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Conform Fractional Semi-Dynamical Systems

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Abstract: The aim of this work is to present the notion of a conform semi-dynamical system, unlike the concept of a dynamical system, here we can work with the continuous functions. Some examples are presented to illustrate the result of the autonomous case.

Keywords: *conform dynamical systems; orbit; omega-set limit; autonomous system.*

Mathematics Subject Classification (2010): 37-XX, 37Cxx, 34Cxx, 34Dxx.

1 Introduction

Fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, starting from some speculations of G.W. Leibeniz (1967) and L. Euler (1730) and since then, it has continued to be developed up to nowadays. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true for problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering. We recall that the fractional partial derivatives are difficult to handle analytically, especially those describing real world processes, and the researchers sometimes have to rely on the numerical methods to solve these equations. One of the well-known fractional derivatives is the Riemann-Liouville fractional order derivative, which is not always appropriate for modeling real world problems. The second one is the so-called Caputo derivative, this one is opposite with relation to displaying physical field complications and has been intensively used for this purpose.

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However, new derivatives should be proposed in order to deal better with the dynamics of the complex systems [9, 11]. In [8] the authors give a new definition of fractional derivative and fractional integral. The form of the definition shows that it is the most natural definition, and the most fruitful one. The definition for $0 < \alpha < 1$ coincides with the classical definitions on polynomials (up to a constant). Further, if $\alpha = 1$, the definition coincides with the classical definition of the first derivative. They presented some applications to fractional differential equations. A. Atanagana in [3] investigated in more detail some new properties of the conform derivative and has proved some useful related theorems. Dynamical systems as a generalization of solutions of ordinary differential equations are already a classical subject in the mathematical literature. Its systematic generalization to systems with nonunique solutions was developed by Barbashin [7]. We note that this study depends on the nature of derivative, our objective is to give the analogue with the conform derivative, in order to weaken the hypothesis of class C functions in continuous functions.

The paper is organized as follows. After this introductory section, we will present and demonstrate some properties concerning the conform derivative in Section 2. The definitions of α-semi-dynamical system, orbit and omega-set and their properties are given in Section 3. The last Section 4 contains qualitative studies of autonomous system in dimension 2.

2 Conform Fractional Derivative

In this section, we will give some definitions and properties concerning the new derivative important in the following.

Definition 2.1 (see [8]) Let $\alpha \in (n, n+1]$ and $f : [0, \infty) \to \mathbb{R}$ be n-differentiable at $t > 0$, then the conformable fractional derivative of f of order α is defined by

$$\left\{ f^{(\alpha)}(t) = \lim_{\epsilon\to 0} \frac{f^{(n)}(t+\epsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\epsilon}, f^{(\alpha)}(0) = \lim_{t\to 0} f^{(\alpha)}(t). \right.$$

Remark 2.1 (see [8]) As consequence of the previous definition, one can easily show that

$$f^{(\alpha)}(t) = t^{n+1-\alpha} f^{(n+1)}(t),$$

where $\alpha \in (n, n+1]$, and f is $(n+1)$-differentiable at $t > 0$.

In [3] we find the following proposition.

Proposition 2.1 *[3] We have the following properties:*

1. $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$,
2. $(fg)^{(\alpha)} = f^{(\alpha)}g + fg^{(\alpha)}$,
3. $(t^p)^{(\alpha)} = pt^{p-\alpha}$,
4. $\left(\frac{f}{g}\right)^{(\alpha)} = \frac{f^{(\alpha)}g - fg^{(\alpha)}}{g^2}$,
5. *If* $c \in \mathbb{R}$, $c^{(\alpha)} = 0$.

Proposition 2.2 *If x is a continuous map, then $t \to x^{(\alpha)}(t)$ is a continuous map.*

Proof. Since x is a continuous map, $t \in \mathbb{R}_+^* \to x\left(t+\epsilon t^{1-\alpha}\right)$ is continuous, thus $\forall \beta > 0$, $\exists \alpha > 0$,

$$|\frac{x\left(t+\epsilon t^{1-\alpha}\right) - x\left(t_0+\epsilon t_0^{1-\alpha}\right)}{\epsilon}| \leq \beta,$$

whenever $|t - t_0| \leq \alpha$, by passing to the limit $\epsilon \to 0$ we get $|x^{(\alpha)}(t) - x^{(\alpha)}(t_0)| \leq \beta$ as desired.

Proposition 2.3 *Let $f : X \to X$ be a Lipschitziane map, i.e., $|f(x) - f(y)| \leq k|x - y|$, $\forall x, y \in X$ and $k \in]0, 1[$. The Cauchy problem*

$$\begin{cases} x^{(\alpha)}(t) = f\left(x(t)\right), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

has a unique solution.

Proof. By Proposition 2.2 x is continuous, the sequence $x_{x+1} = f(x_n)$ is a Cauchy sequence, since $\mathbb{R}$ is a complete space, then x_n converges to the unique solution of (1).

Definition 2.2 (see [8]) Let $\alpha \in (1, 2]$, $(I^\alpha f)(t) = \int_0^t s^{\alpha-2} f(s)ds$.

Theorem 2.1 *(see [8]) $\left(I^\alpha f\right)^{(\alpha)}(t) = f(t)$ for $t \geq 0$.*

Example 2.1

$$I^\alpha(\sin(t)) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\alpha}}{(2n+\alpha)(2n+1)!}$$

where $\alpha \in (1, 2)$.

Definition 2.3 (see [6]) Let $\alpha > 0$. For a Banach space X, a family $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X, X)$ is called a fractional α-semigroup if

1. $T(0) = I$.
2. $T\left((s+t)^{\frac{1}{\alpha}}\right) = T\left(s^{\frac{1}{\alpha}}\right) T\left(t^{\frac{1}{\alpha}}\right)$, for all $s, t \in [0, \infty)$.

Example 2.2 Let A be a bounded linear operator on X. Define $T(t) = e^{2\sqrt{t}A}$. Then $T(t)_{t\geq 0}$ is a $\frac{1}{2}$ semigroup. Indeed,

1. $T(0) = e^{0A} = I$.
2. $\forall s, t \in [0, \infty)$, $T\left((s+t)^2\right) = e^{2(t+s)A} = e^{2tA}e^{2sA} = T(s)T(t)$.

Definition 2.4 (see [6]) An α-semigroup $T(t)$ is called a c_0-semigroup if, for each fixed $x \in X$, $T(t)x \to x$ as $t \to 0^+$.

The conformable α-derivative of $T(t)$ at $t = 0$ is called the α-infinitesimal generator of the fractional α-semigroup $T(t)$, with the domain equal to $\left\{x \in X : \lim\limits_{t\to 0} T(t)x \text{ exists }\right\}$.

3 Conform Fractional Dynamical Systems

3.1 Definition and examples

In this subsection we will introduce the notion of α-dynamical system.

Definition 3.1 Let X be a complete metric space. An α-semi-dynamical system is a couple (X, π_α), where X state space of the system. Each point of X is a state of the system, and $\pi_\alpha : \mathbb{R}_+ \times X \longrightarrow X$ satisfies

$$\begin{cases} \pi_\alpha(0, x) = x, \quad \forall x \in X, \\ \pi_\alpha\big(t, \pi_\alpha(s, x)\big) = \pi_\alpha(t + s, x), \quad \forall x \in X, t, s > 0. \end{cases}$$

Example 3.1 1. Here X is a Banach space. Let $f : X \to X$, be a Lipschitzian map, i.e., $|f(x) - f(y)| \leq k|x - y|$, $\forall x, y \in X$ and $k \in]0, 1[$. The Cauchy problem

$$\begin{cases} x^{(\alpha)}(t) = f\left(x(t)\right), \quad t > 0, \\ x(0) = x_0. \end{cases}$$

has unique solution. The mapping $\pi_\alpha(t, x_0) = x\left(t^{\frac{1}{\alpha}}\right)$ defines an α-semi-dynamical system.

2. Here $X = \mathcal{C}\big([0, 1], \mathbb{R}\big)$, X is a Banach space. For all $f \in X$, we define $\pi_\alpha(t, f)$ by

$$\pi_\alpha(t, f)(s) = f\big(\min\left((t^{\frac{1}{\alpha}} + s), 1\right)\big), \quad 0 \leq s \leq 1.$$

We will demonstrate that this corresponds well to the α-semi-dynamical system.

3.2 Orbit of α-semi-dynamical system

The term orbit generally refers to the image (in the state space) of a solution. It is defined as

$$\mathcal{O}(x) = \big\{\pi_\alpha\left(t, x\right), \ t \geq 0\big\}.$$

Definition 3.2 A point $x \in X$ is said to be a critical point if $\pi_\alpha\left(t, x\right) = x$.

Example 3.2 We take $x^{(\alpha)}(t) = x(t) - x^2(t)$, $x = 0$ and $x = 1$ are two critical points.

Definition 3.3 A point $x \in X$ is said to be a periodic point if there is $\tau > 0$ such that $\pi_\alpha\left(t + \tau, x\right) = \pi_\alpha\left(t, x\right)$.

Remark 3.1 $x \in X$ is a periodic point if and only if $\pi_\alpha\left(\tau, x\right) = x$.

Proposition 3.1 *Let x be a periodical point of π_α, of period τ. It goes through one and only one periodic solution, of period τ, defined on $\mathbb{R}$.*

Proof. The mapping defined by

$$u(t) = \begin{cases} \pi_\alpha(t, x), \forall t \geq 0, \\ \pi_\alpha(t + n\tau, x), \forall t \in [-n\tau, (-n + 1)\tau[, \ n \in \mathbb{N}, \end{cases}$$

is a periodic extension of $\pi_\alpha(t, x)$ on $\mathbb{R}$. We get $u(t + s) = \pi_\alpha(s, u(t))$, $\forall t, s \in \mathbb{R}$.

3.3 Omega-limit set

We will discuss some properties of the Omega-limit set related to the orbit. We will start by the following definition.

Definition 3.4 Let $x \in X$. The Omega-limit set, denoted $\omega(x)$, is defined as

$$\omega(x) = \left\{ y = \lim_{n\to\infty} \pi_\alpha(t_n, x) : \ t_n \to \infty, \text{ such that } \pi_\alpha(t_n, x) converge \right\}.$$

Remark 3.2 We can write

$$\omega(x) = \bigcap_{t>0} \overline{\pi_\alpha([t_n, \infty[, X)},$$

where $\overline{\pi_\alpha([t_n, \infty[, X)}$ is the closure of $\pi_\alpha([t_n, \infty[, X)$.

We will etablish several properties of the Omega-limit set.

Lemma 3.1 *$\omega(x)$ is closed, and satisfies*

$$y \in \omega(x) \Rightarrow \pi_\alpha(t, y) \in \omega(x).$$

Proof. It is closed because there is an intersection of closed parts.
For all $s \geq 0$, we have

$$\begin{aligned} \pi_\alpha\left(s, \omega(x)\right) &\subset \bigcap \pi_\alpha\Big(s, \overline{\pi_\alpha([t, \infty[, X)}\Big) \\ &\subset \bigcap \overline{\pi_\alpha([t, \infty[, X)} \\ &\subset \omega(x). \end{aligned}$$

Lemma 3.2 *If $\mathcal{O}(x)$ is precompact, then $\omega(x) \neq \emptyset$.*

Proof. The sets $\pi_\alpha([t, \infty[, X)$ are compacts, whose finite intersection can not be avoided, thus $\omega(x) \neq \emptyset$.

Proposition 3.2 *If $\mathcal{O}(x)$ is precompact, then $\omega(x)$ is a compact susbset, connected.*

Proof. By Proposition 3.2 then $\omega(x)$ is an intersection of compacts, thus it is a compact. It remains to demonstrate that it is connected.

Theorem 3.1 *If $\mathcal{O}(x)$ is precompact without double point (i.e., $\forall t_1 < t_2$, $\pi_\alpha(t_1, x) \neq \pi_\alpha(t_2, x)$), then $\omega(x) \setminus \mathcal{O}(x)$ is dense in $\omega(x)$.*

Proof. We have the following alternative: $\mathcal{O}(x) \cap \omega(x) = \emptyset$, in this case the result is clear, or $\mathcal{O}(x) \cap \omega(x) \neq \emptyset$, in this case we get $\pi_\alpha([\tau, \infty[, X) \subset \omega(x)$, for some $\tau > 0$. On the other hand we can write

$$\omega(x) \setminus \mathcal{O}(x) = \bigcap_n \omega(x) \setminus \pi_\alpha([0, n], x),$$

each $\omega(x) \setminus \pi_\alpha([0, n], x)$ is open in $\omega(x)$ which is compact, then it is complete. Using the Baire theorem we conclude that each $\omega(x) \setminus \pi_\alpha([0, n], x)$ is dense for all $n \in \mathbb{N}$.
Since for all $y \in \omega(x)$, and $\epsilon > 0$, there is $t > \max n, \tau$, such that $d\left(\pi_\alpha(t, x), y\right) \leq \epsilon$, it becomes that $\pi_\alpha(t, x) \in \omega(x) \setminus \prime(x)$, which proves the density of $\pi_\alpha([0, n], x)$ in $\omega(x)$.

4 Fractional Autonomous Differential Systems

The purpose of this section is to study the following system in dimension 2, first we begin with some notion

$$x^{(\alpha)}(t) = f\left(x(t)\right), \tag{2}$$

where $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, Ω is open and f is a Lipschitzian map on Ω and differentiable at 0.

4.1 Definitions and notations

Definition 4.1 An equilibrium point of (2) is a point x_0 such that $f(x_0) = 0$.

Remark 4.1 If x_0 is an equilibrum point of (2) then $t \to x_0$ is a solution of (2).

Definition 4.2 Let x_0 be an equilibrum point of (2). We say that:

1. x_0 is stable if for all $\epsilon > 0$ there is $\eta > 0$ such that, if x is a solution of (2) which for t_0 satisfies $|x(t_0) - x_0| < \eta$, we have
   - x is defined for all $t \geq t_0$,
   - $|x(t) - x_0| < \epsilon$ for all $t \geq t_0$.
2. x is not stable if x_0 is stable,
3. x is asymptotically stable if
   - x_0 is stable,
   - $\lim_{t\to\infty} x(t) = x_0$.

4.2 Qualitative study of linear systems in dimension 2

In this subsection we consider the following differential system:

$$x^{(\alpha)}(t) = Ax(t), \tag{3}$$

where $x : \mathbb{R} \to \mathbb{R}^2$ and A is a constant matrix in $\mathcal{M}_2(\mathbb{R})$.

Remark 4.2 $x_0 = 0$ is a stable point of (3).

Before we study the above mentioned system, let us first solve the following equation:

$$x^{(\alpha)}(t) = ax(t), \tag{4}$$

where $a \in \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}$.

We put $\phi_0(t) = e^{\frac{a}{\alpha}t^{\alpha}}$, it is clear that ϕ_0 is a solution of (4).
Let ϕ be another solution, we get

$$\left(\frac{\phi}{\phi_0}\right)^{(\alpha)}(t) = \frac{\phi^{(\alpha)}(t)\phi_0(t) - \phi(t)\phi_0^{(\alpha)}(t)}{\left(\phi_0(t)\right)^2} = \frac{a\phi(t)\phi_0(t) - a\phi(t)\phi_0(t)}{\left(\phi_0(t)\right)^2} = 0.$$

Then $\frac{\phi}{\phi_0}$ is constant, which implies that $\phi(t) = ce^{\frac{a}{\alpha}t^{\alpha}}$, where $c \in \mathbb{R}$. The trajectories of the system depend on the nature of the eigenvalues of the matrix.

Case 1. Let v_1 and v_2 be two eigenvectors of A associated, respectively, with two eigenvalues λ_1 and λ_2, note that (v_1, v_2) is a basis of $\mathbb{R}^2$. Let P be the transit matrix from the canonical basis to (v_1, v_2) and we put $x = Py$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$, we get $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. The system (4) is written as

$$y^{(\alpha)} = Dy,$$

where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. It becomes $\begin{cases} y_1(t) = y_1^0 e^{\frac{\lambda_1}{\alpha} t^\alpha}, \\ y_1(t) = y_2^0 e^{\frac{\lambda_2}{\alpha} t^\alpha}. \end{cases}$

Example 4.1

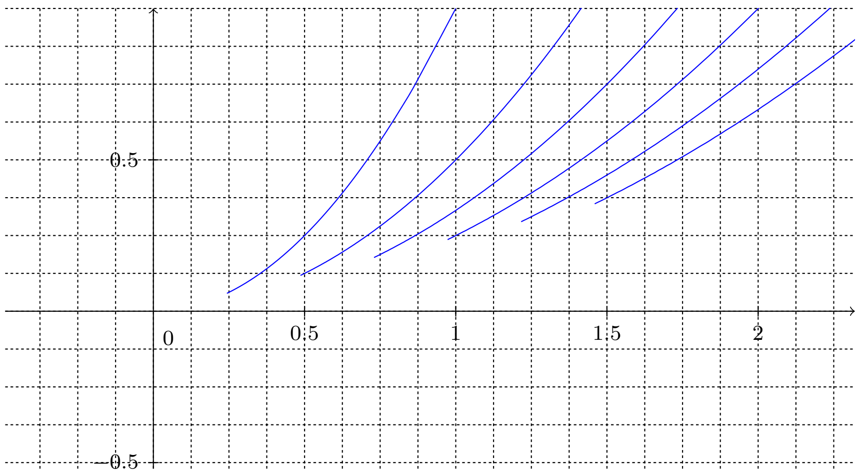

Figure 1. $\lambda_2 < \lambda_1 < 0$. The equilibrium point is asymptotically stable.

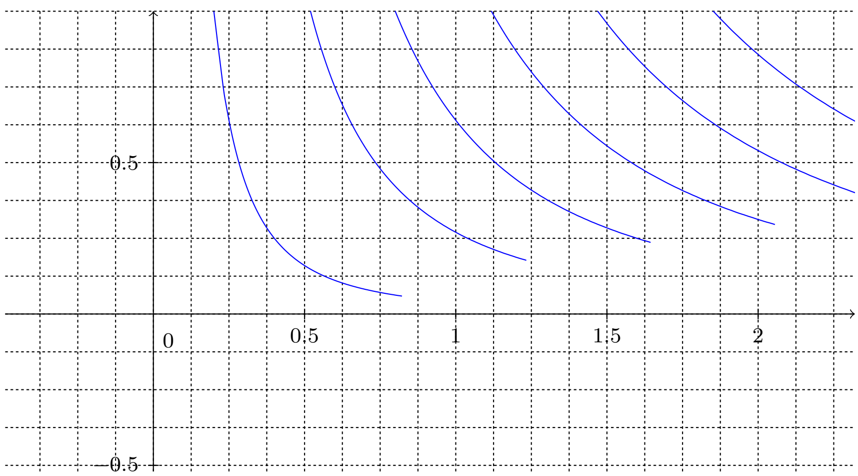

Figure 2. $\lambda_2 > \lambda_1 > 0$. The equilibrium point is not stable.

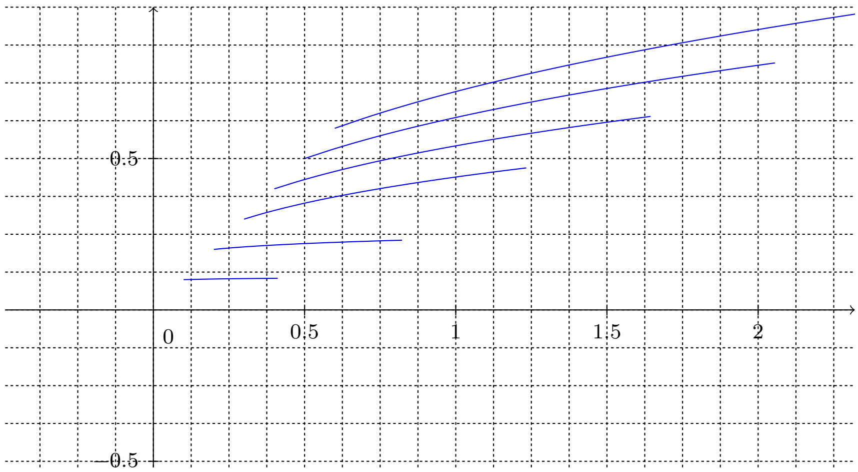

Figure 3. $\lambda_2 > 0 > \lambda_1$. Saddle point.

The trajectories described in Figures 1,2,3 are the curves given in parametric coordinates $y_1(t) = y_1^0 e^{\frac{\lambda_1}{\alpha} t^\alpha}$ and $y_1(t) = y_2^0 e^{\frac{\lambda_2}{\alpha} t^\alpha}$.

Case 2. Let $Z = u + iv$ be a complex eigenvector of A associated with the eigenvalue $\lambda = \eta - i\delta$. We have $Au = \eta u + \delta v$ and $Av = -\delta u + \eta v$. Thus (u, v) is a basis in which the matrix is writen as follows: $\begin{pmatrix} \alpha & -\delta \\ \delta & \alpha \end{pmatrix}$. If we denote by P the transit matrix from the canonical basis to (u, v), and we put $x = Py$, we get

$$\begin{cases} y_1(t) = Re^{\frac{\eta}{\alpha} t^\alpha} \cos\left(\frac{\delta}{\alpha} t^\alpha - \varphi\right), \\ y_2(t) = Re^{\frac{\eta}{\alpha} t^\alpha} \sin\left(\frac{\delta}{\alpha} t^\alpha - \varphi\right). \end{cases}$$

Example 4.2 In this example we take $\alpha = \frac{1}{2}$, $\delta = \frac{1}{2}$, $\varphi = 0$.

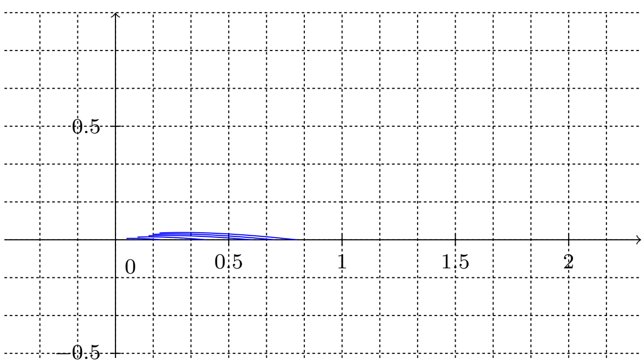

Figure 4: $\eta < 0$. The equilibrium point is stable.

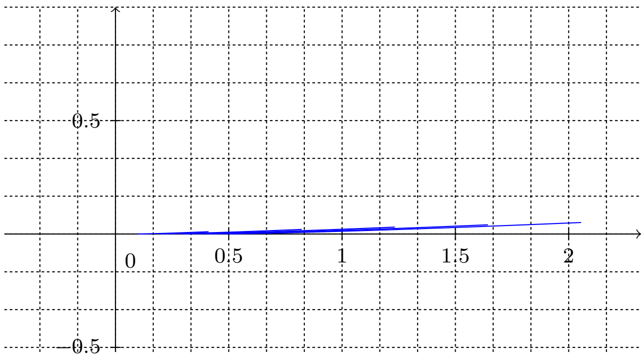

Figure 5: $\eta > 0$. The equilibrium point is not unstable.

Remark 4.3 If $\eta = 0$, then $y_1^2 + y_2^2 = R^2$.

5 Conclusion

This study is a basic idea for beginning the study of dynamical system in the conform frame.

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