma\phi\overline{P}^2 - (a\gamma + b\eta)\overline{P} + bv = 0. \tag{3b}$$

Clearly, the existence of interior equilibria for the system (1) depends on the coefficients of a quadratic polynomial in (3a) (or (3b)). It can be easily verified that the given system admits a unique interior equilibrium whenever

$$v < \frac{a\eta}{\phi}, \tag{4}$$

two interior equilibria whenever

$$\frac{a\eta}{\phi} < v < \frac{(a\gamma + b\eta)^2}{4b\gamma\phi}, \tag{5}$$

and the interior equilibrium does not exist whenever

$$v > \max\{\frac{a\eta}{\phi}, \frac{(a\gamma + b\eta)^2}{4b\gamma\phi}\}. \tag{6}$$

The two interior equilibria of the system are given by

$$\left(\frac{a\gamma - b\eta + \sqrt{(a\gamma - b\eta)^2 - 4b\gamma(v\phi - a\eta)}}{2b\gamma}, \frac{a\gamma + b\eta - \sqrt{(a\gamma + b\eta)^2 - 4\phi\gamma(bv)}}{2\phi\gamma}\right) \tag{7a}$$

$$\& \left(\frac{a\gamma - b\eta - \sqrt{(a\gamma - b\eta)^2 - 4b\gamma(v\phi - a\eta)}}{2b\gamma}, \frac{a\gamma + b\eta + \sqrt{(a\gamma + b\eta)^2 - 4\phi\gamma(bv)}}{2\phi\gamma}\right). \tag{7b}$$

The unique interior equilibrium of the system (if it exists) is the one given in (7a).

From Eqn.(2a) and Eqn.(2b) we observe that the resource stock decreases as the stock of pollution increases. Further, the survival of the resource depends on the stock of pollution, and the existence of pollution (in the environment) mainly depends on the inflow rate and the rate of natural degradation. Hence, it is noteworthy to give more attention to a crucial role played by the vital parameters v, η for the existence of an interior equilibrium for the system (1). This is shown in Figure 1, wherein the ηv-parameter space is divided into three regions I, II, and III. The figure provides a base for not only highlighting the dependence of the existence of interior equilibrium on the vital parameters η, v but also to study the qualitative behavior associated with the system under consideration.

To investigate the stability behavior of the equilibria, the Jacobian matrix associated with the system under study is given by

$$J(x, P) = \begin{pmatrix} a - 2bx - \phi P & -\phi x \\ -\gamma P & -\eta - \gamma x \end{pmatrix}. \tag{8}$$

Evaluating the Jacobian matrix at the axial equilibrium $\left(0, \frac{v}{\eta}\right)$ gives

$$J(0, \frac{v}{\eta}) = \begin{pmatrix} \frac{a\eta - \phi v}{\eta} & 0 \\ -\frac{\gamma v}{\eta} & -\eta \end{pmatrix}. \tag{9}$$

From the eigen values of the Jacobian matrix in (9) (which are $\frac{a\eta - \phi v}{\eta}$ and $-\eta$) we observe that the axial equilibrium is locally stable whenever

$$v > \frac{a\eta}{\phi}, \tag{10}$$

and a saddle otherwise. The Jacobian matrix at the interior equilibrium $(\overline{x}, \overline{P})$ gives

$$J(\overline{x}, \overline{P}) = \begin{pmatrix} -b\overline{x} & -\phi\overline{x} \\ -\gamma\overline{P} & -\frac{v}{\overline{P}} \end{pmatrix},$$

and its characteristic equation is

$$r^2 + (b\overline{x} + \frac{v}{\overline{P}})r + b\overline{x}(\frac{v}{\overline{P}}) - \gamma\phi\overline{P}\overline{x} = 0. \tag{11}$$

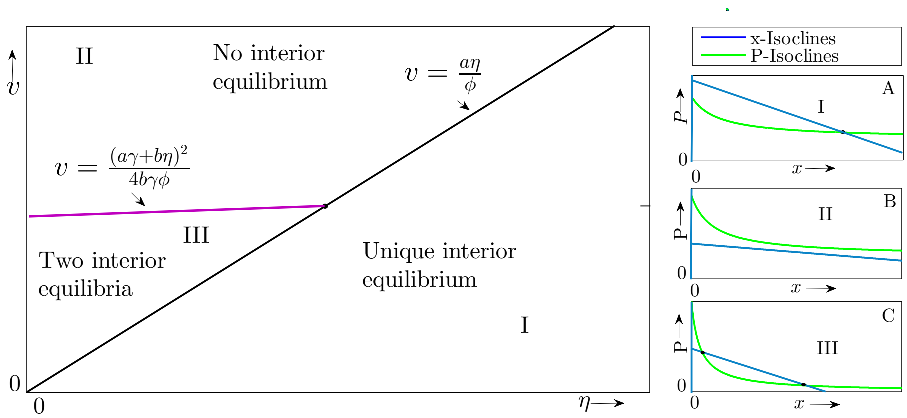

Figure 1: This figure highlights the influence of vital parameters η and v on the existence of interior equilibrium for the system (1). The positive quadrant of the ηv-parameter space is divided into three regions I, II and III (based on the values of (η, v) for which the system (1) admits two interior equilibria, a unique interior equilibrium and no interior equilibrium). For each (η, v) in region I, the relation $v < \frac{a\eta}{\phi}$ holds, and hence the system under study admits a unique interior equilibrium (by (4)). For (η, v) in region III, the relation $\frac{a\eta}{\phi} < v < \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$ holds and hence the system admits two interior equilibria (by (5)). Finally, for each (η, v) in region II, the relation $v > \max\{\frac{a\eta}{\phi}, \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}\}$ is satisfied, and hence the system admits no interior equilibrium (by (6)). The structure of isoclines for the system in question whenever (η, v) belongs to the regions I-III, is presented in the frames A-C, respectively, which are located on the right-hand side of the main figure.

The interior equilibrium $(\overline{x}, \overline{P})$ is locally stable or unstable depending on the roots of a quadratic polynomial in (11). In fact, it is locally stable whenever

$$\overline{P}^2 \leq \frac{bv}{\gamma\phi}, \tag{12}$$

and a saddle otherwise. We have the following theorems pertaining to the system (1) which can be easily established.

Theorem 2.1 *The system* (1) *admits an axial equilibrium and at most two interior equilibria. The axial equilibrium is globally asymptotically stable in the absence of the interior equilibrium, a saddle in the presence of a unique interior equilibrium and locally stable in the presence of two interior equilibria. The unique interior equilibrium is locally stable, and the stability nature in the case of two interior equilibria depends on their proximity to the axial equilibrium interms of their x-component. The closer one is a saddle and the farther one is locally stable.*

Theorem 2.2 *If the associated parameters satisfy the relation $\frac{a}{\phi} > \frac{v}{\eta}$, the system* (1) *admits a unique interior equilibrium which is globally asymptotically stable.*

Now we shall make use of Figure 1 to bring forward various bifurcations that take place in the system under consideration due to variations in the vital parameters v and η. Observe that the two curves $v = \frac{a\eta}{\phi}$ and $v = \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$ given in the figure represent two bifurcation curves; the former one represents a transcritical bifurcation curve between the axial equilibrium and an interior equilibrium, while the latter one represents a saddle-node bifurcation curve of the interior equilibrium. Below we briefly present an overview of the occurrence of various bifurcations in the system (1) as the parameters (η, v) move from one region to another in the figure.

As parameters (η, v) move from region I into II crossing the line $v = \frac{a\eta}{\phi}$, the unique asymptotically stable interior equilibrium and the unstable (saddle) axial equilibrium get closer as the parameters approach the line $v = \frac{a\eta}{\phi}$, collide with each other on the line causing exchange of stability between them due to the occurrence of transcritical bifurcation.

As parameters (η, v) move from region II into III crossing the line $v = \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$, the system experiences a saddle-node bifurcation. Hence there is the emergence of two interior equilibria for the system, where one of them is a saddle, and the other is locally stable. The nature of the axial equilibrium continues to be locally stable. Here, the stable manifold of the saddle interior equilibrium divides the positive quadrant of the xP-space into two invariant regions, each being the region of attraction of the stable equilibrium that it contains.

As parameters (η, v) move from region III into I crossing the line $v = \frac{a\eta}{\phi}$, the saddle interior equilibrium that exists (when the parameters are in region III) moves closer to the locally stable axial equilibrium and collide with each other on the line $v = \frac{a\eta}{\phi}$ causing a transcritical bifurcation between them resulting in the axial equilibrium becoming saddle and retention of only one interior equilibrium when the parameters are in region I.

We have the following observations on the system (1). If the inflow rate of pollutants v is sufficiently low such that $v < \frac{a\eta}{\phi}$ is satisfied, then we are assured of the stable coexistence in the system. If the inflow parameter satisfies $\frac{a\eta}{\phi} < v < \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$, the system admits two interior equilibria (one is locally stable and the other is unstable). In this case, the survival of the resource depends on its initial position. Unless the initial position is in the region of attraction of the interior equilibrium, the resource is likely to go extinct. On the other hand, if the inflow rate is sufficiently large so that $v > \max\{\frac{a\eta}{\phi}, \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}\}$, the resource can not survive in such environment.

3 The Influence of Investing in Pollutant Inflow Reduction on the Harvest

In Section 2, we observed that a reduction in the stock of pollution helps to improve the survival and productivity of the resource, and preventing the inflow of pollutants is a feasible alternative to reduce pollution. Hence, it seems to be reasonable to allocate a part of the total effort capacity towards pollutant inflow reduction to improve the yield. Thus, we consider a revised version of (1) wherein the available effort capacity is divided into two parts: harvesting the resource and pollutant inflow reduction. This will enable us to investigate the influence of investing in inflow reduction on the yield as well as the stock.

Suppose the sole owner has the total effort capacity of M units (measured in terms of money) to invest in harvesting and pollutant inflow reduction. Now, the aim is to find an optimal effort allocation to maximize the sustainable yield. Let E and $M - E$ be the efforts allocated towards harvesting and inflow reduction. Then, the resource and pollution dynamic equations can be given by (ref. [12, 19])

$$\frac{dx}{dt} = r(P)x\big(1 - \frac{x}{K(P)}\big) - q\alpha Ex, \quad x(0) = x_0 > 0, \tag{13a}$$

$$\frac{dP}{dt} = v - \beta(M - E) - \gamma Px - \eta P, \quad P(0) = P_0 > 0, \tag{13b}$$

$$0 \le E \le M. \tag{13c}$$

The parameters $r(P)$ and $K(P)$ (in Eqn.(13a)) are as defined in Section 2 and the term $q\alpha Ex$ stands for the harvest rate, where αE represents the effort in physical terms (measured in vessel units). In Eqn.(13b) the term $\beta(M - E)$ (measured in ton per unit time) stands for the amount of inflow rate reduced by the de-pollution effort $M-E$, where the constants $\alpha > 0, \beta > 0$ are conversion factors, and q is the catchability coefficient.

For each harvest effort $E \in [0, M]$, the system (13a)-(13b) admits an axial equilibrium $(0, \frac{1}{\eta}(v-\beta(M-E))$ (provided $v-\beta(M-E) > 0$), and the interior equilibrium (x_E, P_{M-E}) whose components are given by

$$x_E = \frac{1}{b}\big(a - \phi P_{M-E} - \alpha qE\big), \tag{14a}$$

$$P_{M-E} = \frac{v - \beta(M - E)}{\gamma x_E + \eta}, \tag{14b}$$

provided that $a - \phi P_{M-E} - \alpha qE > 0$, $v - \beta(M - E) > 0$. Note that the subscripts E and $M - E$ for the resource and pollution variables at equilibrium, are meant to indicate that the equilibrium is a result of allocating the efforts E towards harvesting the resource and $M - E$ towards inflow reduction, respectively. The following equations are derived from (14)

$$\gamma bx_E^2 - \big((a - \alpha qE)\gamma - b\eta\big)x_E + \phi(v - \beta(M - E)) - (a - \alpha qE)\eta = 0, \tag{15a}$$

$$\gamma\phi P_{M-E}^2 - \big((a - \alpha qE)\gamma + b\eta\big)P_{M-E} + b(v - \beta(M - E)) = 0. \tag{15b}$$

Clearly, the existence of interior equilibrium for the system under consideration depends on the coefficients of a quadratic polynomial in (15a) (or (15b)). Further, the system admits at most two interior equilibria as in the case of the system (1), but here the equilibria are functions of an additional parameter E. To be more specific, for each E in (13c) the system (13a)-(13b) admits a unique interior equilibrium whenever

$$v - \beta(M - E) < \frac{1}{\phi}(a - \alpha qE)\eta, \tag{16}$$

it admits two interior equilibria whenever

$$\frac{1}{\phi}(a - \alpha qE)\eta < v - \beta(M - E) < \frac{((a - \alpha qE)\gamma + b\eta)^2}{4b\gamma\phi}, \tag{17}$$

and the interior equilibrium does not exist whenever

$$v - \beta(M - E) > \max\Big\{\frac{1}{\phi}(a - \alpha qE)\eta, \frac{((a - \alpha qE)\gamma + b\eta)^2}{4b\gamma\phi}\Big\}. \tag{18}$$

The role of parameter E for the existence of interior equilibrium for the system under consideration can be understood from Figure 2. Hence, we have the following theorems pertaining to the system (13a)-(13b) which can be easily established.

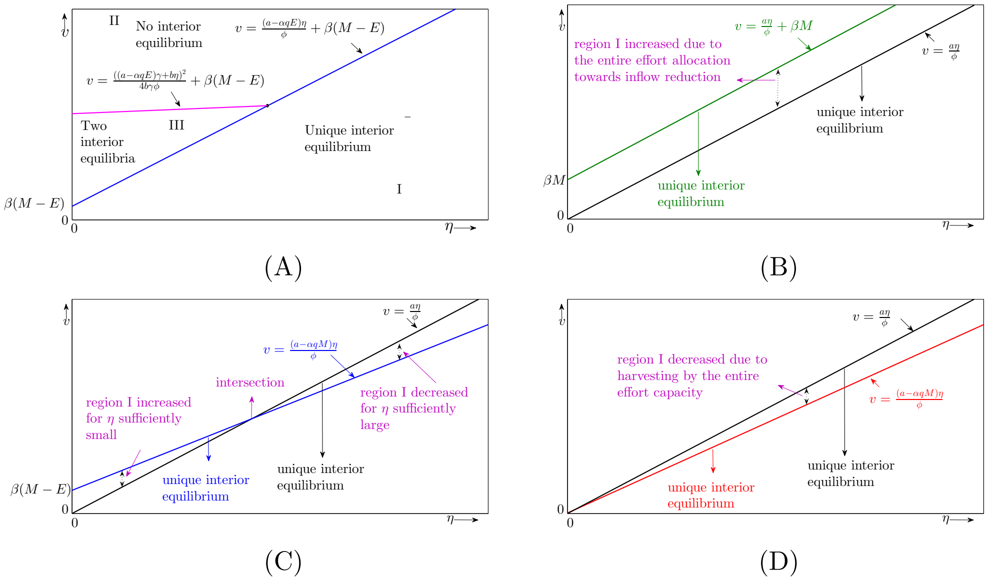

Figure 2: This figure highlights the comparison between systems (1) and (13a)-(13b) based on the values of parameter E. For a fixed $E \in [0, M]$ (in Frame A) the positive quadrant of ηv-space is divided into three regions (I, II and III) based on the values of (η, v) for which the system (13a)-(13b) admits a unique interior equilibrium, no interior equilibrium and two interior equilibria. The region under a line segment $v = \frac{a\eta}{\phi}$ (in frames B-D) represents the set of (η, v) for which the system (1) admits a unique interior equilibrium whereas the regions under the line segments $v = \frac{a\eta}{\phi} + \beta M$(in Frame B where $E = 0$), $v = \frac{1}{\phi}(a - \alpha qE)\eta + \beta(M - E)$ (in Frame C where $0 < E < M$) and $v = \frac{1}{\phi}(a - \alpha qM)\eta$ (in Frame D where $E = M$) represent the set of (η, v) for which the system (13a)-(13b) admits a unique interior equilibrium. In Frame B, the region for unique interior equilibrium is increased (when the entire effort is allocated towards inflow reduction, and no harvesting). In Frame D, the region for unique interior equilibrium is decreased (when the entire effort is allocated towards harvesting with no inflow reduction). In Frame C, the region for unique interior equilibrium is increased for η sufficiently small, and it is reduced for large η.

Theorem 3.1 *For each $E \in [0, M]$, the system* (13a)-(13b) *admits an axial equilibrium (provided that $v - \beta(M - E) > 0$) and at most two interior equilibria. The axial equilibrium is globally asymptotically stable in the absence of the interior equilibrium, it is a saddle in the presence of a unique interior equilibrium, and it is locally stable in the presence of two interior equilibria. The unique interior equilibrium is locally stable, and the stability nature in the case of two interior equilibria depends on their proximity to the axial equilibrium in terms of their x-component. The closer one is a saddle, and the farther one is locally stable.*

Theorem 3.2 *For each $E \in [0, M]$, if the associated parameters satisfy the relation $v - \beta(M - E) < \frac{(a-\alpha qE)\eta}{\phi}$, then system* (13a)-(13b) *admits a unique interior equilibrium which is globally asymptotically stable.*

We observe that it is noteworthy to give more attention to the existence of a unique interior equilibrium in the system as it assures not only the stable coexistence in the system but also the equilibrium with a larger resource level and lower level of pollution. From (14a)-(14b), it follows that the resource x_E decreases and pollution P_{M-E} increases as the effort E increases. Further, the highest resource level and the lowest level of pollution occur at the equilibrium (x_0, P_M) (where $E = 0$). Similarly, the lowest resource level and the highest level of pollution occur at the equilibrium (x_M, P_0) (where $E = M$).

Having seen the highest (lowest) possible levels for the resource stock as well as the stock of pollution, here we are interested in determining the effort allocation that maximizes the sustainable yield. Observe that increasing the harvest effort may not improve the catch in general. The reason being, increasing the harvest effort E causes a decrease in the effort $M - E$ towards the inflow reduction, and this results in a reduction of the stock level x. Therefore, it is useful to find the proper effort allocation to maximize the sustainable yield. For the given harvest effort $E \in [0, M]$ and the stock level x_E, the yield expression is

$$Y(E) = q\alpha E x_E, \tag{19}$$

where the stock level x_E is a function of E (which is the x-component of a unique interior equilibrium (x_E, P_{M-E}) of the system (13a)-(13b)). Clearly, $Y(E)$ is continuous on the interval $[0, M]$, and hence it attains its maximum in $[0, M]$ giving rise to maximum sustainable yield Y_{MSY}. This yield may occur either at the boundary $E = M$ or in the interior of $[0, M]$. If $E_{MSY} \in (0, M)$, then it solves the equation

$$\frac{dY}{dE} = 0, \tag{20}$$

for a concave function $Y(E)$ on $[0, M]$.

4 Optimal Harvest Problem

Following the dynamics of harvested renewable resource surviving in a polluted environment, and having seen the influence of allocating a part of the effort capacity towards inflow reduction on the yield, now we wish to construct an optimal harvesting strategy. Assume that the sole owner has twofold benefit from the resource: benefits from harvesting and the stock. The latter one represents the benefit of the stock in its natural place (such as tourism) [18, 19]. Therefore, it is crucial to ensure a reasonable level of stock in the environment to reap the stock benefit in addition to the revenue from harvesting.

The gross harvesting benefit is assumed to increase with an increase in the harvest, and it decreases with increasing pollution. Further, the marginal (negative) impact of pollution on the gross harvesting benefit is assumed to increase. Hence the gross harvesting benefit (which we denote by $W(h, P)$) may have the following properties (ref. [19])

$$W_h > 0, W_P < 0, W_{PP} < 0. \tag{21}$$

Hence, we consider the following explicit form of the function $W(h, P)$

$$W(h, P) = \tau(1 - \epsilon P^2)h, \tag{22}$$

where $h = q\alpha Ex$ is the harvest rate and $\tau(1 - \epsilon P^2)$ represents the pollution dependent price per unit harvest (where $0 \le 1 - \epsilon P^2 \le 1$). Clearly, τ represents the price per unit catch in the absence of pollution.

The stock benefit is assumed to be affected by the presence of pollution in the environment. Naturally, it decreases with increasing pollution (as it may detour the tourists from visiting). We also assume that the marginal benefit of the stock decreases, and the marginal (negative) impact of pollution increases. Thus, the stock benefit (measured in terms of money) denoted by $S(x,P)$ may have the following properties:

$$S_x > 0, S_{xx} < 0, S_P < 0, S_{PP} < 0, \text{for } 0 < x \le x_s, S_x = 0 \quad \text{for} \quad x > x_s, \tag{23}$$

where x_s denotes the saturation level in the function. Here, we assume the saturation level x_s to be the environmental carrying capacity. The assumption $S_x = 0$ for each $x > x_s$ is due to the observation that the "small" deviation from some "large" stock levels do not necessarily reduce the human in situ benefits [19]. Thus, we consider the following explicit form of the function S:

$$S(x,P) = \begin{cases} \frac{\rho(1-\epsilon P^2)x}{\sigma+x} & \text{for}, \quad 0 < x \le x_s, \\ \frac{\rho(1-\epsilon P^2)x_s}{\sigma+x_s} & \text{for}, \quad x > x_s, \end{cases} \tag{24}$$

where ρ denotes the maximum achievable benefit from the stock in the absence of pollution and σ is a half saturation constant.

The instantaneous net revenue, which is the sum of benefits from harvesting and the stock, is given by

$$R(x,P,E) = (1-\epsilon P^2)\big(\tau q \alpha E x + \frac{\rho x}{\sigma + x}\big) - M. \tag{25}$$

Hence, the present value (denoted by PV) of the total net revenues on the infinite horizon is given by

$$PV = \int_0^\infty e^{-\delta t} R(x,P,h)dt, \tag{26}$$

where δ is the instantaneous discount rate. Now, the objective is to find an optimal effort allocation between harvesting and inflow reduction so that the integral in (26) is as large as possible. Precisely formulated, the problem is as follows:

$$\text{Maximize } PV \tag{27a}$$

$$\text{Subject to: } \frac{dx}{dt} = r(P)x\big(1 - \frac{x}{K(P)}\big) - \alpha q E x, \quad x(0) > 0, \tag{27b}$$

$$\frac{dP}{dt} = v - \beta(M-E) - \gamma P x - \eta P, \quad P(0) > 0, \tag{27c}$$

$$0 \le E \le M. \tag{27d}$$

This is an optimal control problem on an infinite horizon with two state variables (the resource stock x and the stock of pollution P) and one control variable (the effort E towards harvesting).

Solving the problem (27) is amounting to finding out the optimal effort $E_0(t)$ towards harvesting (and hence $M - E_0(t)$ towards inflow reduction) so that the present value in (27a) is maximum. Equivalently, we need to find the path traced out by $(x_0(t), P_0(t))$ with this optimal effort allocation so that if the resource stock and the stock of pollution are kept along this path, then we are assured of achieving the objective of the sole owner.

As per the maximum principle [5, 9, 15], the Hamiltonian $H(x,P,E,\mu_1,\mu_2)$ is given by

$$H(x,P,E,\mu_1,\mu_2) = e^{-\delta t}[(1-\epsilon P^2)(\alpha q\tau Ex + \frac{\rho x}{\sigma + x}) - M] + \\ \mu_1\big[r(P)x(1-\frac{x}{K(P)}) - \alpha qEx\big] + \\ \mu_2[v - \gamma Px - \eta P - \beta(M-E)]. \tag{28}$$

And the associated adjoint variables μ_1, μ_2 satisfy the differential equations

$$\frac{d\mu_1}{dt} = -e^{-\delta t}(1-cP^2)[\tau\alpha qE + \frac{\rho\sigma}{(\sigma+x)^2}] - \mu_1\big(a - 2bx - \phi P - \alpha qE\big) + \gamma\mu_2 P,$$
$$\frac{d\mu_2}{dt} = 2\epsilon e^{-\delta t}P(\tau\alpha qEx + \frac{\rho x}{\sigma+x}) + \mu_2(\eta + \gamma x) + \mu_1\phi x.$$

Because of the presence of the term $e^{-\delta t}$ no steady state is possible for the above system. Hence, we consider the following transformation:

$$\lambda_i(t) = \mu_i(t)e^{\delta t}, \quad i = 1,2 \text{ and } \mathcal{H} = He^{\delta t}, \tag{30}$$

where $\mathcal{H}$ is known as the current value Hamiltonian and λ_1,λ_2 are the current value adjoint variables. From the above transformation we can easily obtain the following adjoint differential equations:

$$\frac{d\lambda_1}{dt} = \lambda_1\delta - \lambda_1\big(a - 2bx - \phi P - \alpha qE\big) - (1-cP^2)[\tau\alpha qE + \frac{\rho\sigma}{(\sigma+x)^2}] + \gamma\lambda_2 P, \tag{31a}$$
$$\frac{d\lambda_2}{dt} = \lambda_2\delta + \lambda_2(\eta+\gamma x) + \lambda_1 xd + 2\epsilon P(\tau\alpha qEx + \frac{\rho x}{\sigma+x}). \tag{31b}$$

Since the problem under consideration is linear in the control variable, the optimal control shall be a combination of bang-bang and singular controls. First, we investigate the singular solution to the problem. Differentiating the current value Hamiltonian with respect to E gives us $\mathcal{H}_E = \tau(1-\epsilon P^2)\alpha qx - \lambda_1\alpha qx + \lambda_2\beta$ with the switching function being

$$s(t) = \tau(1-\epsilon P^2)\alpha qx - \lambda_1\alpha qx + \lambda_2\beta.$$

It is known that along the singular solution $s(t) = 0$, i.e.,

$$\tau(1-\epsilon P^2)\alpha qx - \lambda_1\alpha qx + \lambda_2\beta = 0. \tag{32}$$

Now substituting the interior steady state solution $(\widetilde{x}_E, \widetilde{P}_{M-E}, \widetilde{\lambda}_1, \widetilde{\lambda}_2)$ of the four dimensional dynamical systems (27b), (27c), (31a) and (31b) into (32) we obtain the following equation (which is purely in E):

$$\tau(1-\epsilon\widetilde{P}^2_{M-E})\alpha q\widetilde{x}_E - \alpha q\widetilde{\lambda}_1\widetilde{x}_E + \beta\widetilde{\lambda}_2 = 0. \tag{33}$$

If the solution $\widehat{E}$ of (33) satisfies the condition $0 < \widehat{E} < M$, then $\widehat{E}$ becomes the optimal singular control and $(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}})$ is the associated optimal singular solution [4,6].

After identifying the optimal singular solution $(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}})$, now it remains to reach this solution optimally starting from the given initial state $(x(0), P(0))$. Since the problem under consideration is linear in the control variable, the singular solution $(x_{\widehat{E}}, P_{M-\widehat{E}})$

can be reached by a bang-bang control [15]. If we denote this control by $\overline{E}(t)$, then we have

$$\overline{E}(t) = \begin{cases} 0, & \text{if } \; s(t) < 0, \\ M, & \text{if } \; s(t) > 0, \end{cases} \tag{34}$$

where $s(t) = \tau(1 - \epsilon P^2)\alpha qx - \lambda_1 \alpha qx + \lambda_2 \beta$. Suppose $\mathcal{T}$ represents the time required to reach the singular solution optimally from the given initial state $(x(0), P(0))$ (by a bang-bang control $\overline{E}(t)$). Then the optimal control $E_0(t)$ for the problem under consideration is given by

$$E_0(t) = \begin{cases} \overline{E}(t), & \text{for } \; 0 \le t < \mathcal{T}, \\ \widehat{E}, & \text{for } \; t \ge \mathcal{T}. \end{cases} \tag{35}$$

If $\big(\overline{x}(t), \overline{P}(t)\big)$ represents the trajectory (corresponding to a bang-bang control $\overline{E}(t)$) from the given initial state $(x(0), P(0))$ to the singular solution $\big(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}}\big)$, then the optimal path, traced out by $\big(x_o(t), P_o(t)\big)$, is given by

$$\big(x_0(t), P_0(t)\big) = \begin{cases} \big(\overline{x}(t), \overline{P}(t)\big) & \text{for } \; 0 \le t \le \mathcal{T}, \\ \big(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}}\big) & \text{for } \; t \ge \mathcal{T}. \end{cases} \tag{36}$$

If the singular harvesting effort $\widehat{E}$ is employed right from the given initial state $(x(0), P(0))$, then by the global asymptotic stability of the singular solution $\big(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}}\big)$, the corresponding stock path approaches this solution asymptotically.

5 Applications

This example represents the dynamics of a fish population in a polluted lake environment, where the considered biological and economic parameters are related to actual values one might have in a fishery (ref. [2]). The inflow rate of pollutants (in wastewater effluents) is measured yearly to be 50 tonnes per year. The values assigned to the associated parameters and constants in the problem are given in Table 1. With the given values in the table, it can be easily seen that the condition $\phi(v - \beta(M - E)) - (a - \alpha qE)\eta < 0$ holds for each $E \in [0, M]$, and hence the system (13a)-(13b) admits a unique interior equilibrium $(\widetilde{x}_E, \widehat{P}_{M-E})$ which is globally asymptotically stable.

For each effort $E \in [0, M]$ the unique interior equilibrium $(\widetilde{x}_E, \widehat{P}_{M-E})$ for the system (13a)-(13b) is evaluated and this can be seen in Figure 3. This figure also highlights the relations among the effort E, the resource x, pollution P and yield Y. The highest resource stock and the lowest stock of pollution in the system, respectively, are $x_0 = 3.49 \times 10^4$ and $P_M = 105.9$ (in tons) which are obtained for $E = 0$. Similarly, the lowest resource stock and the highest stock of pollution in the system, respectively, are $x_M = 1.05 \times 16^4$ and $P_0 = 432.2$ (in tons) which are obtained for $E = M$.

The sustainable yield $Y(E)$ corresponding to each $E \in [0, M]$ is evaluated using (19) and this is presented in Figure 3. The maximum sustainable yield is $Y_{MSY} = 3.02 \times 10^3$ (in tons) which is obtained at the critical effort level $E_{MSY} = 8.62 \times 10^5$ (in US\$). The resource stock and stock pollution corresponding to the maximum sustainable yield are given by $(x_{E_{MSY}}, P_{M-E_{MSY}}) = (1.75 \times 10^4, 251.75)$ (in tons).

The interior steady state solution of the four dimensional dynamical systems (27b), (27c), (31a) and (31b) is given by $(1.63 \times 10^4, 272.65, 5.96 \times 10^3, -6.898 \times 10^3)$, where the unique solution $\widehat{E}$ of (33) is $\widehat{E} = 9.21 \times 10^5$ (in US\$). Observe that the solution

Table 1: The values of parameters and their units.

Parameters	Symbol	Values	units
Intrinsic growth rate	a	0.35	1/year
Intraspecific competition	b	1×10^{-5}	1/ton/year
Death of the resource per unit of pollution	ϕ	1×10^{-5}	1/ton/year
Uptake of pollutant per unit of the resource	γ	1×10^{-5}	1/ton/year
Inflow rate of pollutants	v	50	ton/year
Available effort capacity	M	1.2×10^{6}	\$/year
Conversion parameter	α	1×10^{-3}	vessel/\$/year
Conversion parameter	β	1×10^{-5}	ton/\$
Natural degradation rate of pollution	η	1×10^{-2}	1/year
Catchability coefficient	q	0.0002	1/vessel/year
Discount rate	δ	0.025	1/year
Price per unit catch in absence of pollution	τ	6×10^{3}	\$/ton
Maximum stock benefit in absence of pollution	ρ	8×10^{4}	\$/year
Half saturation constant	σ	1×10^{4}	ton

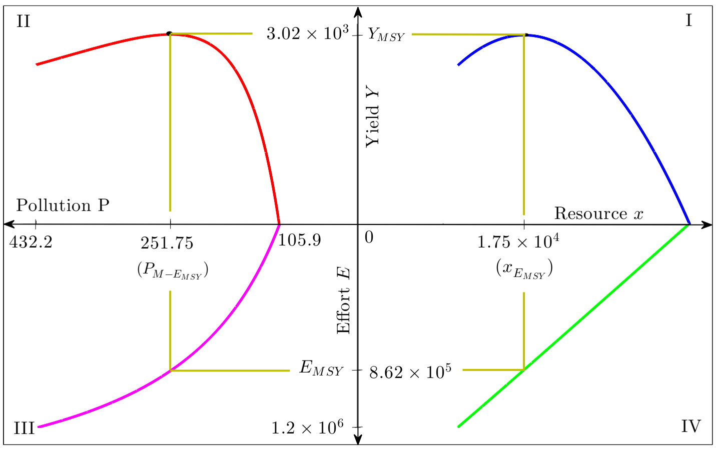

Figure 3: This four-quadrant figure presents the resource and pollution stocks (x_E, P_{M-E}), and the yield $(Y(E))$ associated with each harvest effort $E \in [0, M]$ (and hence the effort $M - E$ for inflow reduction). Quadrant I depicts the relationship between the resource x_E and the yield Y, quadrant II represents the relationship between pollution P_{M-E} and the yield Y, quadrant III represents the relationship between the effort E and pollution P_{M-E} and quadrant IV gives the relationship between the effort E and the resource x_E. For each $E \in [0, M]$, Y can be seen either in quadrant I (through the point (E, x_E)) or in quadrant II (through (E, P_{M-E})). The figure also highlights the critical effort level E_{MSY} that gives the maximum sustainable yield $Y(E_{MSY})$.

$\widehat{E}$ satisfies the relation $0 < \widehat{E} < M$, and hence it becomes an optimal singular effort for harvesting (and hence the effort $M - 9.21 \times 10^5$ goes for inflow reduction). The optimal singular solution $(x_{\widehat{E}}, P_{M-\widehat{E}})$ is given by $(1.63 \times 10^4, 272.65)$ (in tons), and the corresponding yield is $Y(\widehat{E}) = 3.00 \times 10^3$ (in tons). Note that the yield $Y(\widehat{E})$ associated

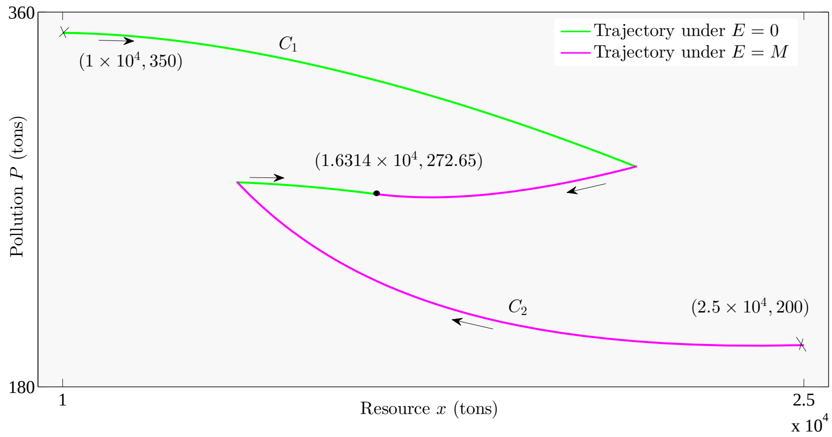

Figure 4: This figure presents the optimal approach path starting from two different initial states $(x(0), P(0)) = (1 \times 10^4, 350)$ (in tons) and $(x(0), P(0)) = (2.5 \times 10^4, 200)$ (in tons) to the singular solution $(1.63 \times 10^4, 272.65)$. The optimal approach path C_1 starts from the initial state $(1 \times 10^4, 350)$ and it takes a time period $\mathcal{T} = 7.65$ (in years) to reach the singular solution by a bang-bang control $\overline{E}(t) = 0$ for $0 \leq t < 4.05$ and $\overline{E}(t) = M$ for $4.05 \leq t < 7.65$. Similarly, the optimal approach C_2 starts from the initial state $(2.5 \times 10^4, 200)$ and it takes a time period $\mathcal{T} = 10.45$ (in years) to reach the singular solution by a bang bang control $\overline{E}(t) = M$ for $0 \leq t < 9.5$ and $\overline{E}(t) = 0$ for $9.5 \leq t < 10.45$.

with the optimal effort $(\widehat{E})$ is less than the maximum sustainable yield (Y_{MSY}). The optimal approach path from the given initial position $(x(0), P(0)) = (1 \times 10^4, 350)$ to the singular solution $(1.63 \times 10^4, 272.65)$ (in tons) takes the time period $t = 7.65$ (in years) under a bang-bang control

$$\overline{E}(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t < 4.05, \\ M & \text{for} \quad 4.05 \leq t < 7.65. \end{cases}$$

Therefore, the optimal control to the given problem is

$$E_o(t) = \begin{cases} \overline{E}(t) & \text{for } 0 \leq t < 7.65, \\ \hat{E} & \text{for } t \geq 7.65. \end{cases}$$

The corresponding optimal approach path is shown in Figure 4. The figure also presents the optimal approach path from another initial position $(2.5 \times 10^4, 200)$ (in tons) to the singular solution $(1.63 \times 10^4, 272.65)$.

6 Concluding Remarks

In this paper, we have presented the bio-economics of a renewable resource in the presence of pollution. A decline in the revenue due to pollutants (in the environment) is a major driving force for investing a part of the effort capacity towards pollution reduction, and pollutant inflow reduction was considered as a feasible alternative to reduce environmental pollution.

The presence of pollution is assumed to affect both the resource growth rate and the quality of the catch. The influence of pollutants on the resource growth and the quality of the catch are captured through its regeneration function and the revenue function, respectively. By investigating the resource-pollution dynamics (in the absence of harvesting and pollutant inflow reduction), we observed that the species goes to extinction whenever the inflow of pollutants is sufficiently large. Furthermore, a criterion is formulated that assures the stable coexistence of the species and pollution.

By incorporating resource harvesting and pollutant inflow reduction into the resource and pollution dynamic equations, respectively, we have investigated the influence of investing in pollutant inflow reduction on the yield. We observed that, by proper allocation of the available effort capacity between harvesting and pollutant inflow reduction, it is possible not only to improve the revenue but also the survival rate of the species. Further, we have observed that there is an optimal effort allocation between harvesting and pollutant inflow reduction that maximizes the yield.

Finally, by considering the pollution dependent revenue obtained from both harvesting and the stock benefit, we have studied an optimal harvest problem. We observed that variation in the stock benefit affects the optimal harvesting strategy. In particular, an increase in the stock benefit results in a reduction in the harvest effort (and hence a rise in pollution reduction effort). Consequently, the resource stock increases, and the stock of pollution decreases.

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The Modified Fractional Power Series Method for Solving Fractional Undamped Duffing Equation with Cubic Nonlinearity

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Abstract: In this paper, the strongly nonlinear fractional undamped Duffing equation for undamped oscillators is studied. The physical and the mathematical model of nonlinear fractional Duffing equation for undamped oscillators is presented. The modified fractional power series (MFPS) method is employed to compute an approximation to the solution of this problem. The validity of the MFPS method is ascertained by comparing our results with numerical results and other methods in the literature. The results reveal that the proposed analytical method can achieve excellent results in predicting the solutions of such problems. The existence of the solution is proved. In addition, the convergence of the proposed method is investigated.

Keywords: *fractional Duffing equation; nonlinear boundary value problem; modified fractional power series method.*

Mathematics Subject Classification (2010): 76A05, 76W05, 76Z99, 65L05.

1 Introduction

In 1918, George Duffing presented the Duffing equation in his publication entitled "Erzwungene Schwingungen bei veranderlicher Eigenfrequenz und ihre technische Bedeuting". Duffing simplified the mathematical model of

$$x''(t) + a^2 x(t) - \beta x^2(t) - \gamma x^3(t) = k \sin \omega t \tag{1}$$

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and calculated the first term $H \ \sin \omega t$ of the periodic solution. Duffing considered the simplified version of equation (1) for describing the motion of the symmetrical pendulum of the form

$$x''(t) + \alpha \ x(t) - \gamma x^3(t) = 0 \tag{2}$$

and the unsymmetrical pendulum of the form

$$x''(t) + \alpha \ x(t) - \gamma x^2(t) = 0. \tag{3}$$

From that time, the differential equation with polynomial type of nonlinearity is called the Duffing equation. The nonlinear differential equation for the cubic free undamped Duffing oscillator of the form

$$x''(t) + \alpha \ x(t) + \beta x^3(t) = 0 \tag{4}$$

is subject to

$$x(0) = A, x'(0) = 0. \tag{5}$$

Many researchers discussed this problem numerically. He [3] used the homotopy perturbation method to solve the Duffing equation, while Belendez et al. [1] used the modified homotopy perturbation method. Ramos, Syam, Chhetri, Wazwaz used the variational iteration method to solve this problem [7, 9–13], while Ghosh et al. [2] used the Adomian decomposition method. In addition, Ramos [8] and Sabeg [14] used the artificial parameter decomposition and He's parameter expanding method, respectively.

Several analytical solutions for the Duffing problem were developed. For the small non-linearity, many analytical approaches were used to solve this problem, namely, the monotone method, the Krylov-Bogolubov method, the straightforward expansion, and the generalized Taylor power series method. For the case of strong cubic non-linearity, see [4, 5].

In this paper, we study the generalization of the problem (4)-(5) of the form

$$D^{2\alpha}x(t) + \beta x(t) + \gamma x^3(t) = 0, \ \frac{1}{2} < \alpha \leq 1, \ 0 < t < T \tag{6}$$

subject to

$$x(0) = A, D^{\alpha}x(0) = 0. \tag{7}$$

The derivative in Eq. (6) is in the Caputo derivative sense. We write the definition and some preliminary results of the Caputo fractional derivatives, as well as the definition of the fractional power series and one of its properties.

Definition 1.1 A real function $f(t), t > 0$,is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, \ m \in \mathbb{N}$.

Definition 1.2 For $\delta > 0, \ m - 1 < \delta < m, \ m \in \mathbb{N}, t > 0$, and $f \in C_{-1}^m$, the left Caputo fractional derivative is defined by

$$D^\delta f(t) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \int_0^t (t-s)^{m-1-\delta} f^{(m)}(s) ds, & \delta > 0, \\ f'(t), & \delta = 0, \end{cases} \tag{8}$$

where Γ is the well-known Gamma function.

The Caputo fractional derivative satisfies the following properties for $\alpha > 0$, see [15].

1. $D^{\alpha}c = 0$, where c is constant,
2. $D^{\alpha}t^{\gamma} = \left\{ \begin{array}{cc} 0, & \gamma < \alpha, \gamma \in \{0,1,2,...\} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}, & \text{otherwise} \end{array} \right\}.$

Next, we write the definition and one of the properties of the fractional power series which are used in this paper. More details can be found in [16].

Definition 1.3 A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + ...$$

is called a fractional power series FPS about $t = t_0$.

Suppose that f has a fractional FPS representation at $t = t_0$ of the form

$$g(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha}, \quad t_0 \le t < t_0 + \beta.$$

If $D^{m\alpha}g(t), \quad m = 0, 1, 2, ..$ are continuous on $\mathbb{R}$, then $c_m = \frac{D^{m\alpha}g(t_0)}{\Gamma(1+m\alpha)}$.

We organize this paper as follows. In Section 2, we present a numerical technique for solving the second order nonlinear fractional boundary value problem using the MFPS method. Convergence of the presented method is given in this section. Some numerical results are presented in Section 3 to illustrate the efficiency of the presented method. Finally, we conclude with some comments and conclusions in Section 4.

2 MFPS Method for Solving Fractional Undamped Duffing Equation with Cubic Nonlinearity

In this section, we discuss how to solve the following class of second-order fractional undamped Duffing equations with cubic nonlinearity using the MFPS method:

$$D^{2\alpha}x(t) + \beta x(t) + \gamma x^3(t) = 0, \ \frac{1}{2} < \alpha \le 1, \ 0 < t < T \tag{9}$$

subject to

$$x(0) = A, D^{\alpha}x(0) = 0. \tag{10}$$

The MFPS method proposes the solution of the problem in the form of fractional power series as

$$x(t) = \sum_{n=0}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{11}$$

To obtain the approximate values of the above series (11), we consider its k-th truncated series $x_k(t)$ which has the form

$$x_k(t) = \sum_{n=0}^{k} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{12}$$

Since $x(0) = f_0 = A$ and $D^{\alpha}x(0) = f_1 = 0$, we rewrite Eq. (12) as

$$x_k(t) = A + \sum_{n=2}^{k} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \qquad k = 2, 3, ..., \tag{13}$$

where $x_1(t) = f_0 + f_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)} = A$ is considered to be the 1^{st} RPS approximate solution of $x(t)$. To find the values of the RPS-coefficients $f_n,\ n = 2, 3, 4, ...$, we solve the fractional differential equation

$$D^{(n-2)\alpha} Res_n(0) = 0,\ n = 2, 3, 4, ...,$$

where $Res_k(t)$ is the k-th residual function and is defined by

$$Res_k(t) = D^{2\alpha}x_k(t) + \beta x_k(t) + \gamma x_k^3(t). \tag{14}$$

To determine the coefficient f_2 in the expansion (12), we substitute the 2^{nd} RPS approximate solution

$$x_2(t) = A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

into Eq.(14) to get

$$\begin{aligned} Res_2(t) &= D^{2\alpha}x_2(t) + \beta x_2(t) + \gamma x_2^3(t) \\ &= f_2 + \beta \left(A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right) + \gamma \left(A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right)^3. \end{aligned} \tag{15}$$

Then, we solve $Res_2(0) = 0$ to get

$$f_2 + \beta A + \gamma A^3 = 0 \tag{16}$$

or

$$f_2 = -\left(\beta A + \gamma A^3\right). \tag{17}$$

To find f_3, we substitute the 3^{rd} RPS approximate solution

$$x_3(t) = A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

into the 3^{rd} residual function $Res_3(t)$ such that

$$\begin{aligned} Res_3(t) &= D^{2\alpha}x_3(t) + \beta x_3(t) + \gamma x_3^3(t) \\ &= f_2 + f_3 \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \beta \left(A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) && (18) \\ &\quad + \gamma \left(A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right)^3. && (19) \end{aligned}$$

Then, we solve $D^{\alpha}Res_3(0) = 0$ to get

$$f_3 = 0. \tag{20}$$

To find f_4, we substitute the 4^{th} RPS approximate solution

$$x_4(t) = A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + f_4 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

into the 4^{th} residual function $Res_4(t)$ such that

$$\begin{aligned} Res_4(t) &= D^{2\alpha}x_4(t)+\beta x_4(t)+\gamma x_4^3(t) \\ &= f_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_4\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad +\beta\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right) \\ &\quad +\gamma\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right)^3. \end{aligned}$$

Then, we solve $D^{2\alpha}Res_4(0)=0$ to get

$$f_4+\beta f_2+3A^2\gamma f_2=0 \tag{21}$$

or

$$\begin{aligned} f_4 &= \left(\beta+3A^2\gamma\right)\left(\beta A+\gamma A^3\right) && (22) \\ &= \beta^2A+\gamma\beta A^3+3A^3\gamma\beta+3A^5\gamma^2. && (23) \end{aligned}$$

To find f_5, we substitute the 5^{th} RPS approximate solution

$$x_5(t)=A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}+f_5\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}$$

into the 5^{th} residual function $Res_5(t)$ such that

$$\begin{aligned} Res_5(t) &= D^{2\alpha}x_4(t)+\beta x_5(t)+\gamma x_5^3(t) \\ &= f_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_4\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_5\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &\quad +\beta\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}+f_5\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right) \\ &\quad +\gamma\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}+f_5\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right)^3. \end{aligned}$$

Then, we solve $D^{3\alpha}Res_5(0)=0$ to get

$$f_5 = 0. \tag{24}$$

To find f_6 , we substitute the 6^{th} RPS approximate solution

$$x_6(t)=A+\sum_{n=2}^{6}f_n\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$$

into the 6^{th} residual function $Res_6(t)$ such that

$$\begin{aligned} Res_6(t) &= D^{2\alpha}x_6(t)+\beta x_6(t)+\gamma x_6^3(t) \\ &= f_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_4\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_5\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_6\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \\ &\quad +\beta\left(A+\sum_{n=2}^{6}f_n\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right)+\gamma\left(A+\sum_{n=2}^{6}f_n\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right)^3. \end{aligned}$$

Then, we solve $D^{4\alpha}Res_6(0) = 0$ to get

$$f_6 + \beta f_4 + \gamma\left(3Af_2^2\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^3} + 3A^2 f_4\right) = 0 \tag{25}$$

or

$$\begin{aligned} f_6 &= -\left(\beta + 3A^2\right) f_4 - 3A\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^3} f_2^2 && (26)\\ &= -\left(\beta + 3A^2\right)\left(\beta^2 A + \gamma\beta A^3 + 3A^3\gamma\beta + 3A^5\gamma^2\right) && (27)\\ &\quad -3A\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^3}\left(\beta A + \gamma A^3\right)^2. && (28)\end{aligned}$$

Using similar argument, we generate $f_7, f_8, f_9, ...$ Thus, the approximate solution is given by

$$x_k(t) = A + \sum_{n=2}^{k} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \qquad k = 2, 3, ... \tag{29}$$

In the next theorem, we study the convergence of the series (2) to the solution of problem (9)-(10).

Theorem 2.1. *Let* $x(t) = \sum_{n=0}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$ *and* $0 < \alpha \leq 1$*. Then, the sequence* $\{x_k(t)\}$ *converges to the solution of problem (9)–(10).*

Proof: First, we want to prove that $\sum_{n=2}^{\infty} f_n \frac{t^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ converges to $D^{2\alpha}x(t)$ when $t > 0$. For any $t > 0$,

$$\begin{aligned} D^{2\alpha}x(t) &= \frac{1}{\Gamma(2-2\alpha)}\int_0^t (t-s)^{1-2\alpha}x''(s)ds\\ &= \frac{1}{\Gamma(2-2\alpha)}\int_0^t (t-s)^{1-2\alpha}\left(\sum_{n=0}^{\infty} f_n \frac{s^{n\alpha}}{\Gamma(1+n\alpha)}\right)'' ds\\ &= \sum_{n=0}^{\infty}\frac{f_n}{\Gamma(1+n\alpha)}\frac{1}{\Gamma(2-2\alpha)}\int_0^t (t-s)^{1-2\alpha}\left(s^{n\alpha}\right)'' ds\\ &= \sum_{n=0}^{\infty}\frac{f_n}{\Gamma(1+n\alpha)} D^{2\alpha}(t^{n\alpha}) = \sum_{n=2}^{\infty}\frac{f_n}{\Gamma(1+(n-2)\alpha)} t^{(n-2)\alpha}.\end{aligned}$$

Thus, $\sum_{n=2}^{\infty} f_n \frac{t^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ converges to $D^{2\alpha}x(t)$ when $t > 0$.

Next, we want to prove the sequence $\{x_k(t)\}$ converges to the solution of problem (9)-(10). Let

$$\begin{aligned} D^{2\alpha}&\left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right) + \beta\left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right)\\ &+\gamma\left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \sum_{n=0}^{\infty}\xi_n t^{n\alpha}\end{aligned}$$

or

$$\sum_{n=2}^{\infty} \frac{f_n}{\Gamma(1+(n-2)\alpha)}\ t^{(n-2)\alpha} + \beta \left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) + \gamma \left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \sum_{n=0}^{\infty} \xi_n t^{n\alpha}.$$

Since $D^{2\alpha}x(t) = \sum_{n=2}^{\infty} f_n \frac{t^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ and $x(t) = \sum_{n=0}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$, we have

$$\sum_{n=2}^{\infty} \xi_n t^{n\alpha} = 0.$$

Let

$$S_k = \sum_{n=k}^{\infty} \xi_n t^{n\alpha}.$$

Then, the sequence $\{S_k\}$ converges to zero. From Eq. (14), we see that

$$Res_k(t) = S_k.$$

Thus,

$$\lim_{k\to\infty} Res_k(t) = \lim_{k\to\infty} S_k = 0.$$

Hence, the sequence $\{x_k(t)\}$ converges to the solution of problem (10)-(11)

3 Results and Discussion

First, we study problem (10)-(11) when $\alpha = 1$. The exact solution of problem (10)-(11) is not known. Therefore, the numerical solutions have been determined by built-in file of MATHEMATICA based on the fully explicit Runge-Kutta method and this solution is used as the standard or reference for comparison. In Tables 1 and 2, we compare our results with the HPM, MHPM, SHPM [6], and the numerical solution for $A = 1, \alpha = 1, \beta = 1$, and $\gamma = 1$ and for $A = 0.75$, $\alpha = 1$, $\beta = 1.5$, and $\gamma = 1.5$, respectively.

t	HPM	MHPM	SHPM	Present results	Numerical results
0.5	0.762476	0.768902	0.768766	0.768802	0.768802
1.0	0.176929	0.233741	0.233680	0.233692	0.233692
2.0	−1.055110	−0.891260	−0.859323	−0.859349	−0.859349
3.5	−0.461650	−0.079433	−0.093034	−0.093013	−0.093013
5.0	2.049041	0.996472	0.947107	0.947130	0.947130

Table 1: The approximate solution for $A = 1$, $\alpha = 1$, $\beta = 1$, and $\gamma = 1$.

t	HPM	MHPM	SHPM	Present results	Numerical results
1	0.056288	0.080176	0.080519	0.0805269	0.080527
2	−0.808192	−0.739174	−0.729000	−0.729018	−0.729018
3	−0.339208	−0.239413	−0.238620	−0.2386259	−0.238626
4	0.891267	0.706827	0.667953	0.6680221	0.668022
5	0.893003	0.395315	0.387550	0.3875509	0.387551

Table 2: The approximate solution for $A = 0.75$, $\alpha = 1$, $\beta = 1.5$, and $\gamma = 1.5$.

Figure 1 shows the comparison between the current method and numerical solutions for $A = 1, \alpha = 1,\ \beta = 0.5$, and $\gamma = 2$ while Figure 2 shows the comparison between the current method and numerical solutions for $A = 1.5,\ \alpha = 1,\ \beta = 1$, and $\gamma = 0.5$.

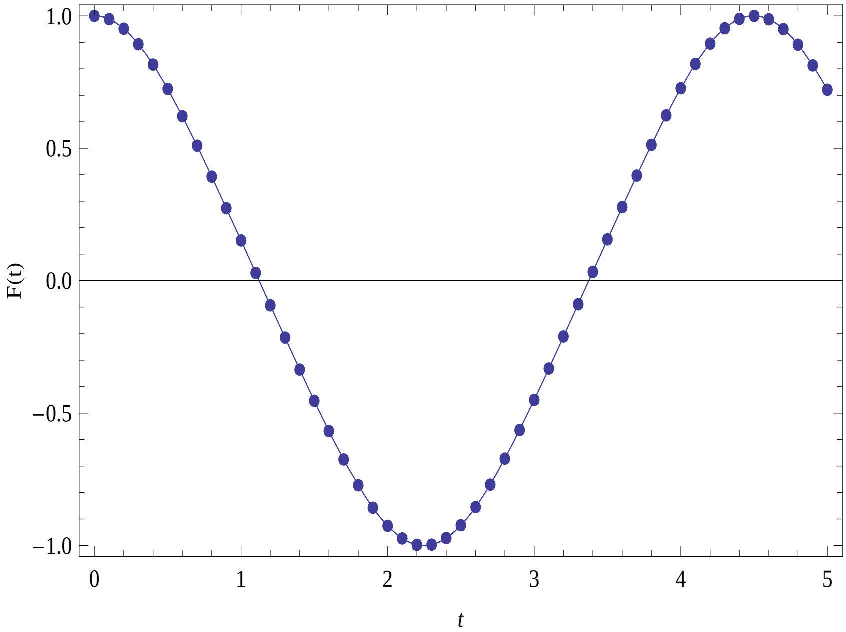

Figure 1. The proposed solution and the numerical solution for $A = 1, \alpha = 1,\ \beta = 0.5$, and $\gamma = 2$.

4 Conclusion

In this paper, the nonlinear differential equation of the cubic free undamped Duffing oscillator of the form

$$D^{2\alpha}x(t) + \beta\ x(t) + \gamma x^3(t) = 0,\ \frac{1}{2} < \alpha \leq 1,\ 0 < t < T, \tag{30}$$

subject to

$$x(0) = A, D^{\alpha}x(0) = 0 \tag{31}$$

is presented. We compare our results with the HPM, MHPM, SHPM [6], and the numerical solution for $A = 1, \alpha = 1$, and $\beta = \gamma = 1$ in Table 1. In Table 2, we compare our results with the HPM, MHPM, SHPM [6], and the numerical solution for $A = 0.75, \alpha = 1$, and $\beta = \gamma = 1.5$. Figure 1 shows the comparison between the current method and numerical solutions for $A = 1, \alpha = 1,\ \beta = 0.5$ and $\gamma = 2$ while Figure 2 shows the comparison between the current method and numerical solutions for $A = 1.5,\ \alpha = 1,\ \beta = 1$, and $\gamma = 0.5$. From the previous section, we can conclude the following:

- From Tables 1 and 2, we see that our results agree exceptionally well with the numerical results and are more accurate than those by the HPM, MHPM, SHPM [6].

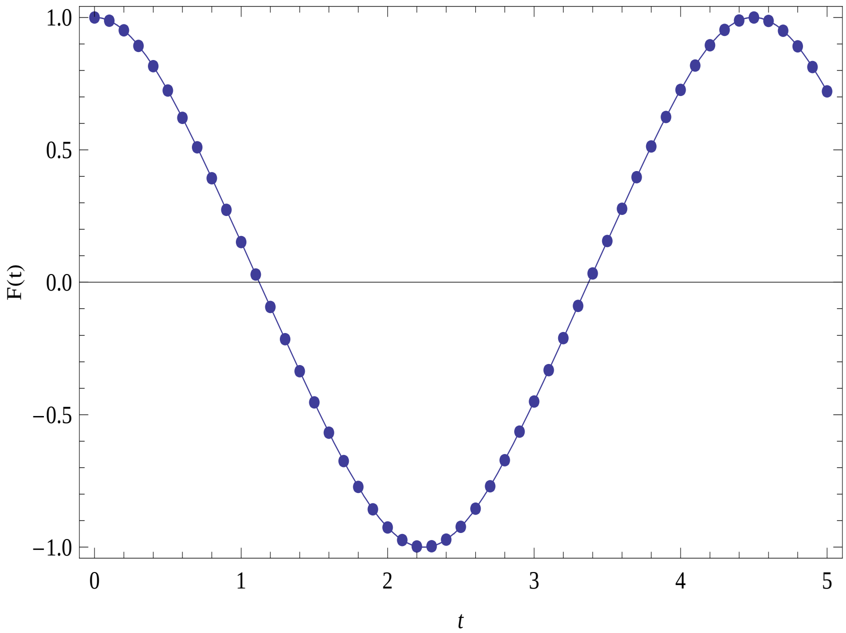

Figure 2. The proposed solution and the numerical solution for $A = 1.5$, $\alpha = 1$, $\beta = 1$, and $\gamma = 0.5$.

- Figure 2 shows the comparison between the current method and numerical solutions for $A = 1, \alpha = 0.5$, and $\beta = 2$. We see that there is agreement between the numerical results and our results.
- The MFPS method is an excellent tool due to the rapid convergent.
- The results in this paper confirm that the MFPS method is a powerful and efficient method for solving nonlinear differential equations in different fields of sciences and engineering.

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