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Bio-Economics of a Renewable Resource in the Presence of Pollution: The Problem of Optimal Effort Allocation

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Abstract: This paper deals with the bio-economics of a renewable resource in a polluted environment. A decline in the revenue due to pollution drives the harvester to allocate a part of the total effort capacity towards pollutant inflow reduction. Hence, the interest is to find an optimal allocation of the available effort capacity between harvesting and pollutant inflow reduction so that the revenue is as large as possible. Therefore, we formulate the optimal harvest problem on an infinite horizon, and it is solved using the standard techniques of optimization. We verify the applicability of the results by considering some practical examples.

Keywords: *bio-economics; renewable resource; pollution; effort allocation; inflow reduction.*

Mathematics Subject Classification (2010): 34H05, 49J15, 92D25.

1 Introduction

The link between renewable resource harvesting and pollution has occupied the attention of researchers and scientists from various disciplines such as economics, biology, engineering, and mathematics. Pollution of water bodies (such as rivers and lakes from the discharge of municipal sewage, septage, industrial chemicals, agricultural run off containing pesticides, etc.) affects the livestock surviving in that environment. Consequently, the economy dependent on the exploitation of such stock suffers as the presence of pollution in the environment affects the health, longevity, and reproductivity of biomass.

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Economic damages to livestock (fisheries), caused by the presence of pollution in the environment have been the main focus in several studies, some of which were undertaken in response to particular pollution incidents [7,8]. The effect of pollutants on the survival of biological species is studied by different authors [10,11,16,20]. Dubey and Hussain [11] studied the survival of species dependent on a resource in a polluted environment. Shukla et al. [16] investigated the effects of pollutants emitted from external sources as well as from their precursors on the survival of a resource-dependent species. Hallam [10] studied the crucial role played by pollutant uptake by the resource. Thomas et al. [20] considered the control problem in a polluted environment to investigate the effect of environmental pollution on a single species population.

Considerable investigations have been made to study the connection between renewable resource harvesting and environmental pollution [12, 17, 19]. Siebert [17] presented the dynamic optimization model that combines renewable resource harvesting and pollution. It underlines the effect of pollution on the regeneration of the resource. Tahvonen [19] studied the dynamics of a harvested renewable resource and pollution with the control strategy, where pollutants are assumed to affect both intrinsic per capita growth rate and saturation level of the resources. A study presented in [12] underlines the effectiveness of pollution reduction expenditures under the assumption that there are constant returns to de-pollution efforts. In particular, the pollutant inflow reduction reduces the pollution of water bodies.

In the literature, we come across studies dealing with optimal exploitation of renewable resource and pollution control wherein the rate of emissions (generated from the production sector) and the harvest rate are considered as controllable variables. These models mostly consider benefits from both the natural resource sector and the pollution generating-production sector. In this study, we focus on the optimal allocation problem in a polluted environment, where there is no benefit from other sectors (such as pollution generating-production sectors).

Pollution affects both the resource growth rate and the quality of the catch, resulting in a decline in the revenue. Since environmental pollution is inevitable, the harvester wishes to allocate a part of the available effort capacity towards pollutant inflow reduction with a hope that it would improve the productivity of the resource resulting in enhanced revenues. To address this problem, it is essential to gain an understanding of the interplay between resource dynamics in the presence of harvesting and the dynamics of pollution under pollutant inflow reduction. Hence, we formulate a dynamic optimization model on an infinite horizon, and it is solved using the standard techniques of dynamic optimization [4, 5, 9, 13, 15].

We organize the paper as follows. In Section 2, we consider the resource-pollution dynamics (with no harvesting and inflow reduction) to investigate the effect of pollutants on the survival of the resource. In Section 3, we investigate the influence of investing in pollutant inflow reduction on the harvest. Here, we evaluate the optimal effort allocation that maximizes the sustainable yield. By including the stock benefit as another source of revenue, we present an optimal harvest problem on an infinite horizon in Section 4. Practical examples that demonstrate the main results and the concluding remarks, respectively, are given in Sections 5 and 6.

2 Resource Dynamics in a Polluted Environment

Consider a renewable resource (fish) that is surviving in a polluted lake environment. Let $x(t)$ and $P(t)$ represent the resource stock and the stock of pollution in the environment at each time t, respectively. Given the inflow rate of pollutants v at each time t, the dynamics of a renewable resource and pollution (in the environment) can be given by (ref. [10, 20])

$$\frac{dx}{dt} = r(P)x\big(1 - \frac{x}{K(P)}\big),\ x(0) = x_0 > 0, \tag{1a}$$

$$\frac{dP}{dt} = v - \gamma Px - \eta P,\ P(0) = P_0 > 0. \tag{1b}$$

The resource x in Eqn.(1a) grows as per Logistic equation where the intrinsic per capita growth rate ($r(P)$) and environmental carrying capacity ($r(P)$) are pollution dependent. These parameters are given by $r(P) = a - \phi P > 0$ and $K(P) = \frac{a-\phi P}{b} > 0$, which are both decreasing functions of P. a and $\frac{a}{b}$ represent the intrinsic per capita growth rate and the environmental carrying capacity, respectively, in the absence of pollution. The expressions γPx and ηP in Eqn.(1b) stand for pollutant uptake by the resource and decay of pollution, respectively. Next, we study the steady-state equilibria and its stability for the system (1).

We know that the resource and pollution stocks are nonnegative, and hence the discussion to come is meaningful only if both the components are nonnegative. Throughout the paper, we use the terms "axial and interior equilibria" to represent the steady-state solutions of (1) that belong to the boundary and interior regions, respectively, of the positive quadrant of xP-space. It can be easily seen that the system (1) admits an axial equilibrium $\big(0, \frac{v}{\eta}\big)$ and the interior equilibrium $(\overline{x}, \overline{P})$ (provided it exists), whose components are given by

$$\overline{x} = \frac{1}{b}(a - \phi\overline{P}), \tag{2a}$$

$$\overline{P} = \frac{v}{\gamma\overline{x} + \eta}. \tag{2b}$$

To express the components ($\overline{x}$ and $\overline{P}$) explicitly, the following equations can be used (which are derived from (2)):

$$\gamma b\overline{x}^2 - (a\gamma - b\eta)\overline{x} + \phi v - a\eta = 0, \tag{3a}$$

$$\gamma\phi\overline{P}^2 - (a\gamma + b\eta)\overline{P} + bv = 0. \tag{3b}$$

Clearly, the existence of interior equilibria for the system (1) depends on the coefficients of a quadratic polynomial in (3a) (or (3b)). It can be easily verified that the given system admits a unique interior equilibrium whenever

$$v < \frac{a\eta}{\phi}, \tag{4}$$

two interior equilibria whenever

$$\frac{a\eta}{\phi} < v < \frac{(a\gamma + b\eta)^2}{4b\gamma\phi}, \tag{5}$$

and the interior equilibrium does not exist whenever

$$v > \max\{\frac{a\eta}{\phi}, \frac{(a\gamma + b\eta)^2}{4b\gamma\phi}\}. \tag{6}$$

The two interior equilibria of the system are given by

$$\left(\frac{a\gamma - b\eta + \sqrt{(a\gamma - b\eta)^2 - 4b\gamma(v\phi - a\eta)}}{2b\gamma}, \frac{a\gamma + b\eta - \sqrt{(a\gamma + b\eta)^2 - 4\phi\gamma(bv)}}{2\phi\gamma}\right) \tag{7a}$$

$$\& \left(\frac{a\gamma - b\eta - \sqrt{(a\gamma - b\eta)^2 - 4b\gamma(v\phi - a\eta)}}{2b\gamma}, \frac{a\gamma + b\eta + \sqrt{(a\gamma + b\eta)^2 - 4\phi\gamma(bv)}}{2\phi\gamma}\right). \tag{7b}$$

The unique interior equilibrium of the system (if it exists) is the one given in (7a).

From Eqn.(2a) and Eqn.(2b) we observe that the resource stock decreases as the stock of pollution increases. Further, the survival of the resource depends on the stock of pollution, and the existence of pollution (in the environment) mainly depends on the inflow rate and the rate of natural degradation. Hence, it is noteworthy to give more attention to a crucial role played by the vital parameters v, η for the existence of an interior equilibrium for the system (1). This is shown in Figure 1, wherein the ηv-parameter space is divided into three regions I, II, and III. The figure provides a base for not only highlighting the dependence of the existence of interior equilibrium on the vital parameters η, v but also to study the qualitative behavior associated with the system under consideration.

To investigate the stability behavior of the equilibria, the Jacobian matrix associated with the system under study is given by

$$J(x, P) = \begin{pmatrix} a - 2bx - \phi P & -\phi x \\ -\gamma P & -\eta - \gamma x \end{pmatrix}. \tag{8}$$

Evaluating the Jacobian matrix at the axial equilibrium $\left(0, \frac{v}{\eta}\right)$ gives

$$J(0, \frac{v}{\eta}) = \begin{pmatrix} \frac{a\eta - \phi v}{\eta} & 0 \\ -\frac{\gamma v}{\eta} & -\eta \end{pmatrix}. \tag{9}$$

From the eigen values of the Jacobian matrix in (9) (which are $\frac{a\eta - \phi v}{\eta}$ and $-\eta$) we observe that the axial equilibrium is locally stable whenever

$$v > \frac{a\eta}{\phi}, \tag{10}$$

and a saddle otherwise. The Jacobian matrix at the interior equilibrium $(\overline{x}, \overline{P})$ gives

$$J(\overline{x}, \overline{P}) = \begin{pmatrix} -b\overline{x} & -\phi\overline{x} \\ -\gamma\overline{P} & -\frac{v}{\overline{P}} \end{pmatrix},$$

and its characteristic equation is

$$r^2 + (b\overline{x} + \frac{v}{\overline{P}})r + b\overline{x}(\frac{v}{\overline{P}}) - \gamma\phi\overline{P}\overline{x} = 0. \tag{11}$$

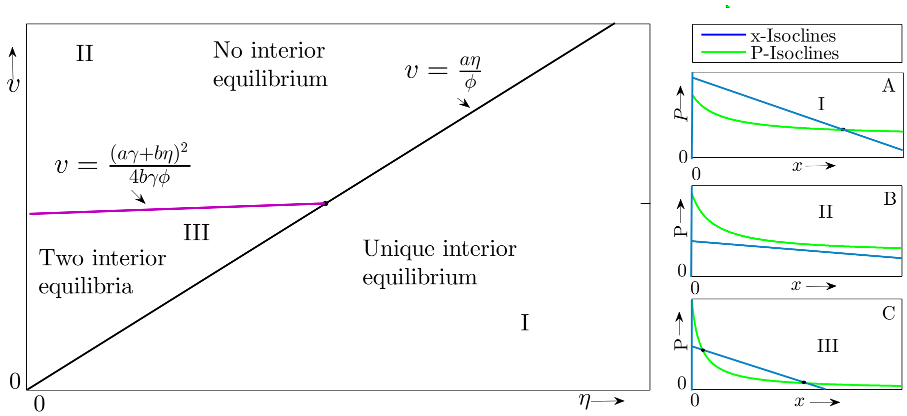

Figure 1: This figure highlights the influence of vital parameters η and v on the existence of interior equilibrium for the system (1). The positive quadrant of the ηv-parameter space is divided into three regions I, II and III (based on the values of (η, v) for which the system (1) admits two interior equilibria, a unique interior equilibrium and no interior equilibrium). For each (η, v) in region I, the relation $v < \frac{a\eta}{\phi}$ holds, and hence the system under study admits a unique interior equilibrium (by (4)). For (η, v) in region III, the relation $\frac{a\eta}{\phi} < v < \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$ holds and hence the system admits two interior equilibria (by (5)). Finally, for each (η, v) in region II, the relation $v > \max\{\frac{a\eta}{\phi}, \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}\}$ is satisfied, and hence the system admits no interior equilibrium (by (6)). The structure of isoclines for the system in question whenever (η, v) belongs to the regions I-III, is presented in the frames A-C, respectively, which are located on the right-hand side of the main figure.

The interior equilibrium $(\overline{x}, \overline{P})$ is locally stable or unstable depending on the roots of a quadratic polynomial in (11). In fact, it is locally stable whenever

$$\overline{P}^2 \leq \frac{bv}{\gamma\phi}, \tag{12}$$

and a saddle otherwise. We have the following theorems pertaining to the system (1) which can be easily established.

Theorem 2.1 *The system* (1) *admits an axial equilibrium and at most two interior equilibria. The axial equilibrium is globally asymptotically stable in the absence of the interior equilibrium, a saddle in the presence of a unique interior equilibrium and locally stable in the presence of two interior equilibria. The unique interior equilibrium is locally stable, and the stability nature in the case of two interior equilibria depends on their proximity to the axial equilibrium interms of their x-component. The closer one is a saddle and the farther one is locally stable.*

Theorem 2.2 *If the associated parameters satisfy the relation $\frac{a}{\phi} > \frac{v}{\eta}$, the system* (1) *admits a unique interior equilibrium which is globally asymptotically stable.*

Now we shall make use of Figure 1 to bring forward various bifurcations that take place in the system under consideration due to variations in the vital parameters v and η. Observe that the two curves $v = \frac{a\eta}{\phi}$ and $v = \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$ given in the figure represent two bifurcation curves; the former one represents a transcritical bifurcation curve between the axial equilibrium and an interior equilibrium, while the latter one represents a saddle-node bifurcation curve of the interior equilibrium. Below we briefly present an overview of the occurrence of various bifurcations in the system (1) as the parameters (η, v) move from one region to another in the figure.

As parameters (η, v) move from region I into II crossing the line $v = \frac{a\eta}{\phi}$, the unique asymptotically stable interior equilibrium and the unstable (saddle) axial equilibrium get closer as the parameters approach the line $v = \frac{a\eta}{\phi}$, collide with each other on the line causing exchange of stability between them due to the occurrence of transcritical bifurcation.

As parameters (η, v) move from region II into III crossing the line $v = \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$, the system experiences a saddle-node bifurcation. Hence there is the emergence of two interior equilibria for the system, where one of them is a saddle, and the other is locally stable. The nature of the axial equilibrium continues to be locally stable. Here, the stable manifold of the saddle interior equilibrium divides the positive quadrant of the xP-space into two invariant regions, each being the region of attraction of the stable equilibrium that it contains.

As parameters (η, v) move from region III into I crossing the line $v = \frac{a\eta}{\phi}$, the saddle interior equilibrium that exists (when the parameters are in region III) moves closer to the locally stable axial equilibrium and collide with each other on the line $v = \frac{a\eta}{\phi}$ causing a transcritical bifurcation between them resulting in the axial equilibrium becoming saddle and retention of only one interior equilibrium when the parameters are in region I.

We have the following observations on the system (1). If the inflow rate of pollutants v is sufficiently low such that $v < \frac{a\eta}{\phi}$ is satisfied, then we are assured of the stable coexistence in the system. If the inflow parameter satisfies $\frac{a\eta}{\phi} < v < \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}$, the system admits two interior equilibria (one is locally stable and the other is unstable). In this case, the survival of the resource depends on its initial position. Unless the initial position is in the region of attraction of the interior equilibrium, the resource is likely to go extinct. On the other hand, if the inflow rate is sufficiently large so that $v > \max\{\frac{a\eta}{\phi}, \frac{(a\gamma+b\eta)^2}{4b\gamma\phi}\}$, the resource can not survive in such environment.

3 The Influence of Investing in Pollutant Inflow Reduction on the Harvest

In Section 2, we observed that a reduction in the stock of pollution helps to improve the survival and productivity of the resource, and preventing the inflow of pollutants is a feasible alternative to reduce pollution. Hence, it seems to be reasonable to allocate a part of the total effort capacity towards pollutant inflow reduction to improve the yield. Thus, we consider a revised version of (1) wherein the available effort capacity is divided into two parts: harvesting the resource and pollutant inflow reduction. This will enable us to investigate the influence of investing in inflow reduction on the yield as well as the stock.

Suppose the sole owner has the total effort capacity of M units (measured in terms of money) to invest in harvesting and pollutant inflow reduction. Now, the aim is to find an optimal effort allocation to maximize the sustainable yield. Let E and $M - E$ be the efforts allocated towards harvesting and inflow reduction. Then, the resource and pollution dynamic equations can be given by (ref. [12, 19])

$$\frac{dx}{dt} = r(P)x\big(1 - \frac{x}{K(P)}\big) - q\alpha Ex, \quad x(0) = x_0 > 0, \tag{13a}$$

$$\frac{dP}{dt} = v - \beta(M - E) - \gamma Px - \eta P, \quad P(0) = P_0 > 0, \tag{13b}$$

$$0 \leq E \leq M. \tag{13c}$$

The parameters $r(P)$ and $K(P)$ (in Eqn.(13a)) are as defined in Section 2 and the term $q\alpha Ex$ stands for the harvest rate, where αE represents the effort in physical terms (measured in vessel units). In Eqn.(13b) the term $\beta(M - E)$ (measured in ton per unit time) stands for the amount of inflow rate reduced by the de-pollution effort $M - E$, where the constants $\alpha > 0, \beta > 0$ are conversion factors, and q is the catchability coefficient.

For each harvest effort $E \in [0, M]$, the system (13a)-(13b) admits an axial equilibrium $(0, \frac{1}{\eta}(v - \beta(M - E))$ (provided $v - \beta(M - E) > 0$), and the interior equilibrium (x_E, P_{M-E}) whose components are given by

$$x_E = \frac{1}{b}\big(a - \phi P_{M-E} - \alpha qE\big), \tag{14a}$$

$$P_{M-E} = \frac{v - \beta(M - E)}{\gamma x_E + \eta}, \tag{14b}$$

provided that $a - \phi P_{M-E} - \alpha qE > 0$, $v - \beta(M - E) > 0$. Note that the subscripts E and $M - E$ for the resource and pollution variables at equilibrium, are meant to indicate that the equilibrium is a result of allocating the efforts E towards harvesting the resource and $M - E$ towards inflow reduction, respectively. The following equations are derived from (14)

$$\gamma bx_E^2 - \big((a - \alpha qE)\gamma - b\eta\big)x_E + \phi(v - \beta(M - E)) - (a - \alpha qE)\eta = 0, \tag{15a}$$

$$\gamma\phi P_{M-E}^2 - \big((a - \alpha qE)\gamma + b\eta\big)P_{M-E} + b(v - \beta(M - E)) = 0. \tag{15b}$$

Clearly, the existence of interior equilibrium for the system under consideration depends on the coefficients of a quadratic polynomial in (15a) (or (15b)). Further, the system admits at most two interior equilibria as in the case of the system (1), but here the equilibria are functions of an additional parameter E. To be more specific, for each E in (13c) the system (13a)-(13b) admits a unique interior equilibrium whenever

$$v - \beta(M - E) < \frac{1}{\phi}(a - \alpha qE)\eta, \tag{16}$$

it admits two interior equilibria whenever

$$\frac{1}{\phi}(a - \alpha qE)\eta < v - \beta(M - E) < \frac{((a - \alpha qE)\gamma + b\eta)^2}{4b\gamma\phi}, \tag{17}$$

and the interior equilibrium does not exist whenever

$$v - \beta(M - E) > \max\Big\{\frac{1}{\phi}(a - \alpha qE)\eta, \frac{((a - \alpha qE)\gamma + b\eta)^2}{4b\gamma\phi}\Big\}. \tag{18}$$

The role of parameter E for the existence of interior equilibrium for the system under consideration can be understood from Figure 2. Hence, we have the following theorems pertaining to the system (13a)-(13b) which can be easily established.

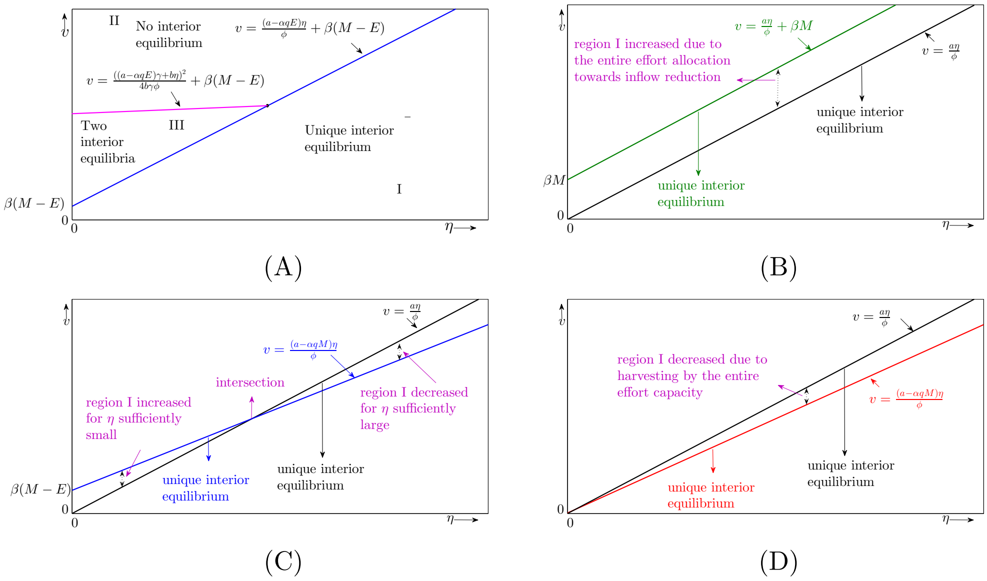

Figure 2: This figure highlights the comparison between systems (1) and (13a)-(13b) based on the values of parameter E. For a fixed $E \in [0, M]$ (in Frame A) the positive quadrant of ηv-space is divided into three regions (I, II and III) based on the values of (η, v) for which the system (13a)-(13b) admits a unique interior equilibrium, no interior equilibrium and two interior equilibria. The region under a line segment $v = \frac{a\eta}{\phi}$ (in frames B-D) represents the set of (η, v) for which the system (1) admits a unique interior equilibrium whereas the regions under the line segments $v = \frac{a\eta}{\phi} + \beta M$(in Frame B where $E = 0$), $v = \frac{1}{\phi}(a - \alpha q E)\eta + \beta(M - E)$ (in Frame C where $0 < E < M$) and $v = \frac{1}{\phi}(a - \alpha q M)\eta$ (in Frame D where $E = M$) represent the set of (η, v) for which the system (13a)-(13b) admits a unique interior equilibrium. In Frame B, the region for unique interior equilibrium is increased (when the entire effort is allocated towards inflow reduction, and no harvesting). In Frame D, the region for unique interior equilibrium is decreased (when the entire effort is allocated towards harvesting with no inflow reduction). In Frame C, the region for unique interior equilibrium is increased for η sufficiently small, and it is reduced for large η.

Theorem 3.1 *For each $E \in [0, M]$, the system* (13a)-(13b) *admits an axial equilibrium (provided that $v - \beta(M - E) > 0$) and at most two interior equilibria. The axial equilibrium is globally asymptotically stable in the absence of the interior equilibrium, it is a saddle in the presence of a unique interior equilibrium, and it is locally stable in the presence of two interior equilibria. The unique interior equilibrium is locally stable, and the stability nature in the case of two interior equilibria depends on their proximity to the axial equilibrium in terms of their x-component. The closer one is a saddle, and the farther one is locally stable.*

Theorem 3.2 *For each $E \in [0, M]$, if the associated parameters satisfy the relation $v - \beta(M - E) < \frac{(a - \alpha q E)\eta}{\phi}$, then system* (13a)-(13b) *admits a unique interior equilibrium which is globally asymptotically stable.*

We observe that it is noteworthy to give more attention to the existence of a unique interior equilibrium in the system as it assures not only the stable coexistence in the system but also the equilibrium with a larger resource level and lower level of pollution. From (14a)-(14b), it follows that the resource x_E decreases and pollution P_{M-E} increases as the effort E increases. Further, the highest resource level and the lowest level of pollution occur at the equilibrium (x_0, P_M) (where $E = 0$). Similarly, the lowest resource level and the highest level of pollution occur at the equilibrium (x_M, P_0) (where $E = M$).

Having seen the highest (lowest) possible levels for the resource stock as well as the stock of pollution, here we are interested in determining the effort allocation that maximizes the sustainable yield. Observe that increasing the harvest effort may not improve the catch in general. The reason being, increasing the harvest effort E causes a decrease in the effort $M - E$ towards the inflow reduction, and this results in a reduction of the stock level x. Therefore, it is useful to find the proper effort allocation to maximize the sustainable yield. For the given harvest effort $E \in [0, M]$ and the stock level x_E, the yield expression is

$$Y(E) = q\alpha E x_E, \tag{19}$$

where the stock level x_E is a function of E (which is the x-component of a unique interior equilibrium (x_E, P_{M-E}) of the system (13a)-(13b)). Clearly, $Y(E)$ is continuous on the interval $[0, M]$, and hence it attains its maximum in $[0, M]$ giving rise to maximum sustainable yield Y_{MSY}. This yield may occur either at the boundary $E = M$ or in the interior of $[0, M]$. If $E_{MSY} \in (0, M)$, then it solves the equation

$$\frac{dY}{dE} = 0, \tag{20}$$

for a concave function $Y(E)$ on $[0, M]$.

4 Optimal Harvest Problem

Following the dynamics of harvested renewable resource surviving in a polluted environment, and having seen the influence of allocating a part of the effort capacity towards inflow reduction on the yield, now we wish to construct an optimal harvesting strategy. Assume that the sole owner has twofold benefit from the resource: benefits from harvesting and the stock. The latter one represents the benefit of the stock in its natural place (such as tourism) [18, 19]. Therefore, it is crucial to ensure a reasonable level of stock in the environment to reap the stock benefit in addition to the revenue from harvesting.

The gross harvesting benefit is assumed to increase with an increase in the harvest, and it decreases with increasing pollution. Further, the marginal (negative) impact of pollution on the gross harvesting benefit is assumed to increase. Hence the gross harvesting benefit (which we denote by $W(h, P)$) may have the following properties (ref. [19])

$$W_h > 0, W_P < 0, W_{PP} < 0. \tag{21}$$

Hence, we consider the following explicit form of the function $W(h, P)$

$$W(h, P) = \tau(1 - \epsilon P^2)h, \tag{22}$$

where $h = q\alpha E x$ is the harvest rate and $\tau(1 - \epsilon P^2)$ represents the pollution dependent price per unit harvest (where $0 \le 1 - \epsilon P^2 \le 1$). Clearly, τ represents the price per unit catch in the absence of pollution.

The stock benefit is assumed to be affected by the presence of pollution in the environment. Naturally, it decreases with increasing pollution (as it may detour the tourists from visiting). We also assume that the marginal benefit of the stock decreases, and the marginal (negative) impact of pollution increases. Thus, the stock benefit (measured in terms of money) denoted by $S(x, P)$ may have the following properties:

$$S_x > 0, S_{xx} < 0, S_P < 0, S_{PP} < 0, \text{for } 0 < x \leq x_s, S_x = 0 \text{ for } x > x_s, \tag{23}$$

where x_s denotes the saturation level in the function. Here, we assume the saturation level x_s to be the environmental carrying capacity. The assumption $S_x = 0$ for each $x > x_s$ is due to the observation that the "small" deviation from some "large" stock levels do not necessarily reduce the human in situ benefits [19]. Thus, we consider the following explicit form of the function S:

$$S(x, P) = \begin{cases} \frac{\rho(1-\epsilon P^2)x}{\sigma+x} & \text{for, } 0 < x \leq x_s, \\ \frac{\rho(1-\epsilon P^2)x_s}{\sigma+x_s} & \text{for, } x > x_s, \end{cases} \tag{24}$$

where ρ denotes the maximum achievable benefit from the stock in the absence of pollution and σ is a half saturation constant.

The instantaneous net revenue, which is the sum of benefits from harvesting and the stock, is given by

$$R(x, P, E) = (1 - \epsilon P^2)\big(\tau q \alpha E x + \frac{\rho x}{\sigma + x}\big) - M. \tag{25}$$

Hence, the present value (denoted by PV) of the total net revenues on the infinite horizon is given by

$$PV = \int_0^\infty e^{-\delta t} R(x, P, h) dt, \tag{26}$$

where δ is the instantaneous discount rate. Now, the objective is to find an optimal effort allocation between harvesting and inflow reduction so that the integral in (26) is as large as possible. Precisely formulated, the problem is as follows:

$$\text{Maximize } PV \tag{27a}$$

$$\text{Subject to: } \frac{dx}{dt} = r(P)x\big(1 - \frac{x}{K(P)}\big) - \alpha q E x, \quad x(0) > 0, \tag{27b}$$

$$\frac{dP}{dt} = v - \beta(M - E) - \gamma P x - \eta P, \quad P(0) > 0, \tag{27c}$$

$$0 \leq E \leq M. \tag{27d}$$

This is an optimal control problem on an infinite horizon with two state variables (the resource stock x and the stock of pollution P) and one control variable (the effort E towards harvesting).

Solving the problem (27) is amounting to finding out the optimal effort $E_0(t)$ towards harvesting (and hence $M - E_0(t)$ towards inflow reduction) so that the present value in (27a) is maximum. Equivalently, we need to find the path traced out by $(x_0(t), P_0(t))$ with this optimal effort allocation so that if the resource stock and the stock of pollution are kept along this path, then we are assured of achieving the objective of the sole owner.

As per the maximum principle [5, 9, 15], the Hamiltonian $H(x, P, E, \mu_1, \mu_2)$ is given by

$$\begin{aligned}H(x,P,E,\mu_1,\mu_2) = e^{-\delta t}[(1-\epsilon P^2)(\alpha q\tau Ex + \frac{\rho x}{\sigma + x}) - M]+\\ \mu_1\big[r(P)x(1-\frac{x}{K(P)}) - \alpha qEx\big]+\\ \mu_2[v-\gamma Px - \eta P - \beta(M-E)].\end{aligned} \tag{28}$$

And the associated adjoint variables μ_1, μ_2 satisfy the differential equations

$$\frac{d\mu_1}{dt} = -e^{-\delta t}(1-cP^2)[\tau\alpha qE + \frac{\rho\sigma}{(\sigma+x)^2}] - \mu_1\big(a - 2bx - \phi P - \alpha qE\big) + \gamma\mu_2 P,$$
$$\frac{d\mu_2}{dt} = 2\epsilon e^{-\delta t}P(\tau\alpha qEx + \frac{\rho x}{\sigma+x}) + \mu_2(\eta + \gamma x) + \mu_1\phi x.$$

Because of the presence of the term $e^{-\delta t}$ no steady state is possible for the above system. Hence, we consider the following transformation:

$$\lambda_i(t) = \mu_i(t)e^{\delta t}, \quad i=1,2 \ \text{ and } \ \mathcal{H} = He^{\delta t}, \tag{30}$$

where $\mathcal{H}$ is known as the current value Hamiltonian and λ_1,λ_2 are the current value adjoint variables. From the above transformation we can easily obtain the following adjoint differential equations:

$$\frac{d\lambda_1}{dt} = \lambda_1\delta - \lambda_1\big(a - 2bx - \phi P - \alpha qE\big) - (1-cP^2)[\tau\alpha qE + \frac{\rho\sigma}{(\sigma+x)^2}] + \gamma\lambda_2 P, \tag{31a}$$
$$\frac{d\lambda_2}{dt} = \lambda_2\delta + \lambda_2(\eta+\gamma x) + \lambda_1 xd + 2\epsilon P(\tau\alpha qEx + \frac{\rho x}{\sigma+x}). \tag{31b}$$

Since the problem under consideration is linear in the control variable, the optimal control shall be a combination of bang-bang and singular controls. First, we investigate the singular solution to the problem. Differentiating the current value Hamiltonian with respect to E gives us $\mathcal{H}_E = \tau(1-\epsilon P^2)\alpha qx - \lambda_1\alpha qx + \lambda_2\beta$ with the switching function being

$$s(t) = \tau(1-\epsilon P^2)\alpha qx - \lambda_1\alpha qx + \lambda_2\beta.$$

It is known that along the singular solution $s(t)=0$, i.e.,

$$\tau(1-\epsilon P^2)\alpha qx - \lambda_1\alpha qx + \lambda_2\beta = 0. \tag{32}$$

Now substituting the interior steady state solution $(\widetilde{x}_E, \widetilde{P}_{M-E}, \widetilde{\lambda}_1, \widetilde{\lambda}_2)$ of the four dimensional dynamical systems (27b), (27c), (31a) and (31b) into (32) we obtain the following equation (which is purely in E):

$$\tau(1-\epsilon\widetilde{P}^2_{M-E})\alpha q\widetilde{x}_E - \alpha q\widetilde{\lambda}_1\widetilde{x}_E + \beta\widetilde{\lambda}_2 = 0. \tag{33}$$

If the solution $\widehat{E}$ of (33) satisfies the condition $0 < \widehat{E} < M$, then $\widehat{E}$ becomes the optimal singular control and $(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}})$ is the associated optimal singular solution [4,6].

After identifying the optimal singular solution $(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}})$, now it remains to reach this solution optimally starting from the given initial state $(x(0), P(0))$. Since the problem under consideration is linear in the control variable, the singular solution $(x_{\widehat{E}}, P_{M-\widehat{E}})$

can be reached by a bang-bang control [15]. If we denote this control by $\overline{E}(t)$, then we have

$$\overline{E}(t) = \begin{cases} 0, & \text{if } s(t) < 0, \\ M, & \text{if } s(t) > 0, \end{cases} \tag{34}$$

where $s(t) = \tau(1 - \epsilon P^2)\alpha q x - \lambda_1 \alpha q x + \lambda_2 \beta$. Suppose $\mathcal{T}$ represents the time required to reach the singular solution optimally from the given initial state $(x(0), P(0))$ (by a bang-bang control $\overline{E}(t)$). Then the optimal control $E_0(t)$ for the problem under consideration is given by

$$E_0(t) = \begin{cases} \overline{E}(t), & \text{for } 0 \leq t < \mathcal{T}, \\ \widehat{E}, & \text{for } t \geq \mathcal{T}. \end{cases} \tag{35}$$

If $\big(\overline{x}(t), \overline{P}(t)\big)$ represents the trajectory (corresponding to a bang-bang control $\overline{E}(t)$) from the given initial state $(x(0), P(0))$ to the singular solution $\big(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}}\big)$, then the optimal path, traced out by $\big(x_o(t), P_o(t)\big)$, is given by

$$\big(x_0(t), P_0(t)\big) = \begin{cases} \big(\overline{x}(t), \overline{P}(t)\big) & \text{for } 0 \leq t \leq \mathcal{T}, \\ \big(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}}\big) & \text{for } t \geq \mathcal{T}. \end{cases} \tag{36}$$

If the singular harvesting effort $\widehat{E}$ is employed right from the given initial state $(x(0), P(0))$, then by the global asymptotic stability of the singular solution $\big(\widetilde{x}_{\widehat{E}}, \widetilde{P}_{M-\widehat{E}}\big)$, the corresponding stock path approaches this solution asymptotically.

5 Applications

This example represents the dynamics of a fish population in a polluted lake environment, where the considered biological and economic parameters are related to actual values one might have in a fishery (ref. [2]). The inflow rate of pollutants (in wastewater effluents) is measured yearly to be 50 tonnes per year. The values assigned to the associated parameters and constants in the problem are given in Table 1. With the given values in the table, it can be easily seen that the condition $\phi(v - \beta(M - E)) - (a - \alpha q E)\eta < 0$ holds for each $E \in [0, M]$, and hence the system (13a)-(13b) admits a unique interior equilibrium $(\widetilde{x}_E, \widehat{P}_{M-E})$ which is globally asymptotically stable.

For each effort $E \in [0, M]$ the unique interior equilibrium $(\widetilde{x}_E, \widehat{P}_{M-E})$ for the system (13a)-(13b) is evaluated and this can be seen in Figure 3. This figure also highlights the relations among the effort E, the resource x, pollution P and yield Y. The highest resource stock and the lowest stock of pollution in the system, respectively, are $x_0 = 3.49 \times 10^4$ and $P_M = 105.9$ (in tons) which are obtained for $E = 0$. Similarly, the lowest resource stock and the highest stock of pollution in the system, respectively, are $x_M = 1.05 \times 16^4$ and $P_0 = 432.2$ (in tons) which are obtained for $E = M$.

The sustainable yield $Y(E)$ corresponding to each $E \in [0, M]$ is evaluated using (19) and this is presented in Figure 3. The maximum sustainable yield is $Y_{MSY} = 3.02 \times 10^3$ (in tons) which is obtained at the critical effort level $E_{MSY} = 8.62 \times 10^5$ (in US\$). The resource stock and stock pollution corresponding to the maximum sustainable yield are given by $(x_{E_{MSY}}, P_{M-E_{MSY}}) = (1.75 \times 10^4, 251.75)$ (in tons).

The interior steady state solution of the four dimensional dynamical systems (27b), (27c), (31a) and (31b) is given by $(1.63 \times 10^4, 272.65, 5.96 \times 10^3, -6.898 \times 10^3)$, where the unique solution $\widehat{E}$ of (33) is $\widehat{E} = 9.21 \times 10^5$ (in US\$). Observe that the solution

Table 1: The values of parameters and their units.

Parameters	Symbol	Values	units
Intrinsic growth rate	a	0.35	1/year
Intraspecific competition	b	1×10^{-5}	1/ton/year
Death of the resource per unit of pollution	ϕ	1×10^{-5}	1/ton/year
Uptake of pollutant per unit of the resource	γ	1×10^{-5}	1/ton/year
Inflow rate of pollutants	v	50	ton/year
Available effort capacity	M	1.2×10^{6}	\$/year
Conversion parameter	α	1×10^{-3}	vessel/\$/year
Conversion parameter	β	1×10^{-5}	ton/\$
Natural degradation rate of pollution	η	1×10^{-2}	1/year
Catchability coefficient	q	0.0002	1/vessel/year
Discount rate	δ	0.025	1/year
Price per unit catch in absence of pollution	τ	6×10^{3}	\$/ton
Maximum stock benefit in absence of pollution	ρ	8×10^{4}	\$/year
Half saturation constant	σ	1×10^{4}	ton

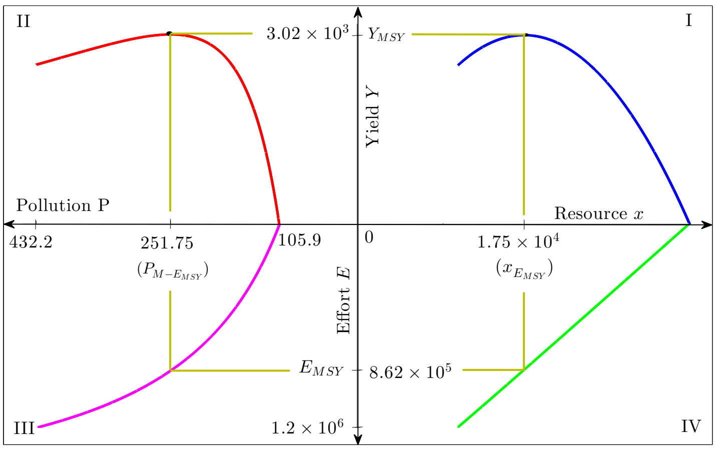

Figure 3: This four-quadrant figure presents the resource and pollution stocks (x_E, P_{M-E}), and the yield $(Y(E))$ associated with each harvest effort $E \in [0, M]$ (and hence the effort $M - E$ for inflow reduction). Quadrant I depicts the relationship between the resource x_E and the yield Y, quadrant II represents the relationship between pollution P_{M-E} and the yield Y, quadrant III represents the relationship between the effort E and pollution P_{M-E} and quadrant IV gives the relationship between the effort E and the resource x_E. For each $E \in [0, M]$, Y can be seen either in quadrant I (through the point (E, x_E)) or in quadrant II (through (E, P_{M-E})). The figure also highlights the critical effort level E_{MSY} that gives the maximum sustainable yield $Y(E_{MSY})$.

$\widehat{E}$ satisfies the relation $0 < \widehat{E} < M$, and hence it becomes an optimal singular effort for harvesting (and hence the effort $M - 9.21 \times 10^5$ goes for inflow reduction). The optimal singular solution $(x_{\widehat{E}}, P_{M-\widehat{E}})$ is given by $(1.63 \times 10^4, 272.65)$ (in tons), and the corresponding yield is $Y(\widehat{E}) = 3.00 \times 10^3$ (in tons). Note that the yield $Y(\widehat{E})$ associated

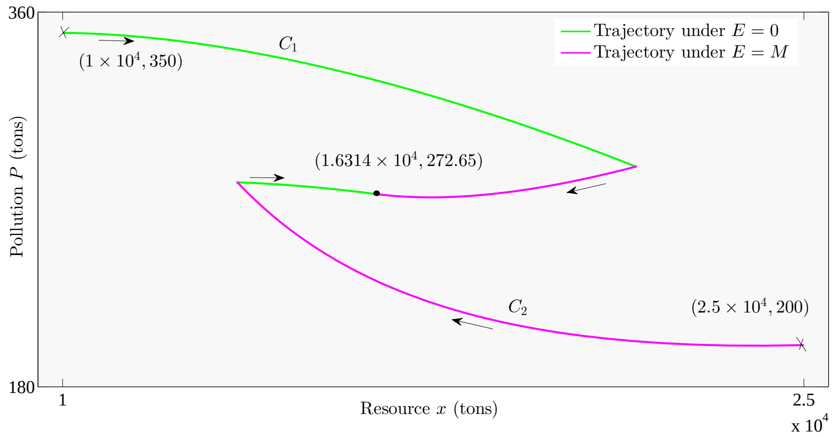

Figure 4: This figure presents the optimal approach path starting from two different initial states $(x(0), P(0)) = (1 \times 10^4, 350)$ (in tons) and $(x(0), P(0)) = (2.5 \times 10^4, 200)$ (in tons) to the singular solution $(1.63 \times 10^4, 272.65)$. The optimal approach path C_1 starts from the initial state $(1 \times 10^4, 350)$ and it takes a time period $\mathcal{T} = 7.65$ (in years) to reach the singular solution by a bang-bang control $\overline{E}(t) = 0$ for $0 \leq t < 4.05$ and $\overline{E}(t) = M$ for $4.05 \leq t < 7.65$. Similarly, the optimal approach C_2 starts from the initial state $(2.5 \times 10^4, 200)$ and it takes a time period $\mathcal{T} = 10.45$ (in years) to reach the singular solution by a bang bang control $\overline{E}(t) = M$ for $0 \leq t < 9.5$ and $\overline{E}(t) = 0$ for $9.5 \leq t < 10.45$.

with the optimal effort $(\widehat{E})$ is less than the maximum sustainable yield (Y_{MSY}). The optimal approach path from the given initial position $(x(0), P(0)) = (1 \times 10^4, 350)$ to the singular solution $(1.63 \times 10^4, 272.65)$ (in tons) takes the time period $t = 7.65$ (in years) under a bang-bang control

$$\overline{E}(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t < 4.05, \\ M & \text{for} \quad 4.05 \leq t < 7.65. \end{cases}$$

Therefore, the optimal control to the given problem is

$$E_o(t) = \begin{cases} \overline{E}(t) & \text{for } 0 \leq t < 7.65, \\ \hat{E} & \text{for } t \geq 7.65. \end{cases}$$

The corresponding optimal approach path is shown in Figure 4. The figure also presents the optimal approach path from another initial position $(2.5 \times 10^4, 200)$ (in tons) to the singular solution $(1.63 \times 10^4, 272.65)$.

6 Concluding Remarks

In this paper, we have presented the bio-economics of a renewable resource in the presence of pollution. A decline in the revenue due to pollutants (in the environment) is a major driving force for investing a part of the effort capacity towards pollution reduction, and pollutant inflow reduction was considered as a feasible alternative to reduce environmental pollution.

The presence of pollution is assumed to affect both the resource growth rate and the quality of the catch. The influence of pollutants on the resource growth and the quality of the catch are captured through its regeneration function and the revenue function, respectively. By investigating the resource-pollution dynamics (in the absence of harvesting and pollutant inflow reduction), we observed that the species goes to extinction whenever the inflow of pollutants is sufficiently large. Furthermore, a criterion is formulated that assures the stable coexistence of the species and pollution.

By incorporating resource harvesting and pollutant inflow reduction into the resource and pollution dynamic equations, respectively, we have investigated the influence of investing in pollutant inflow reduction on the yield. We observed that, by proper allocation of the available effort capacity between harvesting and pollutant inflow reduction, it is possible not only to improve the revenue but also the survival rate of the species. Further, we have observed that there is an optimal effort allocation between harvesting and pollutant inflow reduction that maximizes the yield.

Finally, by considering the pollution dependent revenue obtained from both harvesting and the stock benefit, we have studied an optimal harvest problem. We observed that variation in the stock benefit affects the optimal harvesting strategy. In particular, an increase in the stock benefit results in a reduction in the harvest effort (and hence a rise in pollution reduction effort). Consequently, the resource stock increases, and the stock of pollution decreases.

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