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Reduced Order Multiswitching Synchronization between Two Hyperchaotic Systems of Different Order

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Abstract: In this paper, we have investigated the problem of reduced order multiswitching synchronization using the active control method. Reduced order multiswitching synchronization can be considered as a combination of multi-switching with reduced order synchronization. Apt controllers have been constructed to establish the asymptotically stable synchronized state by using different laws of switching and Lyapunov stability theory. To analyze the proposed methodology, a six-dimensional Lorenz model and four-dimensional hyper-chaotic coupled dynamos system have been considered as a drive and response system, respectively. Theoretical results are validated by numerical simulations performed in MATLAB.

Keywords: *multiswitching synchronization; reduced order synchronization; Lyapunov stability theory; Lorenz model; dynamos system.*

Mathematics Subject Classification (2010): 34D06.

1 Introduction

Nonlinear dynamical systems manifest extreme sensitive dependence on initial conditions [1]. Different aspects of nonlinear dynamical systems such as chaos, stability, bifurcation, Poincare surface and synchronization have many useful applications in the modelling of brain activity [8], secure communication [11], information processing [10], medicine[8, 9], signal processing [10] and chemical networks. This has led to the discovery of various kinds of synchronization such as projective synchronization [6], reduced order synchronization [3], generalized synchronization, lag synchronization [5], phase synchronization [4], complete synchronization [7], anticipated synchronization and increased order synchronization [2].

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Ucar et al. [12] first introduced the concept of multiswitching synchronization. It is an important and interesting extension of the existing synchronization schemes because in this scheme, a greater number of synchronization directions exist as the different states of the drive system are synchronized with the desired states of the response system. Various synchronization schemes have been investigated to achieve multiswitching synchronization such as multiswitching combination synchronization (MSCS), multiswitching combination combination synchronization (MSCCS) and a dual combination multiswitching scheme. Most of the work in multiswitching synchronization, till now, has been restricted to the multiswitching synchronization between the drive system and the response system of same order, here the order means the number of state variables. The multiswitching synchronization problem between chaotic systems of different order is still a relatively unexplored area of research. In recent years, problems related to the reduced-order synchronization of chaotic systems have fascinated researchers because of its occcurence in biological and social sciences. The main feature of the reduced-order synchronization is the synchronization of state variables of the response system with the projections of state variables of the drive system where the order of the response system is less than the order of the drive system. Here all states of the response system will be synchronized during synchronization. This kind of synchronization is required between heart and lungs or between neurons and in ecological systems as these dynamical systems are of different orders thus it makes them a very relevant topic to be investigated. Motivated by the above discussion, in this paper, we have made an effort to study the multiswitching reduced order synchronization between chaotic systems. We have designed appropriate controllers to achieve the reduced-order multi-switching synchronization between a six-dimensional Lorenz model and a four-dimensional hyper-chaotic coupled dynamos system.

2 Problem Formulation

In this section, we explain the reduced order multiswitching synchronization between chaotic systems via the active control method. Consider the hyperchaotic system (the drive system) described as

$$\dot{\xi}(t) = f(\xi(t)), \tag{1}$$

where $\xi = (\xi_1, \xi_2, ...\xi_m) \in \Re^m$ denotes the state variable of the master system and $f(\xi(t)) \in \Re^m$ represents the nonlinear functional vector. Now we consider the following chaotic system as our response system:

$$\dot{\zeta}(t) = g(\zeta(t)) + U(t), \tag{2}$$

where $\zeta = (\zeta_1, \zeta_2, ...\zeta_n) \in \Re^n$ denotes the state variable of the slave system and $g(\zeta(t)) \in \Re^n$ represents the nonlinear functional vector and $U(t) = (u_1, u_2, ...u_n) \in \Re^n$ is the control input to be evaluated which will synchronize the state of the drive and the response system.

Since the order of response system is less than the order of drive system, we have $n < m$, and thus we may select any n variables out of m variables of drive system for the projection because the reduced order synchronization is the problem of synchronizing a response system with the projection of the drive system. Thus, we can divide the drive system into two parts, the projection part and the remaining part, given by

$$\dot{\xi}_p(t) = f_p(\xi(t)), \tag{3}$$

where $\xi_p = (\xi_{p1}, \xi_{p2}, ..., \xi_{pn}) \in \Re^n$ and $f_p : \Re^m \to \Re^n$, and

$$\dot{\xi}_r(t) = f_r(\xi(t)), \tag{4}$$

where $\xi_r = (\xi_{p(n+1),\xi_{p(n+2)},....,\xi_{pm}}) \in \Re^{m-n}$ and $f_r : \Re^m \to \Re^{m-n}$. Clearly, ξ_{pi}'s are the rearrangement of ξ_i's.

To obtain the multiswitching reduced order synchronization between the projection of master system (2) and slave system (3) we need to calculate the controller $U = (u_1, u_2, ..., u_n)$ such that

$$\lim_{t\to\infty} e_{ij} = \lim_{t\to\infty} ||\zeta_j - \xi_{pi}|| = 0. \tag{5}$$

3 System Description

Recently, Carolini C. Felicio and Paulo C. Rech [14] generalized the Lorentz model of $3rd$ order to $6th$ order by adding three more state variables to it. The $6th$ order hyperchaotic Lorentz model (the drive system) is given by

$$\left.\begin{aligned}
\dot{\xi_1} &= a_1(\xi_2 - \xi_1), \\
\dot{\xi_2} &= a_2\xi_1 - \xi_2 - \xi_1\xi_3 + \xi_3\xi_4 - 2\xi_4\xi_6, \\
\dot{\xi_3} &= \xi_1\xi_2 - a_3\xi_3 - \xi_1\xi_5 - \xi_2\xi_6, \\
\dot{\xi_4} &= -(1 + 2a_3)a_1\xi_4 + \frac{a_1}{(1 + 2a_3)}\xi_5, \\
\dot{\xi_5} &= \xi_1\xi_3 - 2\xi_1\xi_6 + a_2\xi_4 - (1 + 2a_3)\xi_5, \\
\dot{\xi_6} &= 2\xi_1\xi_5 + 2\xi_2\xi_4 - 4a_3\xi_6,
\end{aligned}\right\}, \tag{6}$$

where a_1, a_2 and a_3 are constant parameters.

Figures below show the chaotic attractor of drive system for particular values of parameters given by $a_1 = 10, a_2 = 100$ and $a_3 = \frac{8}{3}$.

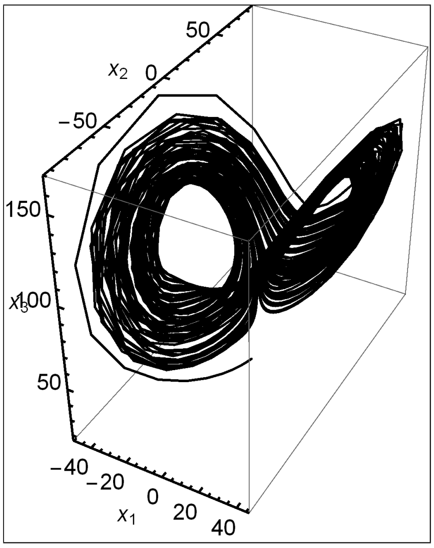

Fig.1: Chaotic behavior of system (6) in ξ_1, ξ_2, ξ_3 plane.

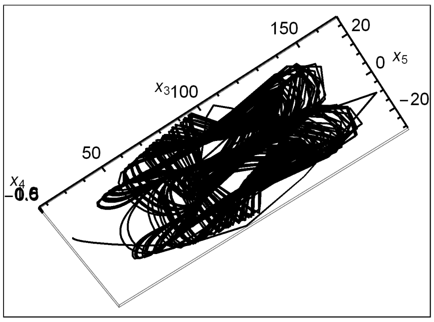

Fig.2: Chaotic behavior of system (6)in ξ_3, ξ_4, ξ_5 plane.

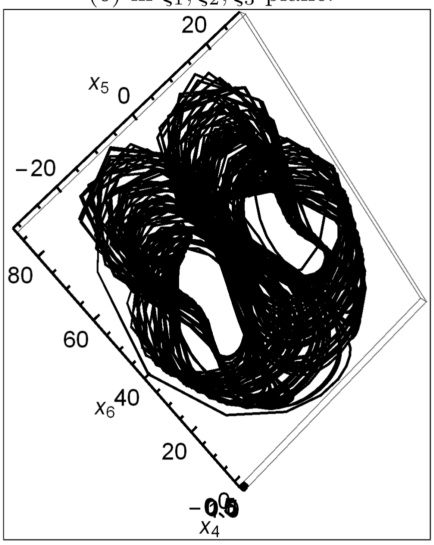

Fig.3: Chaotic behavior of system (6) in ξ_4, ξ_5, ξ_6 plane.

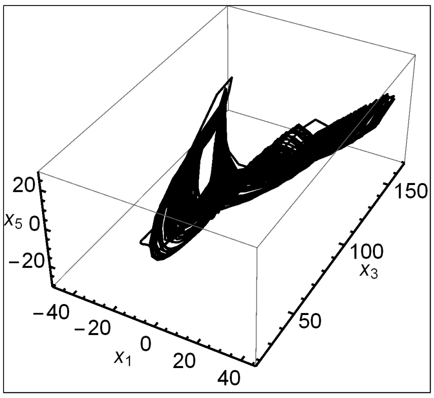

Fig.4: Chaotic behavior of system (6)in ξ_1, ξ_3, ξ_5 plane.

Yanyun Xie and Wenliang Cai[15], in 2017, modified $3rd$ order coupled dynamos system to a new $4th$ order hyperchaotic coupled dynamos (response) system given by

$$\left.\begin{aligned} \dot{\zeta}_1 &= -2\zeta_1 + \zeta_2(\zeta_3 + 3) + \zeta_4 + u_1, \\ \dot{\zeta}_2 &= -2\zeta_2 + \zeta_1(\zeta_3 - 3) + u_2, \\ \dot{\zeta}_3 &= \zeta_3 - \zeta_1\zeta_2 + u_3, \\ \dot{\zeta}_4 &= -m\zeta_2 + u_4, \end{aligned}\right\}, \tag{7}$$

where m is a constant parameter. Figures below show the chaotic attractor of response system for $m = 100$

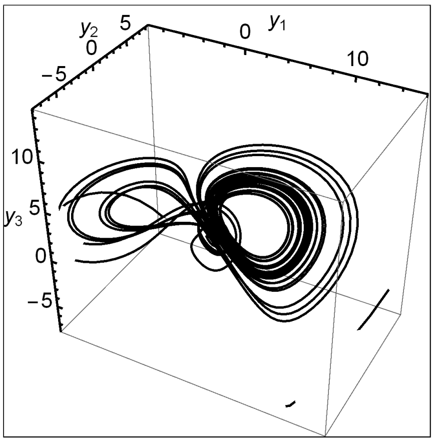

Fig.5: Chaotic behavior of system (7) in $\zeta_1, \zeta_2, \zeta_3$ plane.

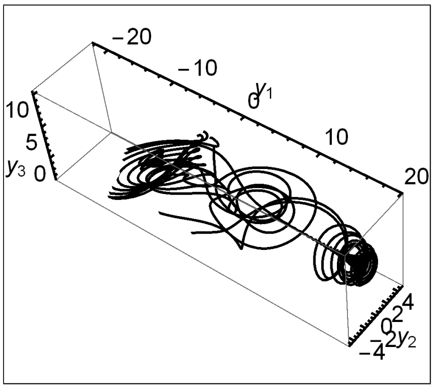

Fig.6: Chaotic behavior of system (7)in $\zeta_2, \zeta_3, \zeta_4$ plane.

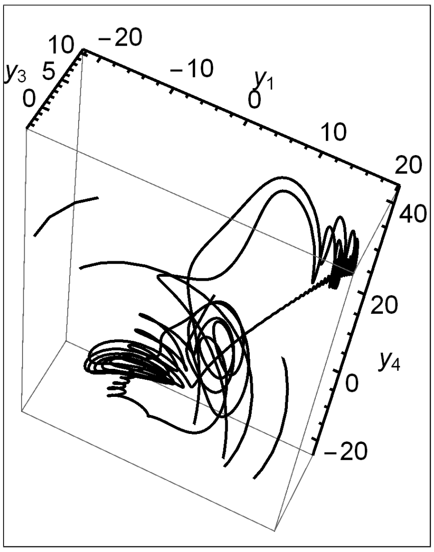

Fig.7: Chaotic behavior of system (8) in $\zeta_1, \zeta_3, \zeta_4$ plane.

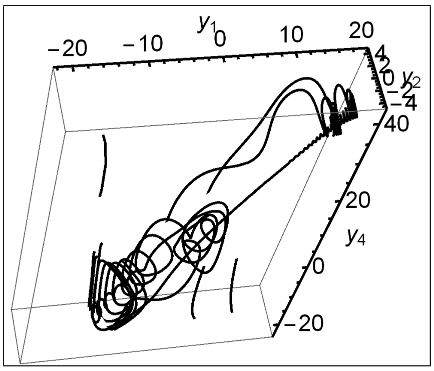

Fig.8: Chaotic behavior of system (9)in $\zeta_1, \zeta_2, \zeta_4$ plane.

4 Illustration

In this section, we present all the possible ways of projection of state variables in the drive system with respect to the order of the response system that is $\binom{6}{4}$ ways. Thus, we have 15 cases for the reduced order multiswitching synchronization and 24 ways of multiswitching for each case. Likewise, we can take a projection to the hyperplane occupied by $\xi_1 - \xi_2 - \xi_3 - \xi_4$, $\xi_2 - \xi_3 - \xi_4 - \xi_5$, $\xi_3 - \xi_4 - \xi_5 - \xi_6$ and so on, which implies we can synchronize these two systems of different order in 360 ways.

First we arbitrarily considered the case of projection of system variables to the hyperplane occupied by $\xi_1 - \xi_3 - \xi_4 - \xi_6$. Again, for this projection we have 24 possible switches given by Switching 1.

The errors are given by

$$\left.\begin{aligned} e_{11} &= \zeta_1(t) - \xi_1(t),\\ e_{12} &= \zeta_2(t) - \xi_3(t),\\ e_{13} &= \zeta_3(t) - \xi_4(t),\\ e_{14} &= \zeta_4(t) - \xi_6(t), \end{aligned}\right\} \tag{8}$$

Switching 2

$$\left.\begin{aligned} e_{21} &= \zeta_1(t) - \xi_1(t),\\ e_{22} &= \zeta_2(t) - \xi_3(t),\\ e_{23} &= \zeta_3(t) - \xi_6(t),\\ e_{24} &= \zeta_4(t) - \xi_4(t), \end{aligned}\right\} \tag{9}$$

Switching 3

$$\left.\begin{aligned} e_{31} &= \zeta_1(t) - \xi_1(t),\\ e_{32} &= \zeta_2(t) - \xi_4(t),\\ e_{33} &= \zeta_3(t) - \xi_3(t),\\ e_{34} &= \zeta_4(t) - \xi_6(t), \end{aligned}\right\} \tag{10}$$

and so on. We now calculate suitable controllers for Switching 1 to achieve the reduced order multiswitching synchronization. Let us define the error dynamics between the drive system (6) and the response system (7) as

$$\left.\begin{aligned} \dot{e}_{11} &= -(1+a_1)e_{11} + a_1\zeta_3 + 2\zeta_3 - \zeta_1\zeta_2 + a_1\xi_2 + u_3,\\ \dot{e}_{12} &= -(2+a_3)e_{12} + \zeta_2\zeta_3 + 3\zeta_2 + \zeta_4 - \xi_1\xi_2 + \xi_1\xi_5\\ &\quad + \xi_2\xi_4 - 2\xi_3 + a_3\zeta_1 + u_1,\\ \dot{e}_{13} &= -(\frac{a_1}{1+2a_3} + a_2)e_{13} - m\zeta_2 - (\frac{a_1}{1+2a_3})\xi_5 + a_2\zeta_4\\ &\quad - a_2\xi_4 + (\tfrac{a_1}{1+2a_3})\zeta_4 + u_4,\\ \dot{e}_{14} &= -(2+a_2)e_{14} + \zeta_1\zeta_3 - 3\zeta_1 - 2\xi_1\xi_5 - 2\xi_2\xi_4 + 4a_3\xi_6\\ &\quad - 2\xi_6 + a_2\zeta_2 - a_2\xi_6 + u_2. \end{aligned}\right\} \tag{11}$$

Next, we give the following proposition for the control parameters based on the error dynamic system (11)

Proposition 4.1 *Considering the error dynamics (8) the reduced order multiswitching synchronization between the drive system (6) and the response system (7) will be achieved if the control functions* u_{11}, u_{12}, u_{13} *and* u_{14} *are chosen as*

$$\left.\begin{aligned} u_{11} &= \xi_1\xi_2 + 2\xi_3 - \xi_1\xi_5 - \xi_2\xi_4 - a_3\zeta_1 - \zeta_2\zeta_3 - 3\zeta_2 - \zeta_4 + A_1, \\ u_{12} &= 2\xi_1\xi_5 + 3\zeta_1 + 2\xi_2\xi_4 + 2\xi_6 + a_2\xi_6 - \zeta_1\zeta_3 - 4a_3\xi_6 - a_2\zeta_2 + A_2, \\ u_{13} &= -a_1\zeta_3 - 2\zeta_3 + \zeta_1\zeta_2 - a_1\xi_2 + A_3, \\ u_{14} &= (\tfrac{a_1}{1+2a_3})\xi_5 + m\zeta_2 + a_2\xi_4 - (\tfrac{a_1}{1+2a_3})\zeta_4 - a_2\zeta_4 + A_4, \end{aligned}\right\} \quad (12)$$

where A_1, A_2, A_3 *and* A_4 *are the functions of* e_{11}, e_{12}, e_{13} *and* e_{14}.

Proof: The error dynamics (11), after using the controllers given by (12), can be written as

$$\left.\begin{aligned} \dot{e}_{11} &= -e_{11}(1 + a_1) + A_3, \\ \dot{e}_{12} &= -e_{12}(2 + a_3) + A_1, \\ \dot{e}_{13} &= -e_{13}(\frac{a_1}{1+2a_3} + a_2) + A_4, \\ \dot{e}_{14} &= -e_{14}(2 + a_2) + A_2. \end{aligned}\right\} \quad (13)$$

We select A_1, A_2, A_3 and A_4 in such a way that the error dynamical system given by (13) gets stabilized, that means the errors will asymptotically tend to zero. Let us consider

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = P \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \end{bmatrix}, \quad (14)$$

where P is a 4×4 matrix whose enteries are selected such that the values of A_1, A_2, A_3 and A_4 will make (13) stable. Let us consider

$$P = \begin{bmatrix} 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & \frac{a_1}{1+2a_3} & 0 \end{bmatrix}. \quad (15)$$

Therefore,

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} a_3e_{12} \\ a_2e14 \\ a_1e_{11} \\ \frac{a_1}{1+2a_3}e_{13} \end{bmatrix}. \quad (16)$$

The error dynamical system (13), after using the values of A_1, A_2, A_3 and A_4, reduced to

$$\left.\begin{aligned} \dot{e}_{11} &= -e_{11}, \\ \dot{e}_{12} &= -2e_{12}, \\ \dot{e}_{13} &= -a_2 e_{13}, \\ \dot{e}_{14} &= -2e_{14}. \end{aligned}\right\} \tag{17}$$

Now, we choose the following Lyapunov function:

$$V_1 = e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2. \tag{18}$$

Clearly, V_1 is positive definite in $\mathbb{R}^4$. After differentiating V_1 with respect to time, we get

$$\frac{dV_1}{dt} = 2e_{11}\dot{e}_{11} + 2e_{12}\dot{e}_{12} + 2e_{13}\dot{e}_{13} + 2e_{14}\dot{e}_{14}. \tag{19}$$

Using (17), we get

$$\frac{dV_1}{dt} = -2e_{11}^2 - 4e_{12}^2 - 2a_2 e_{13}^2 - 4e_{14}^2. \tag{20}$$

Since V_1 is a positive definite function and $\frac{dV_1}{dt}$ is a negative definite function, the Lyapunov stability theory proves that the state of the drive and response systems synchronize asymptotically. Hence the result.

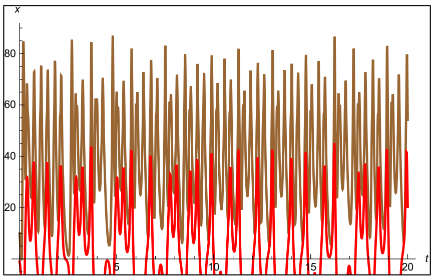

Fig.9: Multiswitching synchronization state between ξ_1, ζ_1.

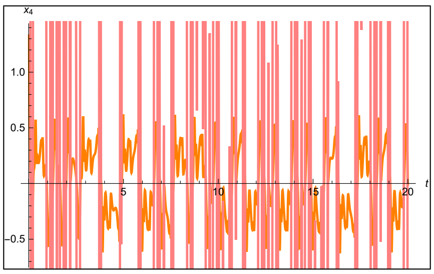

Fig.10: Multiswitching synchronization state between ξ_3, ζ_2.

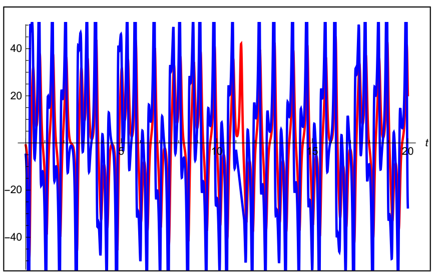

Fig.11: Multiswitching synchronization state between ξ_4, ζ_3.

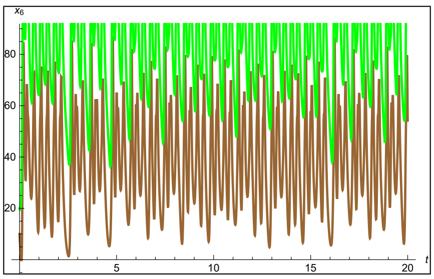

Fig.12: Multiswitching synchronization state between ξ_6, ζ_4.

5 Numerical Simulations

This section presents the synchronization of numerical simulations of the sixth order hyperchaotic Lorentz system and the fourth order hyperchaotic coupled dynamos system. The reduced order multiswitching synchronization is used as an approach to synchronize the systems. The simulations are done in MATLAB. We set the initial values and the parameters as follows: $\xi_1(0) = -1, \xi_2(0) = -5, \xi_3(0) = 20, \xi_4(0) = 5, \xi_5(0) = 3, \xi_6(0) = 10, \zeta_1(0) = -1, \zeta_2(0) = -5, \zeta_3(0) = 20, \zeta_4 = 5, a_1 = 10, a_2 = \frac{8}{3}, a_3 = 100$ and $m = 100$. These figures represent the simulation errors e_1, e_2, e_3, e_4 converging to zero asymptotically, which prove that the hyperchaotic system is synchronized.

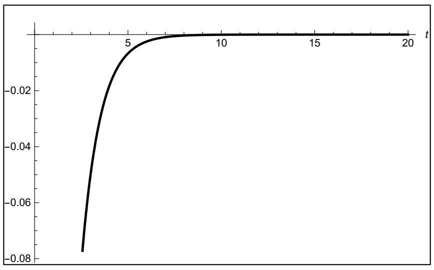

Fig.13: Convergence of error-e_1.

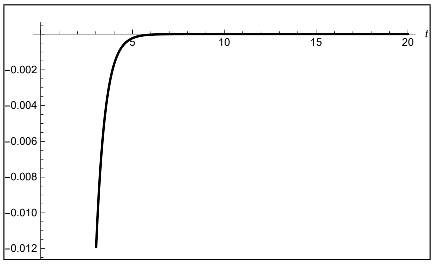

Fig.14: Convergence of error-e_2.

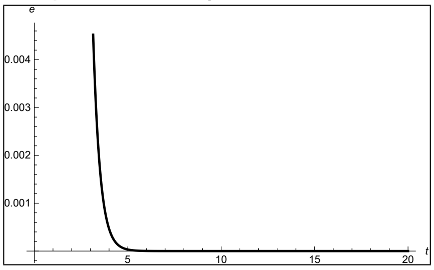

Fig.15: Convergence of error-e_3.

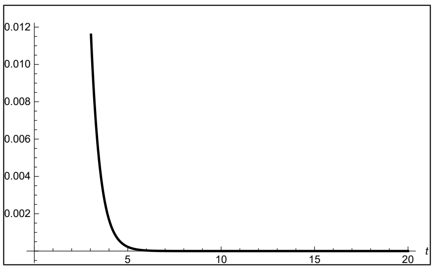

Fig.16: Convergence of error-e_4.

6 Conclusion

In this paper, we have studied the reduced order multiswitching synchronization for two hyperchaotic systems with different order. We have proposed controllers and updating laws based on active control theories. Thus we achieved the error system asymptotically stable. Further, the simulation the results demonstrate the effectiveness and feasibility of results which are performed in MATLAB.

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