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Solution to the Critical Burgers Equation for Small Data in a Bounded Domain

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Abstract: Solvability of Dirichlet's problem for the subcritical fractional Burgers equation is discussed here in the base spaces $D((-\Delta)^{\frac{s}{2}})$, $s \geq 0$ fixed. A unique solution in the *critical case* ($\alpha = \frac{1}{2}$) for small data is obtained next as a limit of the $X^{\frac{1}{2\alpha}}$ solutions to the subcritical equations, when the exponent α of $(-\Delta)^{\alpha}$ tends to $\frac{1}{2}^{+}$.

Keywords: *fractional Burgers equation; global solvability; critical equation.*

Mathematics Subject Classification (2010): 35S11.

1 Introduction

We consider the Dirichlet boundary value problem for the fractional Burgers equation in a bounded interval $I \subset \mathbb{R}$

$$\begin{aligned} &u_t + \frac{1}{2}\nabla u^2 + (-\Delta)^{\alpha} u = 0, \quad x \in I \subset \mathbb{R}, t > 0, \\ &u = 0 \text{ on } \partial I, \\ &u(0,x) = u_0(x), \end{aligned} \tag{1}$$

where $\alpha \in [\frac{1}{2}, 1]$ is a fractional exponent.

In our work we use the following Balakrishnan's definition of the fractional Laplacian (see [14]):

$$(-\Delta)^{\beta} g = \frac{\sin(\beta\pi)}{\pi} \int_0^{\infty} s^{\beta-1}(sI - \Delta)^{-1}(-\Delta) g \, ds, \quad g \in D(-\Delta), \quad \beta \in (0,1).$$

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Equivalence between the semigroup definition, Balakrishnan's formula and Bochner's formula is a general result, see [14]. The above definition can be used to study problems both in the case of a bounded and unbounded domain.

Over the last three decades a number of papers devoted to the Burgers equation with fractional dissipation in $\mathbb{R}$ have been published (see [2, 3, 11, 12, 15]). In paper [12], Kiselev, Nazarov and Shterenberg have conducted an extensive study for the 1-dimensional Burgers equation in the periodic setting, which concerned the subcritical cases $\frac{1}{2} < \alpha < 1$, the critical case $\alpha = \frac{1}{2}$, as well as the supercritical cases $0 < \alpha < \frac{1}{2}$. Karch, Miao and Xu investigated the asymptotics for the subcritical case in [11] whereas Alibaud, Imbert and Karch studied the asymptotics for the critical as well as supercritical case in [2]. In paper [15], the authors made use of the modulus of the continuity method and Fourier localization technique to prove the global well-posedness of the critical Burgers equation in critical Besov spaces $\dot{B}^{\frac{1}{p}}_{p,1}(\mathbb{R})$ with $p \in [1, \infty)$.

The global in time solvability of one-dimensional subcritical Burgers equation in bounded domain was studied recently in [10] in two base spaces $L^2(I)$ and $D((-\Delta)^{\frac{s}{2}})$ with $s > \frac{1}{2}$. Moreover, it was shown there that the solutions to subcritical problems (1) converge to the solution (not necessarily unique) of the critical problem when $\alpha \to \frac{1}{2}^+$.

1.1 Description of the results

This paper is devoted to the global in time solvability and properties of solutions to problem (1) for $\alpha \in [\frac{1}{2}, 1]$ in a bounded domain I. Our aim is to include, in the subcritical case of exponent $\alpha \in (\frac{1}{2}, 1]$, the problem of interest in the framework of semilinear parabolic equations with a sectorial positive operator (see [5, 9]). This offers a simple but formalized proof of local solvability as well as the regularity of solutions. There are different possible choices of the phase spaces for this problem. We choose $D((-\Delta)^{\frac{s}{2}})$ with $s > 0$ as the *base spaces* (in which the equation is fulfilled). The second section of the paper is devoted to the local and then the global in time solvability of the subcritical Burgers equation. Moreover, for small data we obtain a *uniform in* $\alpha \in (\frac{1}{2}, 1]$ estimate of the solutions u_α in $L^\infty(0, T; D((-\Delta)^{\frac{1}{2}}))$ and $L^2(0, T; D((-\Delta)^{\frac{3}{4}}))$, where $T > 0$ is fixed but arbitrarily large. In Section 3, we show that for the small data the solutions to subcritical problems (1) converge to the unique solution of the critical problem when $\alpha \to \frac{1}{2}^+$. It is a consequence of the well known compactness theorems. In this study, we use a technique proposed in our recent publications [6–8, 10].

Notation. Standard notation for Sobolev spaces is used. We indicate the dependence of solution u of (1) on $\alpha \in (\frac{1}{2}, 1]$, calling it u_α. Let r^- denote a number strictly less than r but arbitrarily close to it.

2 Solvability of Subcritical Problem (1), $\alpha \in (\frac{1}{2}, 1]$

Formulation of the problem and its local solvability. Our first task is the local in time solvability of the subcritical problem (1) for $\alpha \in (\frac{1}{2}, 1]$. We will use the standard approach proposed by Dan Henry [9] for semilinear 'parabolic' equations. We start from recalling some usefull facts concerning Henry's approach. So, when we have the abstract Cauchy problem

$$\begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0, \end{cases} \tag{2}$$

where

1. X is a Banach space. The space X is called the *base space*, that is, the space in which the equation is fulfilled,
2. $A\colon D(A) \to X$ is a sectorial positive operator in X,
3. $F\colon X^\gamma \to X$ is Lipschitz continuous on the bounded subset of X^γ for some non-negative $\gamma \in [0,1)$,
4. $u(0) = u_0 \in X^\gamma$,

then by local X^γ solution of this problem we understood the function u, which satisfies the following conditions.

Definition 2.1 The function u is called a local X^γ solution of (2) if, for some real $\tau > 0$, it satisfies

- $u(0) = u_0$,
- $u \in C([0,\tau); X^\gamma)$,
- $u \in C^1((0,\tau); X)$,
- $u(t)$ belongs to $D(A)$ for each $t \in (0,\tau)$,
- the equation $u_t + Au = F(u)$ holds in X for all $t \in (0,\tau)$.

The following theorem concerns the local X^γ solution of the abstract problem (2).

Theorem 2.1 *Let X be a Banach space, $A\colon D(A) \to X$ be a sectorial positive operator in X and $F\colon X^\alpha \to X$ be Lipschitz continuous on the bounded subset of X^γ for some non-negative $\gamma \in [0,1)$. Then for each $u(0) = u_0 \in X^\gamma$, there exists a unique local X^γ solution $u = u(t,u_0)$ of (2) defined on its maximal interval of existence $[0,\tau_{u_0})$.*

Now we use Henry's approach to our problem. There are different possible choices of the base space. We choose $X = D((-\Delta)^{\frac{s}{2}}) \subset H^s(I)$, where $s \geq 0$ is fixed, as the base space. The operator $A_\alpha := (-\Delta)^\alpha$ acting in the Banach space X is equipped with the domain $D(A_\alpha) \subset H^{s+2\alpha}(I)$. The resulting phase space is $X^{\frac{1}{2\alpha}} = [X, D(A_\alpha)]_{\frac{1}{2\alpha}} = D((-\Delta)^{\frac{s+1}{2}}) \subset H^{s+1}(I)$ (since $(A_\alpha)^{\frac{1}{2\alpha}} = (-\Delta)^{\frac{1}{2}}$). Moreover, when Ω is a domain in $\mathbb{R}^N$, then $W^{m,r}(\Omega)$ is the Banach algebra provided $mr > N$ (see [1, p. 115]. Note that in our case, $H^{s+1}(I)$ is a *Banach algebra.*

Working with the *sectorial positive operator* $A_\alpha : D(A_\alpha) \to X$, $\alpha \in (\frac{1}{2}, 1]$, in I with the zero boundary condition (e.g. [5, 9]), we rewrite equation (1) in an abstract form:

$$\begin{aligned} &(u_\alpha)_t + A_\alpha u_\alpha = F(u_\alpha),\ t > 0,\\ &u_\alpha(0,x) = u_0(x), \end{aligned} \tag{3}$$

where

$$F(u_\alpha) = -\frac{1}{2}\nabla u_\alpha^2 \tag{4}$$

is the Nemytskii operator corresponding to a nonlinear term $-\frac{1}{2}\nabla u_\alpha^2$. The following local existence result holds.

Theorem 2.2 *Let $s \geq 0$ be fixed and $\alpha \in (\frac{1}{2}, 1]$. Then for arbitrary $u_0 \in X^{\frac{1}{2\alpha}} = D((-\Delta)^{\frac{s+1}{2}})$, there exists a unique local in time $X^{\frac{1}{2\alpha}}$ solution $u_\alpha(t)$ to the subcritical problem* (3) *defined on its maximal interval of existence* $[0, \tau_{u_0})$. *Moreover,*

$$u_\alpha \in C((0, \tau_{u_0}); X^1) \cap C([0, \tau_{u_0}); X^{\frac{1}{2\alpha}}),\ (u_\alpha)_t \in C((0, \tau_{u_0}); X^\gamma),$$

with arbitrary $\gamma < 1$, ($X^1 = D(A_\alpha) \subset H^{s+2\alpha}(I)$).

Proof. To guarantee the local solvability we need to check if the nonlinearity (4) is Lipschitz continuous on bounded sets as a map from $X^{\frac{1}{2\alpha}}$ into X (see Theorem 2.1; [5], p. 55 for more details), that is, for any $r > 0$ there exists $L(r) > 0$ such that

$$\|F(v) - F(w)\|_X \leq L(r)\|v - w\|_{X^{\frac{1}{2\alpha}}}$$

for all $v, w \in B(r)$, where $B(r)$ denotes an open ball in $X^{\frac{1}{2\alpha}}$ centered at zero of radius r. Since $H^{s+1}(I)$ is the Banach algebra, for $v, w \in B(r)$, we get

$$\begin{aligned}\|F(v) - F(w)\|_{H^s(I)} &= \frac{1}{2}\|\nabla(v^2 - w^2)\|_{H^s(I)} \leq c\|v^2 - w^2\|_{H^{s+1}(I)} \\ &\leq \|v + w\|_{H^{s+1}(I)}\|v - w\|_{H^{s+1}(I)}.\end{aligned}$$

Consequently, we obtain

$$\|F(v) - F(w)\|_{H^s(I)} \leq c'(\|v\|_{H^{s+1}(I)}, \|w\|_{H^{s+1}(I)})\|v - w\|_{H^{s+1}(I)},$$

which proves the local solvability of (1) in the phase space $X^{\frac{1}{2\alpha}}$.

Remark 2.1 The local solution constructed above fulfills Cauchy's integral formula (see [5, Lemma 2.2.1]):

$$u_\alpha(t) = e^{-A_\alpha t}u_0 + \int_0^t e^{-A_\alpha(t-s)}F(u_\alpha(s))ds,\ t \in [0, \tau_{u_0}),$$

where $e^{-A_\alpha t}$ denotes the linear semigroup corresponding to the operator $A_\alpha := (-\Delta)^\alpha$ in $D((-\Delta)^{\frac{s}{2}})$ and $F(u_\alpha) = -\frac{1}{2}\nabla u_\alpha^2$.

Remark 2.2 Note that since the function F is Lipschitz continuous on bounded subsets of $X^{\frac{1}{2\alpha}}$, as a consequence of the embeddings between the fractional power space, it possesses this property as a map from X^β to X for each $\beta \in [\frac{1}{2\alpha}, 1)$. Consequently, for each $\beta \in [\frac{1}{2\alpha}, 1)$ and $u_0 \in X^\beta$, there exists a unique local in time X^β solution to the subcritical problem (3) defined on its maximal interval of existence.

Remark 2.3 Let $\epsilon = 2\alpha - 1 > 0$ and $t_0 > 0$ be chosen arbitrarily close to 0. From Theorem 2.2, we know that $u_\alpha(t_0, \cdot) \in D((-\Delta)^{\frac{s}{2}+\alpha}) \subset H^{s+2\alpha}(I)$. Since ∂I is regular, considering the equation (3) in the base space $D((-\Delta)^{\frac{s+\epsilon}{2}})$ with a new initial condition $u_\alpha(t_0, x) = (u_\alpha)_{t_0}(x)$, we obtain that $u_\alpha(t, \cdot)$ varies continuously in $D((-\Delta)^{\frac{s+\epsilon}{2}+\alpha})$ for $t > t_0$. Next, repeating this procedure n times with $t_n = \sum_{i=0}^n \frac{t_0}{2^i}$ and the base space $D((-\Delta)^{\frac{s+(n+1)\epsilon}{2}})$, we get additional regularity of the solution of (3), that is, $u_\alpha(t, \cdot) \in D((-\Delta)^{\frac{s+(n+1)\epsilon}{2}+\alpha}) \subset H^{s+(n+1)\epsilon+2\alpha}(I)$ for $t > t_n = t_0(2 - \frac{1}{2^n})$. This phenomenon is known in the literature as bootstrapping.

Global solvability. Having obtained the local in time solution of (1), to guarantee its global extensibility we need suitable *a priori estimates.* We start from the Maximum Principle.

Lemma 2.1 *Let $k \in \mathbb{N}$. Then, for a sufficiently regular solution u_α of (1), the following estimates hold:*

$$\|u_\alpha(t,\cdot)\|_{L^{2^k}(I)} \leq \|u_0\|_{L^{2^k}(I)}, \tag{5}$$

$$\|u_\alpha(t,\cdot)\|_{L^{2^k}(I)} \leq \|u_0\|_{L^{2^k}(I)} e^{-2^{1-k}\lambda_1^\alpha t}, \tag{6}$$

where λ_1 is the Poincaré constant (see [7])

$$\lambda_1^\alpha \|\phi\|^2_{L^2(I)} \leq \|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2_{L^2(I)}. \tag{7}$$

Proof. Multiplying (1) by $u_\alpha^{2^k-1}$, $k = 1, 2\ldots$, we get

$$\frac{1}{2^k}\frac{d}{dt}\int_I u_\alpha^{2^k}\, dx + \int_I (-\Delta)^\alpha u_\alpha |u_\alpha|^{2^k-1} \operatorname{sgn} u_\alpha\, dx + \int_I (u_\alpha)_x u^{2^k}\, dx = 0.$$

Using the Kato-Beurling-Deny inequality in the bounded domain [7, Corollary 3.2] with $q = 2^k$, we have

$$\frac{2^k-1}{2^{2k-2}}\int_I \left[(-\Delta)^{\frac{\alpha}{2}}(|u_\alpha|^{2^{k-1}})\right]^2 dx \leqslant \int_I \left[(-\Delta)^\alpha u_\alpha\right] |u_\alpha|^{2^k-1} \operatorname{sgn} u_\alpha\, dx. \tag{8}$$

Since

$$\int_I (u_\alpha)_x u_\alpha^{2^k}\, dx = \frac{1}{2^k+1}\int_I (u_\alpha^{2^k+1})_x\, dx = 0,$$

and $2 \leq \frac{2^k-1}{2^{k-2}}$, thanks to (8) and (7), we obtain

$$\frac{d}{dt}\int_I u_\alpha^{2^k}\, dx \leq \frac{d}{dt}\int_I u_\alpha^{2^k}\, dx + 2\lambda_1^\alpha \int_I |u_\alpha|^{2^k}\, dx \leq 0,$$

which leads to estimates (5) and (6).

Remark 2.4 Let $q \in \mathbb{N}$. Since $u_\alpha(t) \in L^\infty(I)$, the following convergence holds:

$$\lim_{q\to\infty} \|u_\alpha(t,\cdot)\|_{L^q(I)} = \|u_\alpha(t,\cdot)\|_{L^\infty(I)}$$

(see [1, Theorem 2.8]). Consequently, letting $k \to +\infty$ in estimate (5), we obtain

$$\|u_\alpha(t,\cdot)\|_{L^\infty(I)} \leq \|u_0\|_{L^\infty(I)}. \tag{9}$$

Remark 2.5 The constant $\lambda_1^{\alpha-\frac{1}{2}}$ can be estimated independently of $\alpha \in (\frac{1}{2}, 1]$. We have

$$\mu_b := \min\{1, \lambda_1^{\frac{1}{2}}\} \leq \lambda_1^{\alpha-\frac{1}{2}} \leq \max\{1, \lambda_1^{\frac{1}{2}}\} =: \mu_a. \tag{10}$$

Remark 2.6 Multiplying (1) by u_α, due to (7) and Remark 2.5, we obtain a differential inequality of the form

$$0 = \frac{d}{dt}\|u_\alpha\|^2_{L^2(I)} + 2\|(-\Delta)^{\frac{\alpha-\frac{1}{2}}{2}}(-\Delta)^{\frac{1}{4}}u_\alpha\|^2_{L^2(I)} \geq \frac{d}{dt}\|u_\alpha\|^2_{L^2(I)} + 2\mu_b\|(-\Delta)^{\frac{1}{4}}u_\alpha\|^2_{L^2(I)}. \tag{11}$$

Integrating (11) over $(0,T)$, we get

$$\int_0^T \|(-\Delta)^{\frac{1}{4}} u_\alpha\|^2_{L^2(I)} ds = \frac{1}{2\mu_b}\left(\|u_0\|^2_{L^2(I)} - \|u_\alpha(T)\|^2_{L^2(I)}\right) \le \frac{1}{2\mu_b}\|u_0\|^2_{L^2(I)}.$$

This implies a uniform in $\alpha \in (\frac{1}{2}, 1]$ estimate of u_α in $L^2(0,T;D((-\Delta)^{\frac{1}{4}}))$, where $T > 0$ is fixed but arbitrarily large.

The $L^p(I)$ a priori estimates obtained in Lemma 2.1 and Remark 2.4 are, unfortunately, too weak to guarantee the global in time solvability of (3) in $X^{\frac{1}{2\alpha}}$. For this purpose, we need to estimate higher Sobolev norms of the solutions to (3). We will show that $\|u_\alpha\|_{H^{s+1}(I)}$ is bounded on the solutions. Consequently, we will obtain Lipschitz continuity and boundedness of the nonlinear term F as a map from $X^{\frac{1}{2\alpha}}$ to X.

We will start from the $H^1(I)$ a priori estimate.

Lemma 2.2 *For a sufficiently regular solution u_α of* (1)*, the following estimate holds:*

$$\|u_\alpha\|_{H^1(I)} \le c(\|u_0\|_{H^1(I)}, \alpha). \tag{12}$$

Proof. Multiplying (1) by $-(u_\alpha)_{xx}$, we get

$$\frac{1}{2}\frac{d}{dt}\int_I ((u_\alpha)_x)^2\, dx + \int_I [(-\Delta)^{\frac{1+\alpha}{2}} u_\alpha]^2\, dx - \int_I u_\alpha (u_\alpha)_x (u_\alpha)_{xx}\, dx = 0.$$

Since

$$-\int_I u_\alpha (u_\alpha)_x (u_\alpha)_{xx}\, dx = \frac{1}{2}\int_I ((u_\alpha)_x)^3\, dx,$$

we have

$$\frac{d}{dt}\int_I ((u_\alpha)_x)^2\, dx + 2\int_I [(-\Delta)^{\frac{1+\alpha}{2}} u_\alpha]^2\, dx + \int_I ((u_\alpha)_x)^3\, dx = 0. \tag{13}$$

Note that (see [10, p. 63])

$$\|u_\alpha\|^3_{W^{1,3}(I)} \le c(\alpha)\|u_\alpha\|^{3\theta}_{H^{1+\alpha}(I)}\|u_\alpha\|^{3(1-\theta)}_{L^\infty(I)} \tag{14}$$

with $\frac{4}{3(2\alpha+1)} \le \theta < \frac{2}{3}$. Consequently, using the Young inequality, we get

$$\begin{aligned}\frac{d}{dt}\int_I ((u_\alpha)_x)^2\, dx + c\int_I ((u_\alpha)_x)^2\, dx &\le \frac{d}{dt}\int_I ((u_\alpha)_x)^2\, dx + \int_I [(-\Delta)^{\frac{1+\alpha}{2}} u_\alpha]^2\, dx \\ &\le c(\|u_\alpha\|_{L^\infty(I)}, \alpha),\end{aligned}$$

where an equivalent norm in $H^{1+\alpha}(I)$ is used.

Lemma 2.3 *For a sufficiently regular solution u_α of* (1)*, which satisfies the smallest data condition* (17)*, the following uniform in $\alpha \in (\frac{1}{2}, 1]$ estimate*

$$\|(u_\alpha)_x(t)\|_{L^2(I)} \le \|u_0\|_{H^1(I)} e^{-(2\mu_b - C^3\|u_0\|_{L^\infty(I)})t} \tag{15}$$

holds.

Proof. Note that, when the Nirenberg-Gagliardo inequality (and an equivalent norm in $H^{\frac{3}{2}}(I)$)

$$\|u_\alpha\|_{W^{1,3}(I)} \leq c\|u_\alpha\|^{\frac{1}{3}}_{L^\infty(I)}\|u_\alpha\|^{\frac{2}{3}}_{H^{\frac{3}{2}}(I)} \leq C\|u_\alpha\|^{\frac{1}{3}}_{L^\infty(I)}\|(-\Delta)^{\frac{3}{4}}u_\alpha\|^{\frac{2}{3}}_{L^2(I)}$$

is used instead of (14), thanks to the Poincaré inequality (7) and (9), due to Remark 2.5, the estimate (13) extends to

$$\frac{d}{dt}\int_I ((u_\alpha)_x)^2 \, dx + (2\mu_b - C^3\|u_0\|_{L^\infty(I)})\int_I [(-\Delta)^{\frac{3}{4}}u_\alpha]^2 \, dx \leq 0. \tag{16}$$

Consequently, when the data are small

$$\|u_0\|_{L^\infty(I)} < \frac{2\mu_b}{C^3} \tag{17}$$

we obtain the thesis.

Remark 2.7 Under the assumption (17) the estimate (16) implies a uniform in $\alpha \in (\frac{1}{2}, 1]$ estimate of u_α in $L^\infty(0,T;H^1_0(I))$ and $L^2(0,T;D((-\Delta)^{\frac{3}{4}}))$. So, we have

$$\|u_\alpha\|_{L^\infty(0,T;H^1_0(I))} + \|u_\alpha\|_{L^2(0,T;D((-\Delta)^{\frac{3}{4}}))} \leq const, \tag{18}$$

where $T > 0$ is fixed but arbitrarily large and the constant on the right-hand side is independent of α.

Lemma 2.4 *For a sufficiently regular solution u_α of (1), the following estimate holds:*

$$\|\Delta u_\alpha\|_{L^2(I)} \leq c(\|u_0\|_{H^2(I)}, \alpha). \tag{19}$$

Proof. Multiplying (1) by $(-\Delta)^2 u_\alpha$, we get

$$\frac{d}{dt}\|\Delta u_\alpha\|^2_{L^2(I)} + 2\|(-\Delta)^{\frac{2+\alpha}{2}}u_\alpha\|^2_{L^2(I)} + 3\int_I (\Delta u_\alpha)^2 \nabla u_\alpha \, dx = 0.$$

Using the Nirenberg-Gagliardo inequality

$$\|u\|^2_{W^{2,4}(I)} \leq c\|u\|^{\frac{5}{3}}_{H^{\frac{5}{2}}(I)}\|u\|^{\frac{1}{3}}_{H^1(I)} \tag{20}$$

and the Young inequality, we can estimate the nonlinear term as follows:

$$\int_I |\,(\Delta u_\alpha)^2 \nabla u_\alpha|\, dx \leq \|\Delta u_\alpha\|^2_{L^4(I)}\|\nabla u_\alpha\|_{L^2(I)} \leq \frac{\mu_b}{3}\|u_\alpha\|^2_{H^{\frac{5}{2}}(I)} + c\|u_\alpha\|^8_{H^1(I)}.$$

Consequently, thanks to the Poincaré inequality (7), we get

$$\frac{d}{dt}\|\Delta u_\alpha\|^2_{L^2(I)} + \mu_b\|(-\Delta)^{\frac{5}{4}}u_\alpha\|^2_{L^2(I)} \leq c\|u_0\|^8_{H^1(I)}. \tag{21}$$

Remark 2.8 Since for the small data we have uniform in α estimate of solution u_α in $H^1(I)$, we get a uniform in $\alpha \in (\frac{1}{2}, 1]$ estimate

$$\|u_\alpha\|_{L^\infty(0,T;H^2(I))} + \|u_\alpha\|_{L^2(0,T;H^{\frac{5}{2}}(I))} \leq const,$$

where $T > 0$ is fixed but arbitrarily large.

Further we get the $H^l(I)$ estimate of solutions by recurrence.

Lemma 2.5 *Let $l = \frac{k}{2}$, $k \geq 5$. Then, for a sufficiently regular solution of* (1)*, the following estimate holds:*

$$\|u_\alpha\|_{H^l(I)} \leq c(\|u_0\|_{H^{l-\alpha}(I)}, \alpha). \tag{22}$$

Proof. Note first that by (12) we have $\|u_\alpha\|_{H^1(I)} \leq c(\|u_0\|_{H^1(I)}, \alpha)$. Multiplying (1) by $(-\Delta)^l u_\alpha$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_I [(-\Delta)^{\frac{l}{2}} u_\alpha]^2 dx + \int_I [(-\Delta)^{\frac{l+\alpha}{2}} u_\alpha]^2 dx = \int_I (-\Delta)^{\frac{l-\alpha}{2}} (u_\alpha (u_\alpha)_x)(-\Delta)^{\frac{l+\alpha}{2}} u_\alpha dx. \tag{23}$$

Since $H^{l-\alpha}(I)$ is a Banach algebra for $l - \alpha > \frac{1}{2}$, the nonlinear term can be estimated as follows:

$$\begin{aligned}\left|\int_I (-\Delta)^{\frac{l-\alpha}{2}} (u_\alpha (u_\alpha)_x)(-\Delta)^{\frac{l+\alpha}{2}} u_\alpha dx\right| &\leq c\|u_\alpha (u_\alpha)_x\|_{H^{l-\alpha}(I)} \|u_\alpha\|_{H^{l+\alpha}(I)} \\ &\leq c\|u_\alpha\|_{H^{l-\alpha}(I)} \|u_\alpha\|_{H^{l+1-\alpha}(I)} \|u_\alpha\|_{H^{l+\alpha}(I)}.\end{aligned}$$

By the Nirenberg-Gagliardo inequality, we get

$$\|u_\alpha\|_{H^{l+1-\alpha}(I)} \leq c\|u_\alpha\|_{H^{l+\alpha}(I)}^{\frac{1}{2\alpha}} \|u_\alpha\|_{H^{l-\alpha}(I)}^{1-\frac{1}{2\alpha}},$$

hence

$$\left|\int_I (-\Delta)^{\frac{l-\alpha}{2}} (u_\alpha (u_\alpha)_x)(-\Delta)^{\frac{l+\alpha}{2}} u_\alpha dx\right| \leq c\|u_\alpha\|_{H^{l+\alpha}(I)}^{1+\frac{1}{2\alpha}} \|u_\alpha\|_{H^{l-\alpha}(I)}^{2-\frac{1}{2\alpha}}.$$

Consequently, using the Young inequality, we obtain from (23) a differential inequality

$$\frac{d}{dt}\int_I [(-\Delta)^{\frac{l}{2}} u_\alpha]^2 dx + \int_I [(-\Delta)^{\frac{l+\alpha}{2}} u_\alpha]^2 dx \leq c(\|u_\alpha\|_{H^{l-\alpha}(I)}, \alpha).$$

The following global existence result holds.

Theorem 2.3 *The local solution u_α of* (3) *constructed in Theorem 2.2 exists globally in time.*

Lemma 2.6 *Let $\alpha \in (\frac{1}{2}, \frac{3}{4}]$. For solution u_α of* (3) *satisfying the smallest data restriction* (17) *we have a uniform with respect to α estimate*

$$\|(u_\alpha)_t\|_{L^2(0,T;L^2(I))} \leq C(T), \tag{24}$$

where $T > 0$ is fixed but arbitrarily large.

Proof. Since $H^1(I)$ is a Banach algebra from equation (1), thanks to the Poincaré inequality (7), we obtain for $\alpha \in (\frac{1}{2}, \frac{3}{4}]$

$$\|(u_\alpha)_t\|^2_{L^2(I)} \le 2\lambda_1^{\frac{4\alpha-3}{2}} \|(-\Delta)^{\frac{3}{4}} u_\alpha\|^2_{L^2(I)} + c\|u_\alpha^2\|^2_{H^1(I)} \le 2\mu_b^{-1} \|(-\Delta)^{\frac{3}{4}} u_\alpha\|^2_{L^2(I)} + c\|u_\alpha\|^4_{H^1(I)}.$$

Integrating the result over (0,T), due to (18), we get

$$\int_0^T \|(u_\alpha)_t\|^2_{L^2(I)}\, dt \le c(T; \|u_0\|_{H^1(I)}), \tag{25}$$

with a positive constant c independent of α.

3 Critical Problem (1) with $\alpha = \frac{1}{2}$ for Small Data

Passing to the limit in equation (1). Using the Lions-Aubin compactness lemma we will show now that for the small data (the condition (17)) the solutions of subcritical problems (1) converge, as $\alpha \to \frac{1}{2}^+$, to the unique solution of the critical problem. The below lemma will be useful in the limiting procedure.

Lemma 3.1 *For any sequence $\alpha_n \to \frac{1}{2}$ such that $\{\alpha_n : n \in \mathbb{N}\} \subset (\frac{1}{2}, \frac{3}{4}]$ there are a subsequence (denoted in the same way) $\alpha_n \to \frac{1}{2}$ and a function u such that for any $T > 0$*

1. *$u_{\alpha_n} \to u$ weakly in $L^2(0,T; D((-\Delta)^{\frac{3}{4}}))$ and weakly-* in $L^\infty(0,T; D((-\Delta)^{\frac{1}{2}}))$,*
2. *$u_{\alpha_n} \to u$ in $L^2(0,T; D((-\Delta)^{\frac{3}{4}^-}))$,*
3. *$(u_{\alpha_n})_t \to u_t$ weakly in $L^2(0,T; L^2(I))$.*

Proof. Part (1). Note that uniform in α estimate (18) means that any sequence $\{u_{\alpha_n}\}$ is bounded in $L^2(0,T; D((-\Delta)^{\frac{3}{4}}))$. Consequently (see [4, Theorem 3.18]), there exist a subsequence (denoted in the same way) and $u \in L^2(0,T; D((-\Delta)^{\frac{3}{4}}))$ such that $\{u_{\alpha_n}\}$ converges to u weakly when $\alpha_n \to \frac{1}{2}$.

Part (2). Let

$$\mathbf{U} = \left\{ u_\alpha; \quad \alpha \in \left(\frac{1}{2}, \frac{3}{4}\right] \right\} \quad and \quad \frac{\partial \mathbf{U}}{\partial t} = \{(u_\alpha)_t : \quad u_\alpha \in U\}. \tag{26}$$

Since the set $\mathbf{U}$ is bounded in $L^2(0,T; D((-\Delta)^{\frac{3}{4}}))$ and $\frac{\partial \mathbf{U}}{\partial t}$ is bounded in $L^2(0,T; L^2(I))$ (see (18) and (24)), using the Lions-Aubin compactness lemma (see [13], [16, Corollary 4]) we claim that the set $\mathbf{U}$ is relatively compact in the space $L^2(0,T; D((-\Delta)^{\frac{3}{4}^-}))$. Consequently, for any sequence $\{u_{\alpha_n}\}$ there exist a subsequence (denoted in the same way) and $u \in L^2(0,T; D((-\Delta)^{\frac{3}{4}^-}))$ such that $\{u_{\alpha_n}\}$ converges to u strongly.

Part(3) is a consequence of estimate (24) (see [4, Theorem 3.18]).

Remark 3.1 Since the set $\mathbf{U}$ is bounded in $L^\infty(0,T; D((-\Delta)^{\frac{1}{2}}))$ and $\frac{\partial \mathbf{U}}{\partial t}$ is bounded in $L^2(0,T; L^2(I))$ (see (18) and (24)), using the Corollary 4 from [16] we claim that the set $\mathbf{U}$ is also relatively compact in the space $C(0,T; D((-\Delta)^{\frac{1}{2}^-}))$.

Theorem 3.1 *Let $\{\alpha_n : n \in \mathbb{N}\} \subset (\frac{1}{2}, \frac{3}{4}]$ and let u_α be the solution of the subcritical problem (1) (constructed in Theorem 2.3 in $D((-\Delta)^{\frac{s}{2}})$) corresponding to the initial* *condition* $u_0 \in D((-\Delta)^{\frac{s+1}{2}})$ *satisfying the smallest data restriction* (17)*. Then, passing over a subsequence (denoted in the same way), with* α_n *to* $\frac{1}{2}$ *in equation* (1)*, we get a weak solution* u *(not necessarily unique) to the critical problem* ($\alpha = \frac{1}{2}$) *satisfying a.e. in each time interval* $[0, T]$ *the equality*

$$\frac{d}{dt} < u, \phi > + \frac{1}{2} < \nabla u^2, \phi > + < (-\Delta)^{\frac{1}{2}} u, \phi > = 0,$$

for every function $\phi \in H^1_0(I)$*, where* $< \cdot, \cdot >$ *is a scalar product in* $L^2(I)$ *and* $\frac{d}{dt}$ *stands for the distributional derivative.*

Proof. Multiplying equation (1) by a 'test function' $\phi \in H^1_0(I)$ (note, $H^1_0(I) \subset L^\infty(I)$, $N = 1$), we obtain

$$\int_I (u_\alpha)_t \, \phi \, dx + \int_I (-\Delta)^\alpha u_\alpha \phi \, dx = -\frac{1}{2} \int_I \nabla u_\alpha^2 \phi \, dx.$$

Next for each smooth scalar test function $\eta \in D((0, T))$, we get

$$\int_0^T \int_I (u_\alpha)_t \, \phi \, dx \, \eta \, dt + \int_0^T \int_I (-\Delta)^\alpha u_\alpha \phi \, dx \, \eta \, dt = -\frac{1}{2} \int_0^T \int_I \nabla u_\alpha^2 \phi \, dx \, \eta \, dt.$$

We will discuss now the convergence of components in the above equality one by one. In the term containing the time derivative $(u_\alpha)_t$, thanks to [18, Lemma 1.1, Chapt.III], we have

$$\int_0^T < (u_\alpha)_t, \phi > \eta \, dt = \int_0^T \frac{d}{dt} < u_\alpha, \phi > \eta \, dt = -\int_0^T < u_\alpha, \phi > \eta' \, dt$$

for all $\phi \in H^1_0(I)$. Since

$$\begin{aligned}\int_0^T \int_I |u_\alpha - u||\phi| \, |\eta'| \, dx \, dt &\le \int_0^T \|u_\alpha - u\|_{L^2(I)} \|\phi\|_{L^2(I)} |\eta'| \, dt \\ &\le \|u_\alpha - u\|_{L^2(0,T;L^2(I))} \|\phi\|_{L^2(I)} \|\eta'\|_{L^2(0,T)},\end{aligned}$$

using part (2) of Lemma 3.1, we obtain

$$\int_0^T < u_\alpha, \phi > \eta' \, dt \to \int_0^T < u, \phi > \eta' \, dt.$$

For the linear term

$$\int_0^T \int_I (-\Delta)^\alpha u_\alpha \phi \, dx \, \eta \, dt = \int_0^T \int_I (-\Delta)^{\frac{1}{2}} u_\alpha (-\Delta)^{\alpha - \frac{1}{2}} \phi \, dx \, \eta \, dt \tag{27}$$

we get

$$\begin{aligned}&\left| \int_0^T \int_I (-\Delta)^{\frac{1}{2}} u_\alpha (-\Delta)^{\alpha - \frac{1}{2}} \phi \, dx \, \eta \, dt - \int_0^T \int_I (-\Delta)^{\frac{1}{2}} u \phi \, dx \, \eta \, dt \right| \\ &\le \int_0^T \int_I \left| (-\Delta)^{\frac{1}{2}} u_\alpha \right| \left| \left((-\Delta)^{\alpha - \frac{1}{2}} - I \right) \phi \right| \, dx \, |\eta| \, dt \\ &+ \int_0^T \int_I |\phi| \left| (-\Delta)^{\frac{1}{2}} (u_\alpha - u) \right| \, dx \, |\eta| \, dt \\ &\le \|u_\alpha\|_{L^2(0,T;D((-\Delta)^{\frac{1}{2}})} \| \left((-\Delta)^{\alpha - \frac{1}{2}} - I \right) \phi \|_{L^2(I)} \|\eta\|_{L^2(0,T)} \\ &+ \|\phi\|_{L^2(I)} \|\eta\|_{L^2(0,T)} \|u_\alpha - u\|_{L^2(0,T;D((-\Delta)^{\frac{1}{2}}))}.\end{aligned} \tag{28}$$

Passing to the limit, by [14, Theorem 3.1.6] and part (2) of Lemma 3.1, we obtain the convergence

$$\int_0^T \int_I (-\Delta)^{\frac{1}{2}} u_\alpha (-\Delta)^{\alpha-\frac{1}{2}} \phi \, dx \, \eta \, dt \to \int_0^T \int_I (-\Delta)^{\frac{1}{2}} u \phi \, dx \, \eta \, dt. \tag{29}$$

Next, for the nonlinear term, since $H^1(I)$ is Banach algebra, we prove that

$$\left| \int_0^T \int_I \nabla u_\alpha^2 \phi \eta - \nabla u^2 \phi \eta \, dx \, dt \right| \leq \int_0^T \|u_\alpha^2 - u^2\|_{H^1(I)} \|\phi\|_{L^2(I)} \, |\eta| \, dt$$
$$\leq c \|u_\alpha - u\|_{L^2(0,T;D((-\Delta)^{\frac{1}{2}}))} \|u_\alpha + u\|_{L^\infty(0,T;D((-\Delta)^{\frac{1}{2}}))} \|\phi\|_{L^2(I)} \|\eta\|_{L^2(0,T)}.$$

By collecting all the limits together, we find the form of the limit *critical equation*

$$\int_0^T \frac{d}{dt} < u, \phi > \eta \, dt + \frac{1}{2} \int_0^T < \nabla u^2, \phi > \eta \, dt + \int_0^T < (-\Delta)^{\frac{1}{2}} u, \phi > \eta \, dt = 0.$$

Properties of the weak solutions to the critical fractional Burgers equation. We will start from collecting the properties inherited by the solution u of the critical problem(1) in the process of passing to the limit. We have the following results

Corollary 3.1 *For arbitrary $T > 0$ we have*

- $u \in L^2(0,T; D((-\Delta)^{\frac{3}{4}})) \cap L^\infty(0,T; D((-\Delta)^{\frac{1}{2}}))$,
- $u_t \in L^2(0,T; L^2(I))$,
- $u \in C_w(0,T; D((-\Delta)^{\frac{1}{2}}))$.

Proof. Using the properties of the weak limit, due to Lemma 3.1 (1) and (3), we obtain the first two regularies. Next, the Corollary 2.1 from [17] implies that there exists a weakly continuous function on $[0,T]$ with the values in $D((-\Delta)^{\frac{1}{2}})$ which is equal to u almost everywhere.

We will show now that the local solutions of the critical fractional Burgers equation obtained in Theorem 3.1, are locally unique.

Lemma 3.2 *The solution of the critical fractional Burgers equation satisfying*

$$u \in L^\infty([0,\tau); H^1(I))$$

is locally unique.

Proof. Let $U = u_1 - u_2$, where u_1 and u_2 are the local in time solutions of the critical problem (1) (in the above class) corresponding to the same initial condition u_0. Then U satisfies

$$U_t + u_1 \nabla U + \nabla u_2 U + (-\Delta)^{\frac{1}{2}} U = 0, \quad x \in I \subset \mathbb{R}, \, t > 0,$$
$$U = 0 \text{ on } \partial I,$$
$$U(0,x) = 0.$$

Multiplying the above equation in $L^2(I)$ by U, thanks to the integration by parts, we obtain

$$\frac{d}{dt} \int_I U^2 \, dx + \int_I \nabla u_2 U^2 \, dx + 2 \int_I \left[(-\Delta)^{\frac{1}{4}} U \right]^2 \, dx = 0.$$

From the Hölder and the Nirenberg-Gagliardo inequality the nonlinear term can be transformed as follows:

$$\int_I \nabla u_2 U^2 \, dx \leq \|\nabla u_2\|_{L^2(I)} \|U\|^2_{L^4(I)} \leq \|\nabla u_2\|_{L^2(I)} \|U\|_{L^2(I)} \|U\|_{H^{\frac{1}{2}}(I)}.$$

Since $\|u_2\|_{H^1(I)}$ is bounded, using the Cauchy inequality, we get a differential inequality of the form

$$\begin{aligned} &\frac{d}{dt}\|U(t,\cdot)\|^2_{L^2(I)} \leq c(\|u_2\|_{H^1(I)})\|U(t,\cdot)\|^2_{L^2(I)}, \\ &U(0,x) = 0, \end{aligned}$$

having only a zero solution on $[0, \tau)$.

Theorem 3.2 *The solution of the critical fractional Burgers equation obtained in Theorem 3.1, is unique.*

4 Conclusion

This paper is devoted to the global in time solvability and properties of solutions to the critical problem (1) ($\alpha = \frac{1}{2}$) in a bounded domain I. For this purpose we constructed first the local and then the global in time $X^{\frac{1}{2\alpha}}$ solution u_α of the subcritical fractional Burgers equation ($\alpha \in (\frac{1}{2}, 1]$) in the base spaces $D((-\Delta)^{\frac{s}{2}})$, $s \geq 0$ fixed. Moreover, for small data we obtained a *uniform in* $\alpha \in (\frac{1}{2}, 1]$ estimate of the solutions u_α in $L^\infty(0, T; D((-\Delta)^{\frac{1}{2}}))$ and $L^2(0, T; D((-\Delta)^{\frac{3}{4}}))$, where $T > 0$ is fixed but arbitrarily large. Using the Lions-Aubin compactness lemma, thanks to the above uniform in α estimates, we showed that, for the small data (the condition (17)), the solutions of subcritical problems (1) converge, as $\alpha \to \frac{1}{2}^+$, to the unique solution of the critical problem. For any data, the uniqueness of the solution to the critical problem (1) is an open problem.

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