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Almost Periodic Solutions for a Class of Nonlinear Duffing System with Time-Varying Coefficients and Stepanov-Almost Periodic Forcing Terms

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Abstract: In this paper, we study the existence of almost periodic solutions for a class of nonlinear Duffing system with time-varying coefficients and Stepanov-almost periodic forcing terms. Some sufficient conditions for the existence and uniqueness of an almost periodic solution of the system are established. We provide an example to illustrate the main result.

Keywords: *nonlinear Duffing system; almost periodic; Stepanov-almost periodic; contraction mapping principle.*

Mathematics Subject Classification (2010): 34C27, 34K14, 34A34.

1 Introduction

In recent years, various kinds of dynamic behaviors of nonlinear Duffing equations have been investigated by many authors due to its applications in many fields such as physics, mechanics, engineering and other scientific fields, for example, see [4, 5, 14]. In such applications, the existence of almost periodic solutions for nonlinear Duffing equations is an important topic. Many authors have studied the existence of periodic and almost periodic solutions of nonlinear differential equations, for more details we refer, [1–3, 6, 9–11, 13, 15–18] and the references cited therein.

Peng and Wang [13] considered the following model for a nonlinear Duffing equation with deviating argument

$$u''(t) + cu'(t) - au(t) + bu^m(t - \phi(t)) = \psi(t), \tag{1}$$

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where $\phi(t)$ and $\psi(t)$ are almost periodic functions on $\mathbb{R}$, $m > 1$ is an integer, and a, b, c are constants. By considering

$$v = u' + \xi u - Q_1(t),\ Q_2(t) = \psi(t) + (\xi - c)Q_1(t) - Q_1'(t), \tag{2}$$

where $Q_1(t)$ is a continuous and differentiable function on $\mathbb{R}$ and $\xi > 1$ is a constant, Peng and Wang [13] transformed (1) into the following system of differential equations

$$\begin{cases} u'(t) = -\xi u(t) + v(t) + Q_1(t), \\ v'(t) = -(c-\xi)v(t) + (a + \xi(c-\xi))u(t) - bu^m(t - \phi(t)) + Q_2(t), \end{cases} \tag{3}$$

and then proved the existence of positive almost periodic solutions of (1), and (3). Xu [15] extended the system (3) to the following nonlinear Duffing system with time-varying coefficients and delay

$$\begin{cases} u'(t) = -\delta_1(t)u(t) + v(t) + Q_1(t), \\ v'(t) = \delta_2(t)v(t) + [\mu(t) - \delta_2^2(t)]u(t) - \nu(t)u^m(t - \phi(t)) + Q_2(t), \end{cases} \tag{4}$$

where $\mu(t), \nu(t), \phi(t), \delta_1(t), \delta_2(t), Q_1(t), Q_2(t)$ are all almost periodic functions on $\mathbb{R}$, m is an integer with $m > 1$, $\mu(t) > 0, \nu(t) \neq 0$, and established some sufficient conditions for the existence of almost periodic solutions of (4).

In this paper, we extend the systems (3) and (4) to the following Duffing system

$$\begin{cases} u'(t) = -f_1(t)u(t) + v(t) + F_1(t), \\ v'(t) = -f_2(t)v(t) + [\alpha(t) + f_2^2(t)]u(t) - \beta(t)u^m(t - \phi(t)) + F_2(t), \end{cases} \tag{5}$$

where $f_1(t)$ is a bounded continuous function on $\mathbb{R}$ with $\inf_{t\in\mathbb{R}} f_1(t) > 0$; $f_2(t)$, $\alpha(t)$, $\beta(t)$, $\phi(t)$ are almost periodic functions on $\mathbb{R}$; $F_1(t)$, $F_2(t)$ are Stepanov-almost periodic continuous functions on $\mathbb{R}$, and $\inf_{t\in\mathbb{R}} f_2(t) > 0$, $m > 1$ is an integer.

In [15, 16, 18], the authors considered the following almost periodic system

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}, \tag{6}$$

where $f : \mathbb{R} \to \mathbb{R}^n$ is an almost periodic function and $A(t)$ is an $n \times n$ almost periodic matrix defined on $\mathbb{R}$, to prove the existence of almost periodic solutions for a class of nonlinear Duffing systems. In this paper, we first study the existence of almost periodic solutions of (6) when $f : \mathbb{R} \to \mathbb{R}^n$ is a Stepanov-almost periodic continuous function and $A(t)$ is an $n \times n$ bounded continuous matrix defined on $\mathbb{R}$ satisfying some suitable conditions, and using these results we find sufficient conditions for the existence and uniqueness of almost periodic solution of (5). Finally, we provide an example to illustrate the results.

2 Preliminaries

In this section we give some basic definitions, notations, and results. In the rest of this paper $\mathbb{R}$ stands for a set of real numbers. We let $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ to denote a column vector, in which the symbol $()^T$ denotes the transpose of a vector, and if $x \in \mathbb{R}^n$, then we define $\|x\| = \max_{1\le i\le n} |x_i|$.

Definition 2.1 A continuous function $u : \mathbb{R} \to \mathbb{R}^n$ is said to be almost periodic if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\|u(t+\tau) - u(t)\| < \epsilon \quad \forall t \in \mathbb{R}.$$

Lemma 2.1 *If $u : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ are almost periodic functions, then $u(\cdot - g(\cdot)) : \mathbb{R} \to \mathbb{R}^n$ is also an almost periodic function.*

For a detailed proof of the above lemma see [7, Lemma 2.4].

Throughout the rest of the paper we fix $p, 1 \leq p < \infty$. Denote by $L^p_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ the space of all functions from $\mathbb{R}$ into $\mathbb{R}^n$ which are locally p-integrable in the Bochner-Lebesgue sense. We say that a function, $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ is p-Stepanov bounded (S^p-bounded) if

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s)\|^p ds \Big)^{1/p} < \infty.$$

We indicate by $L^p_s(\mathbb{R};\mathbb{R}^n)$ the set of all S^p-bounded functions $\mathbb{R}$ into $\mathbb{R}^n$.

Definition 2.2 A function $f \in L^p_s(\mathbb{R};\mathbb{R}^n)$ is said to be almost periodic in the sense of Stepanov (S^p-almost periodic) if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s+\tau) - f(s)\|^p ds \Big)^{1/p} < \epsilon.$$

Lemma 2.2 ([12]) *Let $A(t) = (a_{ij})$ be an $n \times n$ continuous matrix defined on $\mathbb{R}$. If*

(i) *$A(t)$ is bounded,*

(ii) $|\det A(t)| \geq \kappa$ *on $\mathbb{R}$ for some $\kappa > 0$,*

(iii) $a_{ii}(t) \leq 0$ *for $i = 1, 2, \cdots, n$ and for all $t \in \mathbb{R}$,*

(iv) $|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ji}|$ *for all $i = 1, 2, \cdots, n$ and for all $t \in \mathbb{R}$,*

then there exist positive constants M, γ, and the fundamental solution matrix $X(t)$ of the linear system

$$x'(t) = A(t)x(t), \quad t \in \mathbb{R}, \tag{7}$$

satisfying

$$\|X(t)X^{-1}(s)\| \leq Me^{-\gamma(t-s)} \quad \text{for} \quad t \geq s. \tag{8}$$

In the rest of the paper, we assume that $A(t)$ satisfies all conditions given in Lemma 2.2.

Lemma 2.3 ([8]) *Let $f : \mathbb{R} \to \mathbb{R}^n$ be a continuous function. Then the solution $x : \mathbb{R} \to \mathbb{R}^n$ of (6) is given by*

$$x(t) = X(t)X^{-1}(a)x(a) + \int_a^t X(t)X^{-1}(s)f(s)ds, \quad t \geq a, \ a \in \mathbb{R}. \tag{9}$$

Lemma 2.4 *If $f : \mathbb{R} \to \mathbb{R}^n$ is an S^p-almost periodic continuous function, then the function $\Lambda : \mathbb{R} \to \mathbb{R}^n$ defined by*

$$\Lambda(t) = \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds, \quad t \in \mathbb{R},$$

is an almost periodic function.

Proof. We consider

$$\Lambda_k(t) := \int_{t-k}^{t-k+1} X(t)X^{-1}(s)f(s)ds, \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

Then

$$\begin{aligned}
\|\Lambda_k(t)\| &\le \int_{t-k}^{t-k+1} \|X(t)X^{-1}(s)f(s)\|ds \\
&\le \int_{t-k}^{t-k+1} \|X(t)X^{-1}(s)\|\|f(s)\|ds \\
&\le M \int_{t-k}^{t-k+1} e^{-\gamma(t-s)}\|f(s)\|ds \\
&\le M\Big(\int_{t-k}^{t-k+1} e^{-q\gamma(t-s)}ds\Big)^{1/q}\Big(\int_{t-k}^{t-k+1} \|f(s)\|^p ds\Big)^{1/p} \\
&= M\frac{e^{-\gamma k}\sqrt[q]{e^{\gamma q}-1}}{\sqrt[q]{q\gamma}}\|f\|_{S^p}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} e^{-\gamma k}$ is convergent, from the Weierstrass test it follows that the sequence of functions $\sum_{k=1}^{n} \Lambda_k(t)$ is uniformly convergent on $\mathbb{R}$. Thus we have

$$\Lambda(t) = \sum_{k=1}^{\infty} \Lambda_k(t).$$

Let $\epsilon > 0$. Then there exists a number $l > 0$ such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_{t}^{t+1} \|f(s+\tau) - f(s)\|^p ds\Big)^{1/p} \le \epsilon_1,$$

where

$$0 < \epsilon_1 < \frac{\epsilon\sqrt[q]{q\gamma}}{M(e^{\gamma}-1)(\sqrt[q]{e^{q\gamma}-1})}.$$

Now, we consider

$$
\begin{aligned}
&\|\Lambda_k(s+\tau)-\Lambda_k(s)\| \\
&= \Big\| \int_{s+\tau-k}^{s+\tau-k+1} X(s+\tau)X^{-1}(z)f(z)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz \Big\| \\
&= \Big\| \int_{s-k}^{s-k+1} X(s+\tau)X^{-1}(z+\tau)f(z+\tau)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz \Big\| \\
&= \Big\| \int_{s-k}^{s-k+1} X(s)X(\tau)X^{-1}(\tau)X^{-1}(z)f(z+\tau)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz \Big\| \\
&= \Big\| \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z+\tau)dz - \int_{s-k}^{s-k+1} X(s)X^{-1}(z)f(z)dz \Big\| \\
&\le \int_{s-k}^{s-k+1} \|X(s)X^{-1}(z)\|\|f(\tau+z)-f(z)\|dz \\
&\le M \int_{s-k}^{s-k+1} e^{-\gamma(s-z)}\|f(\tau+z)-f(z)\|dz \\
&\le M\Big(\int_{s-k}^{s-k+1} e^{-q\gamma(s-z)}dz\Big)^{1/q}\Big(\int_{s-k}^{s-k+1} \|f(z+\tau)-f(z)\|^p dz\Big)^{1/p} \\
&\le \epsilon_1 M\Big(\int_{s-k}^{s-k+1} e^{-q\gamma(s-z)}dz\Big)^{1/q} = \frac{\epsilon_1 M e^{-\gamma k}(\sqrt[q]{e^{q\gamma}-1})}{\sqrt[q]{q\gamma}} < \epsilon.
\end{aligned}
$$

Therefore,

$$\sum_{k=1}^{\infty}\|\Lambda_k(s+\tau)-\Lambda_k(s)\| \le \frac{\epsilon_1 M(\sqrt[q]{e^{q\gamma}-1})}{\sqrt[q]{q\gamma}}\sum_{k=1}^{\infty} e^{-\gamma k} < \epsilon.$$

Hence, $\Lambda(t)$ is an almost periodic function.

Lemma 2.5 *If $f:\mathbb{R}\to\mathbb{R}^n$ is an S^p-almost periodic continuous function, then the system* (6) *has an almost periodic solution $x:\mathbb{R}\to\mathbb{R}^n$ if and only if*

$$x(t) = \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds, \quad t\in\mathbb{R}. \tag{10}$$

Moreover, the system (6) *has a unique almost periodic solution.*

Proof. Let $x:\mathbb{R}\to\mathbb{R}^n$ be an almost periodic solution of (6). Then

$$x(t) = X(t)X^{-1}(a)x(a) + \int_a^t X(t)X^{-1}(s)f(s)ds, \quad t\ge a,\ a\in\mathbb{R}.$$

For $t\ge a$, we have

$$
\begin{aligned}
\|X(t)X^{-1}(a)x(a)\| &\le Me^{-\gamma(t-a)}\|x(a)\| \\
&\le Me^{-\gamma(t-a)}\sup_{t\in\mathbb{R}}\|x(t)\|.
\end{aligned}
$$

Therefore,

$$\lim_{a\to-\infty}\|X(t)X^{-1}(a)x(a)\| = 0.$$

Hence,

$$x(t) = \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds, \quad t \in \mathbb{R}.$$

Conversely, let $x : \mathbb{R} \to \mathbb{R}^n$ be a function satisfying the integral representation (10). Then, by Lemma 2.4 , x is almost periodic. For $t \geq a$, we have

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} X(t)X^{-1}(s)f(s)ds \\ &= \int_{-\infty}^{a} X(t)X^{-1}(s)f(s)ds + \int_{a}^{t} X(t)X^{-1}(s)f(s)ds \\ &= X(t)X^{-1}(a)\int_{-\infty}^{a} X(a)X^{-1}(s)f(s)ds + \int_{a}^{t} X(t)X^{-1}(s)f(s)ds \\ &= X(t)X^{-1}(a)x(a) + \int_{a}^{t} X(t)X^{-1}(s)f(s)ds. \end{aligned}$$

Hence x is an almost periodic solution of (6). Suppose x and y are two almost periodic solutions of (6), then $u = x - y$ is an almost periodic solution of (7), hence $u = 0$. Thus (6) has a unique almost periodic solution.

3 Main Result

In this section, we prove the existence of almost periodic solution of (5).
Consider the following assumptions:

(H1) $m > 1$ is an integer, $1 \leq p < \infty$ and q is the conjugate index of p.

(H2) $f_1(t)$ is a bounded continuous function defined from $\mathbb{R}$ into $\mathbb{R}$, and $\inf_{t\in\mathbb{R}} f_1(t) > 0$.

(H3) $f_2(t),\ \alpha(t),\ \beta(t),\ \phi(t)$ are all almost periodic functions defined from $\mathbb{R}$ into $\mathbb{R}$, and $\inf_{t\in\mathbb{R}} f_2(t) > 0$.

(H4) $F_1(t),\ F_2(t)$ are S^p-almost periodic continuous functions defined from $\mathbb{R}$ into $\mathbb{R}$.

Consider the following notations:

$$\begin{aligned} &\delta_1 = \inf_{t\in\mathbb{R}} f_1(t),\ \delta_2 = \inf_{t\in\mathbb{R}} f_2(t),\ \delta = \min\{\delta_1, \delta_2\}, \\ &\theta = \max\{\frac{1}{\delta},\ \frac{\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|)}{\delta}\},\ \sigma = \sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + m|\beta(t)|), \\ &\lambda = \frac{\sqrt[q]{(e^{q\delta} - 1)}}{\sqrt[q]{q\delta}(e^{\delta} - 1)} \max\{\|F_1\|_{S^p}, \|F_2\|_{S^p}\}. \end{aligned}$$

We indicate by E the set of all functions of the form $\varphi(t) = (\varphi_1(t), \varphi_2(t))^T$, where $\varphi_1(t), \varphi_2(t)$ are almost periodic functions defined from $\mathbb{R}$ into $\mathbb{R}$. Then E forms a Banach space with respect to the norm $\|\cdot\|_E$ given by

$$\|\varphi\|_E = \max\Big\{\sup_{t\in\mathbb{R}} |\varphi_1(t)|, \sup_{t\in\mathbb{R}} |\varphi_2(t)|\Big\}.$$

Theorem 3.1 *Suppose the assumptions (H1)-(H4) hold, and the positive constants λ, θ and σ satisfy*

$$\max\{1, \sigma\} < \delta, \ \frac{\lambda}{1-\theta} < 1, \tag{11}$$

then there exists a unique almost periodic solution of system (5) *in the region*

$$E^* = \big\{\varphi \in E : \|\varphi - \varphi_0\|_E \leq \frac{\theta\lambda}{1-\theta}\big\},$$

where

$$\varphi_0(t) = \left(\int_{-\infty}^{t} e^{-\int_s^t f_1(z)dz} F_1(s)ds, \int_{-\infty}^{t} e^{-\int_s^t f_2(z)dz} F_2(s)ds\right)^T.$$

Proof. Since $m > 1$, we have

$$\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|) \leq \sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + m|\beta(t)|),$$

hence $\theta \leq \max\{\frac{1}{\delta}, \frac{\sigma}{\delta}\} < 1$. Clearly, E^* is a closed convex subset of E. We consider

$$\begin{aligned}
\|\varphi_0\|_E &= \max\{\sup_{t\in\mathbb{R}} |\int_{-\infty}^{t} e^{-\int_s^t f_1(z)dz} F_1(s)ds|, \sup_{t\in\mathbb{R}} |\int_{-\infty}^{t} e^{-\int_s^t f_2(z)dz} F_2(s)ds|\} \\
&\leq \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t f_1(z)dz} |F_1(s)|ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t f_2(z)dz} |F_2(s)|ds\} \\
&\leq \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \delta dz} |F_1(s)|ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \delta dz} |F_2(s)|ds\} \\
&\leq \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} |F_1(s)|ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} |F_2(s)|ds\} \\
&= \max\{\sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} e^{-\delta(t-s)} |F_1(s)|ds, \\
&\qquad \sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} e^{-\delta(t-s)} |F_2(s)|ds\} \\
&\leq \max\{\sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} (\int_{t-k}^{t-k+1} e^{-q\delta(t-s)} ds)^{1/q} \|F_1\|_{S^p}, \\
&\qquad \sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} (e^{-q\delta(t-s)} ds)^{1/q} \|F_2\|_{S^p}\} \\
&= \max\{\frac{\sqrt[q]{(e^{q\delta}-1)}}{\sqrt[q]{q\delta}(e^{\delta}-1)} \|F_1\|_{S^p}, \frac{\sqrt[q]{(e^{q\delta}-1)}}{\sqrt[q]{q\delta}(e^{\delta}-1)} \|F_2\|_{S^p}\} \\
&= \frac{\sqrt[q]{(e^{q\delta}-1)}}{\sqrt[q]{q\delta}(e^{\delta}-1)} \max\{\|F_1\|_{S^p}, \|F_2\|_{S^p}\} = \lambda.
\end{aligned}$$

Therefore, for any $\varphi \in E^*$, we get

$$\|\varphi\|_E \leq \|\varphi - \varphi_0\|_E + \|\varphi_0\|_E \leq \frac{\theta\lambda}{1-\theta} + \lambda = \frac{\lambda}{1-\theta} < 1. \tag{12}$$

Now, let $\varphi \in E$, and consider the following nonlinear system

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} -f_1(t) & 0 \\ 0 & -f_2(t) \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} \varphi_2(t) + F_1(t) \\ \tilde{\varphi}_1(t) \end{pmatrix}, \tag{13}$$

where $\tilde{\varphi}_1(t) = (\alpha(t) + f_2^2(t))\varphi_1(t) - \beta(t)\varphi_1^m(t - \phi(t)) + F_2(t)$. By Lemma 2.1, $\varphi_1(t - \phi(t))$ is almost periodic, and hence $\tilde{\varphi}_1(t)$ is a Stepanov-almost periodic function. Since the matrix $\begin{pmatrix} -f_1(t) & 0 \\ 0 & -f_2(t) \end{pmatrix}$ satisfies all conditions given in Lemma 2.2, by Lemma 2.5 the system (13) has a unique almost periodic solution and is given by

$$\begin{pmatrix} u^\varphi(t) \\ v^\varphi(t) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{-\infty}^t e^{-\int_s^t f_1(z)dz}[\varphi_2(s) + F_1(s)]ds \\ \displaystyle\int_{-\infty}^t e^{-\int_s^t f_2(z)dz}\tilde{\varphi}_1(s)ds \end{pmatrix}.$$

Therefore, for each $\varphi \in E$, (13) has a unique almost periodic solution $\begin{pmatrix} u^\varphi(t) \\ v^\varphi(t) \end{pmatrix}$.

Define a map $T : E \to E$ by

$$T(\varphi)(t) = \begin{pmatrix} u^\varphi(t) \\ v^\varphi(t) \end{pmatrix}.$$

To prove T is a self-mapping from E^* into E^*, we consider, for any $\varphi \in E^*$,

$$\begin{aligned}
&\|T\varphi - \varphi_0\|_E \\
&= \max\{\sup_{t\in\mathbb{R}} |\int_{-\infty}^t e^{-\int_s^t f_1(z)dz}\varphi_2(s)ds|, \\
&\quad \sup_{t\in\mathbb{R}} |\int_{-\infty}^t e^{-\int_s^t f_2(z)dz}(\alpha(s) + f_2^2(s))\varphi_1(s) - \beta(s)\varphi_1^m(t - \phi(s))ds|\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\delta_1(t-s)}\|\varphi\|_E ds, \\
&\quad \sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\delta_2(t-s)}[(|\alpha(s) + f_2^2(s)|)\|\varphi\|_E + |\beta(s)|\|\varphi\|_E^m]ds\} \\
&\le \max\{\sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\delta_1(t-s)}ds, \sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\delta_2(t-s)}(|\alpha(s) + f_2^2(s)| + |\beta(s)|)ds\}\|\varphi\|_E \\
&\le \max\{\frac{1}{\delta_1}, \frac{\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|)}{\delta_2}\}\|\varphi\|_E \\
&\le \max\{\frac{1}{\delta}, \frac{\sup_{t\in\mathbb{R}}(|\alpha(t) + f_2^2(t)| + |\beta(t)|)}{\delta}\}\|\varphi\|_E \\
&= \theta\|\varphi\|_E \le \frac{\theta\lambda}{1-\theta}.
\end{aligned}$$

Therefore, T maps E^* into itself. Next, we prove that T is a contraction mapping from E^* into itself. For $\varphi, \psi \in E^*$, we have

$$\begin{aligned}
&\|T(\varphi)-T(\psi)\|_E \\
&= \max\{\sup_{t\in\mathbb{R}}|\int_{-\infty}^{t} e^{-\int_s^t f_1(z)dz}[\varphi_2(s)-\psi_2(s)]ds|, \sup_{t\in\mathbb{R}}|\int_{-\infty}^{t} e^{-\int_s^t f_2(z)dz}[(\alpha(s) \\
&\qquad + f_2^2(s))(\varphi_1(s)-\psi_1(s)) - \beta(s)(\varphi_1^m(s-\phi(s)) - \psi_1^m(s-\phi(s)))]ds|\} \\
&\le \max\{\sup_{t\in\mathbb{R}}|\int_{-\infty}^{t} e^{-\int_s^t \delta_1 dz}\|\varphi-\psi\|_E ds, \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t \delta_2 dz}[|\alpha(s)+f_2^2(s)|\times \\
&\qquad |\varphi_1(s)-\psi_1(s)| + |\beta(s)||\varphi_1^m(s-\phi(s)) - \psi_1^m(s-\phi(s))|]ds\} \\
&\le \max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E, \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\delta_2(t-s)}[|\alpha(s)+f_2^2(s)|\|\varphi-\psi\|_E + |\beta(s)|\times \\
&\quad (|\varphi_1(s-\phi(s))-\psi_1(s-\phi(s))|(|\varphi_1^{m-1}(s-\phi(s))| + |\varphi_1^{m-2}(s-\phi(s))|\times \\
&\quad |\psi_1(s-\phi(s))| + \cdots + |\varphi_1(s-\phi(s))||\psi_1^{m-2}(s-\phi(s))| + |\psi_1^{m-1}(s-\phi(s))|))]ds\} \\
&\le \max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E, \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\delta_2(t-s)}[|\alpha(s)+f_2^2(s)|\|\varphi-\psi\|_E + |\beta(s)|\times \\
&\quad \|\varphi-\psi\|_E(\|\varphi\|_E^{m-1} + \|\varphi\|_E^{m-2}\|\psi\|_E + \cdots + \|\varphi\|_E\|\psi\|_E^{m-2} + \|\psi\|_E^{m-1})]ds\} \\
&\le \max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E, \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\delta_2(t-s)}[|\alpha(s)+f_2^2(s)|\|\varphi-\psi\|_E \\
&\qquad\qquad + m|\beta(s)|\|\varphi-\psi\|_E]ds\} \\
&\le \max\{\frac{1}{\delta_1}\|\varphi-\psi\|_E, \frac{1}{\delta_2}\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)| + m|\beta(t)|)\|\varphi-\psi\|_E\} \\
&\le \max\{\frac{1}{\delta}\|\varphi-\psi\|_E, \frac{1}{\delta}\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)| + m|\beta(t)|)\|\varphi-\psi\|_E\} \\
&\le \max\{\frac{1}{\delta}, \frac{1}{\delta}\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)| + m|\beta(t)|)\}\|\varphi-\psi\|_E \\
&= \max\{\frac{1}{\delta}, \frac{\sigma}{\delta}\}\|\varphi-\psi\|_E.
\end{aligned}$$

Since $\max\{\frac{1}{\delta}, \frac{\sigma}{\delta}\} < 1$, T is a contraction map on E^*. Therefore, T has a unique fixed point $\varphi^*(t) = (u^*(t), v^*(t))^T \in E^*$, i.e., $T\varphi^* = \varphi^*$. By (13), φ^* satisfies (5), hence φ^* is an almost periodic solution of the system (5) in E^*.

4 Application

Example 4.1 The following nonlinear Duffing equation with time-varying coefficients

$$\begin{aligned}
&u''(t) + (20+\sin t+\sin t^2)u'(t) + (100.5 + 10\sin t - \sin^2 t + \sin t\sin t^2 + 10\sin t^2 \\
&\qquad + 2t\cos t^2)u(t) + 0.5\cos t(u^3(t-\sin t) - \sin t - 10) - 0.5\sin\sqrt{2}t = 0 \qquad (14)
\end{aligned}$$

has at least one almost periodic solution.

Proof. Consider

$$v(t) = u'(t) + (10+\sin t^2)u(t) - 0.5\cos t, \qquad (15)$$

then we can transform (14) into the following system of differential equations

$$\begin{cases} u'(t) = -(10+\sin t^2)u(t) + v(t) + 0.5\cos t, \\ v'(t) = -(10+\sin t)v(t) + (\sin^2 t - 0.5)u(t) - 0.5\cos t u^3(t-\sin t) \\ \qquad\quad +0.5\sin t + 0.5\sin\sqrt{2}t. \end{cases} \tag{16}$$

Since $f_1(t) = 10+\sin t^2$, $f_2(t) = 10+\sin t$, $\alpha(t) = -100.5 - 20\sin t$, $\beta(t) = 0.5\cos t$, $F_1(t) = 0.5\cos t$, $F_2(t) = 0.5\sin t + 0.5\sin\sqrt{2}t$, $\phi(t) = \sin t$, $p = 2$, $m = 3$, we have $\delta_1 = \delta_2 = \delta = 9$, $q = 2$,

$$\begin{aligned} \theta &= \max\{\frac{1}{\delta}, \frac{\sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)| + |\beta(t)|)}{\delta}\} \\ &= \max\{\frac{1}{9}, \frac{\sup_{t\in\mathbb{R}}(|-0.5+\sin^2 t| + |0.5\cos t|)}{9}\} \\ &= \max\{\frac{1}{9}, \frac{1}{9}\} = \frac{1}{9}. \end{aligned}$$

Since

$$0 < \|F_1\|_{S^2} \le \sup_{t\in\mathbb{R}} |0.5\cos t| = 0.5 \quad \text{and}$$

$$0 < \|F_2\|_{S^2} \le \sup_{t\in\mathbb{R}}(|0.5\sin t + 0.5\sin\sqrt{2}t|) = 1,$$

$$\begin{aligned} 0 < \lambda &= \frac{\sqrt[q]{(e^{q\delta}-1)}}{\sqrt[q]{q\delta}(e^{\delta}-1)} \max\{\|F_1\|_{S^p}, \|F_2\|_{S^p}\} \\ &= \frac{\sqrt{(e^{18}-1)}}{\sqrt{18}(e^9-1)} \max\{\|F_1\|_{S^2}, \|F_2\|_{S^2}\} \le \frac{\sqrt{(e^{18}-1)}}{\sqrt{18}(e^9-1)} \\ \Longrightarrow \lambda^2 &\le \frac{(e^{18}-1)}{18(e^9-1)^2} = \frac{1}{18} + \frac{1}{9(e^9-1)} < \frac{3}{18} \\ \Longrightarrow \lambda &< \sqrt{\frac{3}{18}} < \frac{8}{9} = 1-\theta \Longrightarrow \frac{\lambda}{1-\theta} < 1. \end{aligned}$$

$$\sigma = \sup_{t\in\mathbb{R}}(|\alpha(t)+f_2^2(t)| + m|\beta(t)|) = \sup_{t\in\mathbb{R}}(|-0.5+\sin^2 t| + 3|0.5\cos t|) = 2 < 9 = \delta.$$

Therefore, all the assumptions given in Theorem 3.1 are satisfied, hence (16) has at least one almost periodic solution. Thus, the nonlinear Duffing equation (14) has at least one almost periodic solution.

Remark 4.1 Notice that the function $f_1(t) = 10 + \sin t^2$ is not almost periodic and the coefficient of $u(t)$ in (14) is unbounded. Thus, the results of this paper are substantially extended and improved the main results of [13, 15, 16, 18].

5 Conclusion

In this paper, we considered a class of nonlinear Duffing system (5) with time-varying coefficients and Stepanov-almost periodic forcing terms. We first considered the almost periodic system (6) with a Stepanov-almost periodic continuous forcing function and then studied the existence of almost periodic solutions of (6). Using these results, we established some sufficient conditions for the existence and uniqueness of an almost periodic solution of the system (5). Finally, we provided an example to illustrate the main result.

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