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The Modified Fractional Power Series Method for Solving Fractional Undamped Duffing Equation with Cubic Nonlinearity

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Abstract: In this paper, the strongly nonlinear fractional undamped Duffing equation for undamped oscillators is studied. The physical and the mathematical model of nonlinear fractional Duffing equation for undamped oscillators is presented. The modified fractional power series (MFPS) method is employed to compute an approximation to the solution of this problem. The validity of the MFPS method is ascertained by comparing our results with numerical results and other methods in the literature. The results reveal that the proposed analytical method can achieve excellent results in predicting the solutions of such problems. The existence of the solution is proved. In addition, the convergence of the proposed method is investigated.

Keywords: *fractional Duffing equation; nonlinear boundary value problem; modified fractional power series method.*

Mathematics Subject Classification (2010): 76A05, 76W05, 76Z99, 65L05.

1 Introduction

In 1918, George Duffing presented the Duffing equation in his publication entitled "Erzwungene Schwingungen bei veranderlicher Eigenfrequenz und ihre technische Bedeuting". Duffing simplified the mathematical model of

$$x''(t) + a^2 x(t) - \beta x^2(t) - \gamma x^3(t) = k \sin \omega t \quad (1)$$

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and calculated the first term $H\ \sin\omega t$ of the periodic solution. Duffing considered the simplified version of equation (1) for describing the motion of the symmetrical pendulum of the form

$$x''(t) + \alpha\ x(t) - \gamma x^3(t) = 0 \tag{2}$$

and the unsymmetrical pendulum of the form

$$x''(t) + \alpha\ x(t) - \gamma x^2(t) = 0. \tag{3}$$

From that time, the differential equation with polynomial type of nonlinearity is called the Duffing equation. The nonlinear differential equation for the cubic free undamped Duffing oscillator of the form

$$x''(t) + \alpha\ x(t) + \beta x^3(t) = 0 \tag{4}$$

is subject to

$$x(0) = A, x'(0) = 0. \tag{5}$$

Many researchers discussed this problem numerically. He [3] used the homotopy perturbation method to solve the Duffing equation, while Belendez et al. [1] used the modified homotopy perturbation method. Ramos, Syam, Chhetri, Wazwaz used the variational iteration method to solve this problem [7, 9–13], while Ghosh et al. [2] used the Adomian decomposition method. In addition, Ramos [8] and Sabeg [14] used the artificial parameter decomposition and He's parameter expanding method, respectively.

Several analytical solutions for the Duffing problem were developed. For the small non-linearity, many analytical approaches were used to solve this problem, namely, the monotone method, the Krylov-Bogolubov method, the straightforward expansion, and the generalized Taylor power series method. For the case of strong cubic non-linearity, see [4, 5].

In this paper, we study the generalization of the problem (4)-(5) of the form

$$D^{2\alpha}x(t) + \beta x(t) + \gamma x^3(t) = 0,\ \frac{1}{2} < \alpha \leq 1,\ 0 < t < T \tag{6}$$

subject to

$$x(0) = A, D^{\alpha}x(0) = 0. \tag{7}$$

The derivative in Eq. (6) is in the Caputo derivative sense. We write the definition and some preliminary results of the Caputo fractional derivatives, as well as the definition of the fractional power series and one of its properties.

Definition 1.1 A real function $f(t), t > 0$,is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C^m_μ if $f^{(m)} \in C_\mu,\ m \in \mathbb{N}$.

Definition 1.2 For $\delta > 0,\ m-1 < \delta < m,\ m \in \mathbb{N}, t > 0$, and $f \in C^m_{-1}$, the left Caputo fractional derivative is defined by

$$D^\delta f(t) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \int_0^t (t-s)^{m-1-\delta} f^{(m)}(s)ds, & \delta > 0, \\ f'(t), & \delta = 0, \end{cases} \tag{8}$$

where Γ is the well-known Gamma function.

The Caputo fractional derivative satisfies the following properties for $\alpha > 0$, see [15].

1. $D^{\alpha}c = 0$, where c is constant,
2. $D^{\alpha}t^{\gamma} = \left\{ \begin{array}{cc} 0, & \gamma < \alpha, \gamma \in \{0,1,2,...\} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{otherwise} \end{array} \right\}.$

Next, we write the definition and one of the properties of the fractional power series which are used in this paper. More details can be found in [16].

Definition 1.3 A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + ...$$

is called a fractional power series FPS about $t = t_0$.

Suppose that f has a fractional FPS representation at $t = t_0$ of the form

$$g(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha}, \quad t_0 \le t < t_0 + \beta.$$

If $D^{m\alpha}g(t), \quad m = 0,1,2,..$ are continuous on $\mathbb{R}$, then $c_m = \frac{D^{m\alpha}g(t_0)}{\Gamma(1+m\alpha)}$.

We organize this paper as follows. In Section 2, we present a numerical technique for solving the second order nonlinear fractional boundary value problem using the MFPS method. Convergence of the presented method is given in this section. Some numerical results are presented in Section 3 to illustrate the efficiency of the presented method. Finally, we conclude with some comments and conclusions in Section 4.

2 MFPS Method for Solving Fractional Undamped Duffing Equation with Cubic Nonlinearity

In this section, we discuss how to solve the following class of second-order fractional undamped Duffing equations with cubic nonlinearity using the MFPS method:

$$D^{2\alpha}x(t) + \beta x(t) + \gamma x^3(t) = 0, \ \frac{1}{2} < \alpha \le 1, \ 0 < t < T \tag{9}$$

subject to

$$x(0) = A, D^{\alpha}x(0) = 0. \tag{10}$$

The MFPS method proposes the solution of the problem in the form of fractional power series as

$$x(t) = \sum_{n=0}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{11}$$

To obtain the approximate values of the above series (11), we consider its k-th truncated series $x_k(t)$ which has the form

$$x_k(t) = \sum_{n=0}^{k} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{12}$$

Since $x(0) = f_0 = A$ and $D^{\alpha}x(0) = f_1 = 0$, we rewrite Eq. (12) as

$$x_k(t) = A + \sum_{n=2}^{k} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \qquad k = 2, 3, ..., \tag{13}$$

where $x_1(t) = f_0 + f_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)} = A$ is considered to be the 1^{st} RPS approximate solution of $x(t)$. To find the values of the RPS-coefficients $f_n,\ n = 2, 3, 4, ...$, we solve the fractional differential equation

$$D^{(n-2)\alpha} Res_n(0) = 0,\ n = 2, 3, 4, ...,$$

where $Res_k(t)$ is the k-th residual function and is defined by

$$Res_k(t) = D^{2\alpha}x_k(t) + \beta x_k(t) + \gamma x_k^3(t). \tag{14}$$

To determine the coefficient f_2 in the expansion (12), we substitute the 2^{nd} RPS approximate solution

$$x_2(t) = A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

into Eq.(14) to get

$$\begin{aligned} Res_2(t) &= D^{2\alpha}x_2(t) + \beta x_2(t) + \gamma x_2^3(t) \\ &= f_2 + \beta\left(A + f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) + \gamma\left(A + f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^3. \end{aligned} \tag{15}$$

Then, we solve $Res_2(0) = 0$ to get

$$f_2 + \beta A + \gamma A^3 = 0 \tag{16}$$

or

$$f_2 = -\left(\beta A + \gamma A^3\right). \tag{17}$$

To find f_3, we substitute the 3^{rd} RPS approximate solution

$$x_3(t) = A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

into the 3^{rd} residual function $Res_3(t)$ such that

$$\begin{aligned} Res_3(t) &= D^{2\alpha}x_3(t) + \beta x_3(t) + \gamma x_3^3(t) \\ &= f_2 + f_3\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \beta\left(A + f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) \qquad (18) \\ &\quad + \gamma\left(A + f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^3. \qquad (19) \end{aligned}$$

Then, we solve $D^{\alpha}Res_3(0) = 0$ to get

$$f_3 = 0. \tag{20}$$

To find f_4, we substitute the 4^{th} RPS approximate solution

$$x_4(t) = A + f_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + f_4 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

into the 4^{th} residual function $Res_4(t)$ such that

$$\begin{aligned} Res_4(t) &= D^{2\alpha}x_4(t)+\beta x_4(t)+\gamma x_4^3(t) \\ &= f_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_4\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad +\beta\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right) \\ &\quad +\gamma\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right)^3. \end{aligned}$$

Then, we solve $D^{2\alpha}Res_4(0)=0$ to get

$$f_4+\beta f_2+3A^2\gamma f_2=0 \tag{21}$$

or

$$f_4 = \left(\beta+3A^2\gamma\right)\left(\beta A+\gamma A^3\right) \tag{22}$$

$$= \beta^2 A+\gamma\beta A^3+3A^3\gamma\beta+3A^5\gamma^2. \tag{23}$$

To find f_5, we substitute the 5^{th} RPS approximate solution

$$x_5(t)=A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}+f_5\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}$$

into the 5^{th} residual function $Res_5(t)$ such that

$$\begin{aligned} Res_5(t) &= D^{2\alpha}x_4(t)+\beta x_5(t)+\gamma x_5^3(t) \\ &= f_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_4\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_5\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &\quad +\beta\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}+f_5\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right) \\ &\quad +\gamma\left(A+f_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}+f_5\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right)^3. \end{aligned}$$

Then, we solve $D^{3\alpha}Res_5(0)=0$ to get

$$f_5 = 0. \tag{24}$$

To find f_6 , we substitute the 6^{th} RPS approximate solution

$$x_6(t)=A+\sum_{n=2}^{6}f_n\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$$

into the 6^{th} residual function $Res_6(t)$ such that

$$\begin{aligned} Res_6(t) &= D^{2\alpha}x_6(t)+\beta x_6(t)+\gamma x_6^3(t) \\ &= f_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_4\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_5\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+f_6\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \\ &\quad +\beta\left(A+\sum_{n=2}^{6}f_n\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right)+\gamma\left(A+\sum_{n=2}^{6}f_n\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right)^3. \end{aligned}$$

Then, we solve $D^{4\alpha}Res_6(0) = 0$ to get

$$f_6 + \beta f_4 + \gamma \left(3Af_2^2 \frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^3} + 3A^2 f_4 \right) = 0 \tag{25}$$

or

$$\begin{aligned} f_6 &= -\left(\beta + 3A^2\right) f_4 - 3A\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^3} f_2^2 & (26)\\ &= -\left(\beta + 3A^2\right)\left(\beta^2 A + \gamma\beta A^3 + 3A^3\gamma\beta + 3A^5\gamma^2\right) & (27)\\ &\quad -3A\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^3}\left(\beta A + \gamma A^3\right)^2. & (28) \end{aligned}$$

Using similar argument, we generate $f_7, f_8, f_9, ...$ Thus, the approximate solution is given by

$$x_k(t) = A + \sum_{n=2}^{k} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \qquad k = 2, 3, ... \tag{29}$$

In the next theorem, we study the convergence of the series (2) to the solution of problem (9)-(10).

Theorem 2.1. *Let* $x(t) = \sum_{n=0}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$ *and* $0 < \alpha \le 1$. *Then, the sequence* $\{x_k(t)\}$ *converges to the solution of problem (9)–(10).*

Proof: First, we want to prove that $\sum_{n=2}^{\infty} f_n \frac{t^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ converges to $D^{2\alpha}x(t)$ when $t > 0$. For any $t > 0$,

$$\begin{aligned} D^{2\alpha}x(t) &= \frac{1}{\Gamma(2-2\alpha)} \int_0^t (t-s)^{1-2\alpha} x''(s) ds \\ &= \frac{1}{\Gamma(2-2\alpha)} \int_0^t (t-s)^{1-2\alpha} \left(\sum_{n=0}^{\infty} f_n \frac{s^{n\alpha}}{\Gamma(1+n\alpha)} \right)'' ds \\ &= \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+n\alpha)} \frac{1}{\Gamma(2-2\alpha)} \int_0^t (t-s)^{1-2\alpha} \left(s^{n\alpha}\right)'' ds \\ &= \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+n\alpha)} D^{2\alpha}(t^{n\alpha}) = \sum_{n=2}^{\infty} \frac{f_n}{\Gamma(1+(n-2)\alpha)} t^{(n-2)\alpha}. \end{aligned}$$

Thus, $\sum_{n=2}^{\infty} f_n \frac{t^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ converges to $D^{2\alpha}x(t)$ when $t > 0$.

Next, we want to prove the sequence $\{x_k(t)\}$ converges to the solution of problem (9)-(10). Let

$$\begin{aligned} D^{2\alpha} &\left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) + \beta \left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) \\ &+ \gamma \left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \sum_{n=0}^{\infty} \xi_n t^{n\alpha} \end{aligned}$$

or

$$\sum_{n=2}^{\infty} \frac{f_n}{\Gamma(1+(n-2)\alpha)}\ t^{(n-2)\alpha} + \beta\ \left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right)$$
$$+\gamma\left(A + \sum_{n=2}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \sum_{n=0}^{\infty} \xi_n t^{n\alpha}.$$

Since $D^{2\alpha}x(t) = \sum_{n=2}^{\infty} f_n \frac{t^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ and $x(t) = \sum_{n=0}^{\infty} f_n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$, we have

$$\sum_{n=2}^{\infty} \xi_n t^{n\alpha} = 0.$$

Let

$$S_k = \sum_{n=k}^{\infty} \xi_n t^{n\alpha}.$$

Then, the sequence $\{S_k\}$ converges to zero. From Eq. (14), we see that

$$Res_k(t) = S_k.$$

Thus,

$$\lim_{k\to\infty} Res_k(t) = \lim_{k\to\infty} S_k = 0.$$

Hence, the sequence $\{x_k(t)\}$ converges to the solution of problem (10)-(11)

3 Results and Discussion

First, we study problem (10)-(11) when $\alpha = 1$. The exact solution of problem (10)-(11) is not known. Therefore, the numerical solutions have been determined by built-in file of MATHEMATICA based on the fully explicit Runge-Kutta method and this solution is used as the standard or reference for comparison. In Tables 1 and 2, we compare our results with the HPM, MHPM, SHPM [6], and the numerical solution for $A = 1, \alpha = 1, \beta = 1$, and $\gamma = 1$ and for $A = 0.75$, $\alpha = 1$, $\beta = 1.5$, and $\gamma = 1.5$, respectively.

t	HPM	MHPM	SHPM	Present results	Numerical results
0.5	0.762476	0.768902	0.768766	0.768802	0.768802
1.0	0.176929	0.233741	0.233680	0.233692	0.233692
2.0	−1.055110	−0.891260	−0.859323	−0.859349	−0.859349
3.5	−0.461650	−0.079433	−0.093034	−0.093013	−0.093013
5.0	2.049041	0.996472	0.947107	0.947130	0.947130

Table 1: The approximate solution for $A = 1$, $\alpha = 1$, $\beta = 1$, and $\gamma = 1$.

t	HPM	MHPM	SHPM	Present results	Numerical results
1	0.056288	0.080176	0.080519	0.0805269	0.080527
2	−0.808192	−0.739174	−0.729000	−0.729018	−0.729018
3	−0.339208	−0.239413	−0.238620	−0.2386259	−0.238626
4	0.891267	0.706827	0.667953	0.6680221	0.668022
5	0.893003	0.395315	0.387550	0.3875509	0.387551

Table 2: The approximate solution for $A = 0.75$, $\alpha = 1$, $\beta = 1.5$, and $\gamma = 1.5$.

Figure 1 shows the comparison between the current method and numerical solutions for $A = 1, \alpha = 1,\ \beta = 0.5$, and $\gamma = 2$ while Figure 2 shows the comparison between the current method and numerical solutions for $A = 1.5,\ \alpha = 1,\ \beta = 1$, and $\gamma = 0.5$.

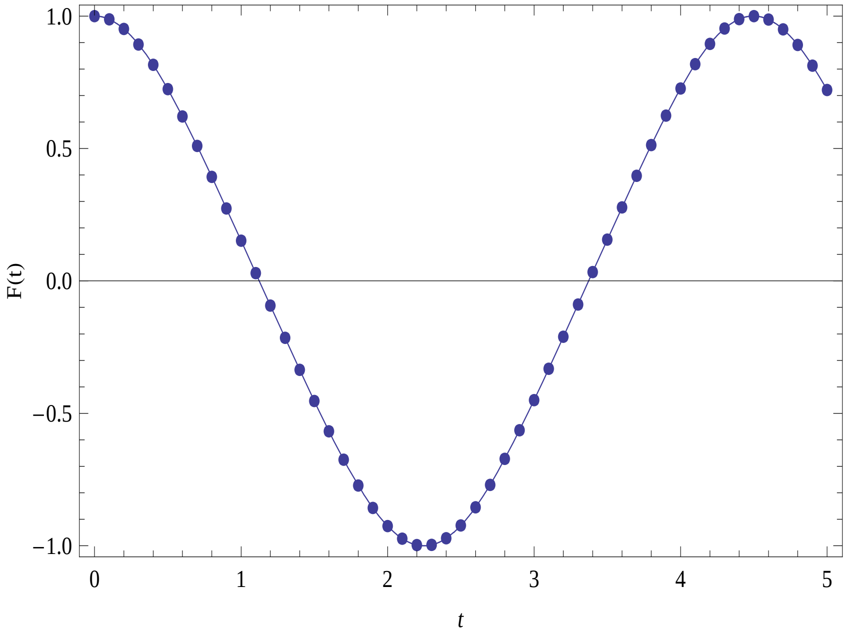

Figure 1. The proposed solution and the numerical solution for $A = 1, \alpha = 1,\ \beta = 0.5$, and $\gamma = 2$.

4 Conclusion

In this paper, the nonlinear differential equation of the cubic free undamped Duffing oscillator of the form

$$D^{2\alpha}x(t) + \beta\ x(t) + \gamma x^3(t) = 0,\ \frac{1}{2} < \alpha \leq 1,\ 0 < t < T, \tag{30}$$

subject to

$$x(0) = A, D^{\alpha}x(0) = 0 \tag{31}$$

is presented. We compare our results with the HPM, MHPM, SHPM [6], and the numerical solution for $A = 1, \alpha = 1$, and $\beta = \gamma = 1$ in Table 1. In Table 2, we compare our results with the HPM, MHPM, SHPM [6], and the numerical solution for $A = 0.75, \alpha = 1$, and $\beta = \gamma = 1.5$. Figure 1 shows the comparison between the current method and numerical solutions for $A = 1, \alpha = 1,\ \beta = 0.5$ and $\gamma = 2$ while Figure 2 shows the comparison between the current method and numerical solutions for $A = 1.5,\ \alpha = 1,\ \beta = 1$, and $\gamma = 0.5$. From the previous section, we can conclude the following:

- From Tables 1 and 2, we see that our results agree exceptionally well with the numerical results and are more accurate than those by the HPM, MHPM, SHPM [6].

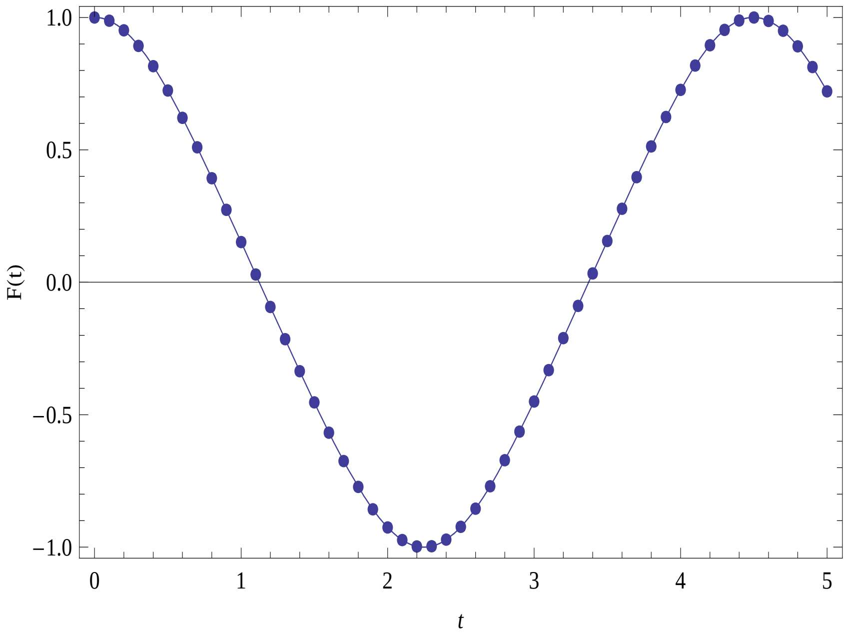

Figure 2. The proposed solution and the numerical solution for $A = 1.5$, $\alpha = 1$, $\beta = 1$, and $\gamma = 0.5$.

- Figure 2 shows the comparison between the current method and numerical solutions for $A = 1$, $\alpha = 0.5$, and $\beta = 2$. We see that there is agreement between the numerical results and our results.
- The MFPS method is an excellent tool due to the rapid convergent.
- The results in this paper confirm that the MFPS method is a powerful and efficient method for solving nonlinear differential equations in different fields of sciences and engineering.

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References

[1] A. Belendez, T. Belendez, A. Marquez and C. Neipp. Application of He's homotopy perturbation method to conservative truly nonlinear oscillators. *Chaos, Solitons & Fractals* **37** (2008) 770–780.

[2] S. Ghosh, A. Roy and D. Roy. An adaptation of Adomian decomposition for numericanalytic integration of strongly nonlinear and chaotic oscillators. *Comput. Methods Appl. Mech. Eng.* **196** (2007) 1133–1153.

[3] J.H. He. Homotopy perturbation method: A new nonlinear analytical technique. *Appl. Math. Comput.* **135** (2003) 73–79.

[4] A. Jaradat, M.S.M. Noorani, M. Alquran, and H.M. Jaradat. A Novel Method for Solving Caputo-Time-Fractional Dispersive Long Wave Wu-Zhang System. *Nonlinear Dynamics and Systems Theory* **18**(2) (2018) 182–190.

[5] Z. Denton and J.D. Ramirez. Monotone Method for Finite Systems of Nonlinear Riemann-Liouville Fractional Integro-Differential Equations. *Nonlinear Dynamics and Systems Theory* **18**(2) (2018) 130–143.

[6] Prcious Sibanda and A. Khidir. A new modification of the HPM for the Duffing equation with cubic nonlinearity. In: *Proc. 2011 International Conference on Applied and Computational Mathematics*, Turkey, 2011, 139–145.

[7] J. I. Ramos. On the variational iteration method and other iterative techniques for nonlinear differential equations. *Appl. Math. Comput.* **199** (2008) 39–69.

[8] J. I. Ramos. An artificial parameterdecomposition method for nonlinear oscillators: Applications to oscillators with odd nonlinearities. *J. Sound Vib.* **307** (2007) 312–329.

[9] P. D. Ariel, M. I. Syam and Q. M. Al-Mdallal. The extended homotopy perturbation method for the boundary layer flow due to a stretching sheet with partial slip. *International Journal of Computer Mathematics* **90**(9) (2013) 1990–2002.

[10] M. I. Syam and H. I. Siyyam. Numerical differentiation of implicitly defined curves. *J. Comput. Appl. Math.* **108**(1-2) (1999) 131–144.

[11] M. Syam. The modified Broyden-variational method for solving nonlinear elliptic differential equations. *Chaos, Solitions and Fractals* **32**(2) (2007) 392–404.

[12] B. Attili, K. Furati, and M.I. Syam. An efficient implicit Runge-Kutta method for second order systems. *Appl. Math. Comput.* **178**(2) (2006) 229–238.

[13] P.G. Chhetri and A.S. Vatsala. Generalized Monotone Method for Riemann-Liouville Fractional Reaction Diffusion Equation with Applications. *Nonlinear Dynamics and Systems Theory* **18**(3) (2018) 259–272.

[14] D.J. Sabeg, R. Ezzati, and K. Maleknejad. Solving Two-Dimensional Integral Equations of Fractional Order by Using Operational Matrix of Two-Dimensional Shifted Legendre Polynomials. *Nonlinear Dynamics and Systems Theory* **18**(3) (2018) 297–306.

[15] K.B. Oldham, and J. Spanier. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Courier Corporation, Massachusetts, USA, 2002.

[16] O. Abu Arqub, A. El-Ajou, A. Bataineh and I. Hashim. A representation of the exact solution of generalized Lane Emden equations using a new analytical method. *Abstract and Applied Analysis* **2013** (2013) 1–10.