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Gradient Optimal Control Problems for a Class of Infinite Dimensional Systems

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Abstract: In this work, we address the issue of optimal control for a class of bilinear systems. The goal is to achieve approximately a desired gradient on the whole domain by seeking the minimum of a function. Next, optimization methods are used to reach the desired subregion gradient at time T. The proposed methods are illustrated by a theoretical approach and algorithm.

Keywords: *infinite dimensional systems; gradient problems; regional problems; algorithm.*

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1 Introduction

Infinite-dimensional systems are present in many problems. The analysis of such systems regroup many concepts such as stability, exact controllability, approximate controllability [2, 4, 5]. Nonlinear dynamics is of interest to mathematicians because most systems are nonlinear in nature. The multiplication of state and control in bilinear dynamics make them an important subclass of nonlinear systems, such nonlinearity appears in many dynamical process, for example, a convective-diffusive fluid problem used in [6] to remove a contaminant from water and control of velocity in a Kirchhoff plate, see [4]. Bichiou et al. in [3] treated an approach for the minimum time control of dynamical systems. Alharbi et al. in [1] studied the immune system using vitamins intervention. Regional controllability is a very important generalization referring to the optimal control problems in which the target is studied particularly on a subregion ω.

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An important real situation that requires such notions, arises when the control is required to attain a level of temperature in a specified zone of furnace, see [5]. El Jai et al. give the most important motivation of regional controllability in [5], proving that there exists a system which is regional controllable but not global controllable. Backgrounds in dynamical systems of linear and semi-linear type are established by Zerrik and Ould Sidi in [10] when studying the control of the gradient state of a regional target.

One of the important motivations are the thermal isolation problems, where the control is maintained to reduce the gradient temperature on the boundary. Very interesting developments of this field are found in [7], in particular the characterization of the control achieving gradient controllability.

The partial analysis of bilinear systems was initiated by Zerrik and Ould Sidi in [11,12] and [13]. Using a minimizing sequence, they study the existence of solutions for the problems governed by such systems. Zine and Ould Sidi in [14,15] and Zine in [16] worked on bilinear hyperbolic distributed systems. Ould Sidi in [8] gives necessary conditions for optimal control problems with more regular control functions.

In this work, we address the issue of optimal control for a class of bilinear systems. The goal is to achieve approximately a desired gradient on the whole domain by seeking the minimum of a function. Next, optimization methods help us to reach the desired subregion gradient at time T. The proposed methods are illustrated by a theoretical approach and algorithm.

2 Gradient Optimal Control Problem

We choose an open bounded domain $\Omega \subset \mathbb{R}^n (n \in \{1, 2, 3\})$ and $\partial\Omega$ is its regular boundary. Let $T > 0$ and $\Theta = \Omega \times]0, T[$, $\Sigma = \partial\Omega \times]0, T[$ and $Q \in L^2(0, T; L^2(\Omega))$ be the control. We consider the following bi-linear equation:

$$\begin{cases} \dfrac{\partial z}{\partial t} + \Delta^2 z = Q(x,t)\nabla z, & \Theta, \\ z(x,0) = z_0(x), & \Omega, \\ z = \dfrac{\partial z}{\partial \nu} = 0, & \Sigma. \end{cases} \tag{1}$$

Δ^2 is the bi-Laplace operator, ∇ is the gradient operator defined by

$$\begin{array}{rcl} \nabla : H^1(\Omega) & \longrightarrow & (L^2(\Omega))^n \\ z & \longrightarrow & \nabla z = (\dfrac{\partial z}{\partial x_1}, ..., \dfrac{\partial z}{\partial x_n}). \end{array}$$

The state space is

$$W = \{z \in L^2(0, T; H_0^1(\Omega)) / \frac{\partial z(x,t)}{\partial t} \in L^2(0, T; H^{-2}(\Omega))\}.$$

For $Q \in L^2(0, T; L^2(\Omega))$ and $z_0(x) \in L^2(\Omega)$, from the results in [9] the equation (1) has a unique solution z_Q in $W \cap L^\infty(0, T; L^2(\Omega))$.

The gradient problem of (1) is

$$\min_{Q \in L^2(0,T;L^2(\Omega))} J_\varepsilon(Q). \tag{2}$$

For $\varepsilon > 0$, the gradient quadratic cost J_ε is defined by

$$\begin{aligned} J_\varepsilon(Q) &= \frac{1}{2}\Big\|\nabla z - z^d\Big\|^2_{(L^2(0,T;L^2(\Omega)))^n} + \frac{\varepsilon}{2}\int_\Theta \Big\|Q(x,t)\Big\|^2_{I\!R^n} dxdt \\ &= \frac{1}{2}\sum_{i=1}^{n}\Big\|\frac{\partial z}{\partial x_i} - z_i^d\Big\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\varepsilon}{2}\int_\Theta \Big\|Q(x,t)\Big\|^2_{I\!R^n} dxdt, \end{aligned} \tag{3}$$

where $z^d = (z_1^d, ...z_n^d)$ is the desired gradient in $L^2(0,T;L^2(\Omega))$.

In the literature, quadratic problems such as (1) command the system state to a desired function, see [4,6]. The main objective of our work is to steer the gradient of (1) to $z^d(x)$ minimizing (3), and to characterize the optimal control $Q \in L^2(0,T;L^2(\Omega))$.

3 Solving Method

This section studies the existence and proposes a solution of (2).

Theorem 3.1
There exists $(z^*, Q^*) \in \mathcal{C}([0,T];H_0^1(\Omega)) \times L^2([0,T])$, where z^* is the solution of

$$\begin{cases} \dfrac{\partial z}{\partial t} + \Delta^2 z = Q^*(x,t)\nabla z, & \Theta, \\ z(x,0) = z_0(x), & \Omega, \\ z = \dfrac{\partial z}{\partial \nu} = 0, & \Sigma, \end{cases} \tag{4}$$

and Q^* is the optimal control of (2).

Proof. The set $\{J_\varepsilon(Q) \mid Q \in L^2(0,T;L^2(\Omega))\}$ is a positive nonempty set of $I\!R$, then it admits a lower bound. We choose $(Q_n)_n$ as a minimum such that

$$J^* = \lim_{n\to+\infty} J(Q_n) = \inf_{Q\in L^2(0,T;L^2(\Omega))} J_\varepsilon(Q).$$

$J_\varepsilon(Q_n)$ is then bounded, it follows that $||Q_n||_{L^2(0,T;L^2(\Omega))} \le C$, for a positive constant C. Using lemma in [16], we can deduce that

$$\begin{array}{ll} Q_n \rightharpoonup Q^*, & L^2(0,T;L^2(\Omega)), \\ z_n \rightharpoonup z^*, & W, \\ \Delta^2 z_n \rightharpoonup \chi, & W, \\ Q_n\nabla z_n \rightharpoonup \Lambda, & W, \\ \dfrac{\partial z_n(x,t)}{\partial t} \rightharpoonup \Psi, & W. \end{array} \tag{5}$$

The limit in $\dfrac{\partial z_n(x,t)}{\partial t} + \Delta^2 z_n = Q_n\nabla z_n$, we get $\dfrac{\partial z^*(x,t)}{\partial t} = \Psi$.

The linearity of the operator $z \mapsto \Delta^2 z$ and the operator ∇ gives $\Delta^2 z^* = \chi$ and $Q^*\nabla z^* = \Lambda$. Hence we obtain

$$\frac{\partial z^*}{\partial t} + \Delta^2 z^* = Q^*(x,t)\nabla z^*.$$

We use the lower semi-continuity of $J_\varepsilon(Q)$:

$$\begin{aligned} J_\varepsilon(Q^*) &= \inf_n \sum_{i=1}^{n}\frac{1}{2}\int_0^T\int_\Omega (\frac{\partial z_n}{\partial x_i} - z_i^d)^2 dx + \frac{\varepsilon}{2}\int_\Theta \Big\|Q_n(t)\Big\|^2_{I\!R^n} dxdt \\ &\le \lim_{n\longrightarrow\infty} J_\varepsilon(Q_n) = \inf_Q J_\varepsilon(Q). \end{aligned} \tag{6}$$

Thus Q^* is a solution of (2). To characterize the solution of problem (2), we study the differential of cost $J_\varepsilon(Q)$

Lemma 3.1 *For the map*

$$\begin{array}{ccc} L^2(0,T;L^2(\Omega)) & \longrightarrow & \mathcal{C}(0,T;H^1(\Omega)), \\ Q & \longrightarrow & z(Q) \end{array}$$

the solution of (4) is differentiable and its differential ψ verifies the system

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = -\Delta^2\psi(x,t) + Q^*(x,t)\nabla\psi + h(x,t)\nabla z, & \Theta, \\ \psi(x,0) = \psi_0(x) = 0, & \Omega, \\ \psi = \dfrac{\partial \psi}{\partial \nu} = 0, & \Sigma, \end{cases} \tag{7}$$

with $z^* = z(Q^*)$, $h \in U$, and $d(z(Q^*))h$ is the differential of $Q \to z(Q)$.

Proof. The solution of the equation (7) verifies

$$||\psi||_W \leq k_1 ||z||_{L^\infty(0,T;H_0^1(\Omega))} ||h||_{L^2(0,T;L^2(\Omega))}.$$

Also,

$$||\psi'||_W \leq k_2 ||z^*||_{L^\infty(0,T;H_0^1(\Omega))} ||h||_{L^2(0,T;L^2(\Omega))}.$$

Thus,

$$||\psi||_{\mathcal{C}([0,T];H_0^1(\Omega))} \leq k_3 ||h||_{L^2(0,T;L^2(\Omega))}.$$

Consequently, we have $h \in L^2(0,T;L^2(\Omega)) \to \psi \in \mathcal{C}((0,T);H_0^1(\Omega))$ is bounded, see [12]. Let $z_h = z(Q^* + h)$ and $\varphi = z_h - z^*$, then φ verifies

$$\begin{cases} \dfrac{\partial \varphi(x,t)}{\partial t} = -\Delta^2\varphi + Q^*(x,t)\nabla\varphi(x,t) + h(x,t)\nabla z_h, & \Theta, \\ \varphi(x,0) = \varphi_0(x) = 0, & \Omega, \\ \varphi = \dfrac{\partial \varphi}{\partial \nu} = 0, & \Sigma. \end{cases} \tag{8}$$

Thus

$$||\varphi||_{L^\infty([0,T];H_0^1(\Omega))} \leq k_4 ||h\;||_{L^2(0,T;L^2(\Omega))}\;.$$

Put $\phi = \varphi - \psi$ which verifies the system

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = -\Delta^2\phi + Q^*(x,t)\nabla\phi(x,t) + h(x,t)\nabla\varphi, & \Theta, \\ \phi(x,0) = 0, & \Omega, \\ \phi = \dfrac{\partial \phi}{\partial \nu} = 0, & \Sigma. \end{cases} \tag{9}$$

$\phi \in \mathcal{C}(0,T;H_0^1(\Omega))$, and we have

$$||\phi||_{\mathcal{C}([0,T];H_0^1(\Omega))} \leq k ||h||^2_{L^2(0,T;L^2(\Omega))}.$$

Consequently,

$$||z(Q^* + h) - z(Q^*) - d(z(Q^*))h||_{\mathcal{C}(0,T;H_0^1(\Omega))} = ||\phi||_{\mathcal{C}([0,T];H_0^1(\Omega))} \leq k ||h||^2_{L^2(0,T;L^2(\Omega))}.$$

Next, we consider the family of optimality systems

$$\begin{cases} -\dfrac{\partial p_i}{\partial t} = -\Delta^2 p_i - Q_\varepsilon^*(x,t)\nabla p_i + (\dfrac{\partial z}{\partial x_i} - z_i^d), & \Theta, \\ p_i(x,T) = 0, & \Omega, \\ p_i = \dfrac{\partial p_i}{\partial \nu} = 0, & \Sigma. \end{cases} \tag{10}$$

The next result gives the differential of $J_\varepsilon(Q)$.

Lemma 3.2 *For $Q_\varepsilon \in L^2(0,T;L^2(\Omega))$, which is the solution of the problem (2), we have*

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon+\beta h) - J_\varepsilon(Q_\varepsilon)}{\beta} &= \sum_{i=1}^n \int_\Omega \int_0^T \frac{\partial \psi(x,t)}{\partial x_i}(\frac{\partial z}{\partial x_i} - z_i^d) dt dx \\ &+ \varepsilon \int_\Omega \int_0^T h Q_\varepsilon dt dx. \end{aligned} \tag{11}$$

Proof. The functional $J_\varepsilon(Q_\varepsilon)$ from (3) can be written in the following form

$$J_\varepsilon(Q_\varepsilon) = \frac{1}{2}\sum_{i=1}^n \int_\Omega \int_0^T (\frac{\partial z}{\partial x_i} - z_i^d)^2 dt dx + \frac{\varepsilon}{2}\int_\Omega \int_0^T Q_\varepsilon^2(t) dt dx. \tag{12}$$

Let $z_\beta = z(Q_\varepsilon+\beta h)$ and $z = z(Q_\varepsilon)$, from (12) we deduce

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon+\beta h) - J_\varepsilon(Q_\varepsilon)}{\beta} &= \lim_{\beta\longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\Omega \int_0^T \frac{(\frac{\partial z_\beta}{\partial x_i} - z_i^d)^2 - (\frac{\partial z}{\partial x_i} - z_i^d)^2}{\beta} dt dx \\ &+ \lim_{\beta\longrightarrow 0} \frac{\varepsilon}{2}\int_\Omega \int_0^T \frac{(Q_\varepsilon+\beta h)^2 - Q_\varepsilon^2}{\beta}(t) dt dx. \end{aligned} \tag{13}$$

Thus

$$\begin{aligned} &\lim_{\beta\longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon+\beta h) - J_\varepsilon(Q_\varepsilon)}{\beta} \\ &= \lim_{\beta\longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\Omega \int_0^T \frac{(\frac{\partial z_\beta}{\partial x_i} - \frac{\partial z}{\partial x_i})}{\beta}(\frac{\partial z_\beta}{\partial x_i} + \frac{\partial z}{\partial x_i} - 2z_i^d) dt dx \\ &+ \lim_{\beta\longrightarrow 0} \int_\Omega \int_0^T (\varepsilon h Q_\varepsilon + \beta\varepsilon h^2) dt dx \end{aligned} \tag{14}$$

$$= \sum_{i=1}^n \int_\Omega \int_0^T \frac{\partial \psi(x,t)}{\partial x_i}(\frac{\partial z(x,t)}{\partial x_i} - z_i^d) dt dx + \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dt dx.$$

We characterize the solution of (2) by the following theorem.

Theorem 3.2 *If $Q_\varepsilon \in L^2(0,T;L^2(\Omega))$ and $z_\varepsilon = z(Q_\varepsilon)$ is the output of (1), then*

$$Q_\varepsilon(t) = \frac{-1}{\varepsilon}(\nabla z^*(x,t))(Div(p)) \tag{15}$$

is the solution of (2), where $p = (p_1 .... p_n)$ and $p_i \in C([0,T];H_0^1(\Omega))$ is the unique solution of (10).

Proof. Let $h \in L^{\infty}(0,T;L^2(\Omega))$ and $Q_\varepsilon + \beta h \in L^2(0,T;L^2(\Omega))$ for $\beta > 0$. The extremal of J_ε is achieved at Q_ε, then

$$0 \leq \lim_{\beta \longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon + \beta h) - J_\varepsilon(Q_\varepsilon)}{\beta}. \tag{16}$$

From Lemma (3.2), we deduce

$$\begin{aligned} 0 &\leq \lim_{\beta \longrightarrow 0} \frac{Q_\varepsilon(u_\varepsilon + \beta h) - Q_\varepsilon(u_\varepsilon)}{\beta} \\ &= \sum_{i=1}^{n} \int_\Omega \int_0^T \frac{\partial \psi(x,t)}{\partial x_i}(\frac{\partial z(x,t)}{\partial x_i} - z_i^d)dtdx + \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dtdx, \end{aligned} \tag{17}$$

and using the system (10), we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} \int_\Omega \int_0^T \frac{\partial \psi(x,t)}{\partial x_i}(-\frac{\partial p_i(x,t)}{\partial t} \\ &+ \Delta^2 p_i(x,t) + Q_\varepsilon^*(x,t)\nabla p_i(x,t))dtdx + \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dtdx \\ &= \sum_{i=1}^{n} \int_\Omega \int_0^T \frac{\partial}{\partial x_i}(\frac{\partial \psi}{\partial t} + \Delta^2 \psi - Q_\varepsilon^*(x,t)\nabla \psi)p_i(x,t)dtdx + \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dtdx \\ &= \sum_{i=1}^{n} \int_\Omega \int_0^T \frac{\partial}{\partial x_i}(h(x,t)\nabla z)p_i dtdx + \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dtdx \\ &= \int_\Omega \int_0^T h(x,t)[\nabla z(\sum_{i=1}^{n} \frac{\partial p_i(x,t)}{\partial x_i}) + \varepsilon Q_\varepsilon dtdx]. \end{aligned} \tag{18}$$

For h = h(t), an arbitrary control with $Q_\varepsilon + \beta h \in L^2(0,T;L^2(\Omega))$, and all small β, we deduce

$$Q_\varepsilon(t) = \frac{-1}{\varepsilon}(\nabla z)(\sum_{i=1}^{n} \frac{\partial p_i}{\partial x_i}) = \frac{-1}{\varepsilon}(\nabla z)(Div(p)). \tag{19}$$

4 Regional Gradient Optimal Control Problem

For $\omega \in \Omega$, we define the restriction operator to ω by

$$\begin{array}{rcl} \chi_\omega : (L^2(\Omega))^n & \longrightarrow & (L^2(\omega))^n \\ z & \longrightarrow & \chi_\omega z = z|_\omega. \end{array}$$

The adjoint of χ_ω is defined by

$$\chi_\omega^* z = \begin{cases} z \ in \ \Omega, \\ 0 \in \Omega \setminus \omega, \end{cases}$$

and

$$\begin{array}{rcl} \widetilde{\chi}_\omega : (L^2(\Omega)) & \longrightarrow & (L^2(\omega)) \\ z & \longrightarrow & \widetilde{\chi}_\omega z = z|_\omega. \end{array}$$

Definition 4.1 A system state is said to be weakly partial gradient controllable on $\omega \subset \Omega$ if for $\forall\ \varepsilon > 0$, we can find a control $Q \in L^2(0,T;L^2(\Omega))$ such that

$$||\chi_\omega \nabla z_Q(T) - z^d||_{(L^2(\omega))^n} \leq \varepsilon,$$

where $z^d = (z_1^d, ..., z_n^d)$ is the desired gradient in $(L^2(\omega))^n$.

Let us consider the partial gradient control problem

$$\min_{Q\ \in L^2(0,T;L^2(\Omega))} J_\varepsilon(Q), \tag{20}$$

where the regional gradient quadratic cost J_ε is defined by

$$\begin{aligned} J_\varepsilon(Q) &= \frac{1}{2}\left\|\chi_\omega \nabla z(T) - z^d\right\|^2_{(L^2(\omega))^n} + \frac{\varepsilon}{2}\int_\Theta \left\|Q(x,t)\right\|^2_{I\!R^n} dxdt \\ &= \frac{1}{2}\sum_{i=1}^n \left\|\widetilde{\chi}_\omega \frac{\partial z(T)}{\partial x_i} - z_i^d\right\|^2_{(L^2(\omega))} + \frac{\varepsilon}{2}\int_\Theta \left\|Q(x,t)\right\|^2_{I\!R^n} dxdt. \end{aligned} \tag{21}$$

Next, we consider the family of optimality systems

$$\begin{cases} -\dfrac{\partial p_i}{\partial t} = -\Delta^2 p_i - Q_\varepsilon^*(x,t)\nabla p_i, & \Theta, \\ p_i(x,T) = (\dfrac{\partial z(T)}{\partial x_i} - \widetilde{\chi}_\omega^* z_i^d), & \Omega, \\ p_i(x,t) = \dfrac{\partial p_i(x,t)}{\partial \nu} = 0, & \Sigma. \end{cases} \tag{22}$$

Lemma 4.1 *If $Q_\varepsilon \in L^2(0,T;L^2(\Omega))$ is the optimal control solution of (20), p_i is the solution of (22), and ψ is the solution of (7), we have*

$$\begin{aligned} &\lim_{\beta \longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon + \beta h) - J_\varepsilon(Q_\varepsilon)}{\beta} \\ = &\sum_{i=1}^n \int_\Omega \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega \left[\int_0^T \frac{\partial p_i}{\partial t}\frac{\partial \psi(x,t)}{\partial x_i} dt + \int_0^T p_i \frac{\partial}{\partial x_i}(\frac{\partial \psi}{\partial t})dt\right] dx \\ + &\int_\Omega \int_0^T \varepsilon h Q_\varepsilon dtdx. \end{aligned} \tag{23}$$

Proof. The quadratic cost $J_\varepsilon(Q_\varepsilon)$ defined by (3), can be written in the following form:

$$J_\varepsilon(Q_\varepsilon) = \frac{1}{2}\sum_{i=1}^n \int_\omega (\widetilde{\chi}_\omega \frac{\partial z}{\partial x_i} - z_i^d)^2 dx + \frac{\varepsilon}{2}\int_\Omega \int_0^T Q_\varepsilon^2(t) dtdx. \tag{24}$$

Let $z_\beta = z(Q_\varepsilon + \beta h)$ and $z = z(Q_\varepsilon)$, using (24), we have

$$\begin{aligned} \lim_{\beta \longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon + \beta h) - J_\varepsilon(Q_\varepsilon)}{\beta} &= \lim_{\beta \longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\omega \frac{(\widetilde{\chi}_\omega \frac{\partial z_\beta}{\partial x_i} - z_i^d)^2 - (\widetilde{\chi}_\omega \frac{\partial z}{\partial x_i} - z_i^d)^2}{\beta} dx \\ &+ \lim_{\beta \longrightarrow 0} \frac{\varepsilon}{2}\int_\Omega \int_0^T \frac{(Q_\varepsilon + \beta h)^2 - Q_\varepsilon^2}{\beta} dtdx. \end{aligned} \tag{25}$$

Consequently,

$$\begin{aligned}
&\lim_{\beta\longrightarrow 0}\frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta}\\
=\;& \lim_{\beta\longrightarrow 0}\sum_{i=1}^{n}\frac{1}{2}\int_\omega \widetilde{\chi}_\omega\frac{(\frac{\partial z_\beta}{\partial x_i}-\frac{\partial z}{\partial x_i})}{\beta}(\widetilde{\chi}_\omega\frac{\partial z_\beta}{\partial x_i}+\widetilde{\chi}_\omega\frac{\partial z}{\partial x_i}-2z_i^d)dx\\
+\;& \frac{1}{2}\int_\Omega\int_0^T(2\varepsilon hQ_\varepsilon+\beta\varepsilon h^2)dtdx \qquad (26)\\
=\;& \sum_{i=1}^{n}\int_\omega \widetilde{\chi}_\omega\frac{\partial\psi(x,T)}{\partial x_i}\widetilde{\chi}_\omega(\frac{\partial z(x,T)}{\partial x_i}-\widetilde{\chi}_\omega^* z_i^d)dx+\int_\Omega\int_0^T\varepsilon hQ_\varepsilon dtdx\\
=\;& \sum_{i=1}^{n}\int_\omega \widetilde{\chi}_\omega\frac{\partial\psi(x,T)}{\partial x_i}\widetilde{\chi}_\omega p_i(x,T)dx+\varepsilon\int_\Omega\int_0^T hQ_\varepsilon dtdx.
\end{aligned}$$

From (10) and (26), we deduce that

$$\begin{aligned}
&\lim_{\beta\longrightarrow 0}\frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta}\\
=\;& \sum_{i=1}^{n}\int_\Omega \widetilde{\chi}_\omega^*\widetilde{\chi}_\omega\left[\int_0^T\frac{\partial p_i}{\partial t}\frac{\partial\psi(x,t)}{\partial x_i}dt+\int_0^T p_i\frac{\partial}{\partial x_i}(\frac{\partial\psi}{\partial t})dt\right]dx \qquad (27)\\
+\;& \int_\Omega\int_0^T\varepsilon hQ_\varepsilon dtdx.
\end{aligned}$$

Theorem 4.1 *Let $Q_\varepsilon\in L^2(0,T;L^2(\Omega))$ and $z_\varepsilon=z(Q_\varepsilon)$ be the associated state solution of (1), we have*

$$Q_\varepsilon(t)=\frac{-1}{\varepsilon}\sum_{i=1}^{n}\widetilde{\chi}_\omega\frac{\partial z(x,t)}{\partial x_i}\widetilde{\chi}_\omega p_i(t) \qquad (28)$$

is a solution of the problem (20).

Proof. Choose $h\in L^\infty(0,T,L^2(\Omega))$ with $Q_\varepsilon+\beta h\in L^2(0,T;L^2(\Omega))$ for $\beta>0$. The critical point of J_ε is Q_ε, we have

$$0\le\lim_{\beta\longrightarrow 0}\frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta}. \qquad (29)$$

Using Lemma (4.1) and replacing $\frac{\partial \psi}{\partial t}$ in the system (7), we have

$$
\begin{aligned}
0 &\leq \lim_{\beta \longrightarrow 0} \frac{Q_\varepsilon(z_\varepsilon + \beta h) - Q_\varepsilon(z_\varepsilon)}{\beta} \\
&= \sum_{i=1}^{n} \int_\Omega \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega \left[\int_0^T \frac{\partial \psi}{\partial x_i} \frac{\partial p_i}{\partial t} dt \right] dx \\
&+ \sum_{i=1}^{n} \int_\Omega \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega \left[\int_0^T (-\Delta^2 \frac{\partial \psi}{\partial x_i} + Q_\varepsilon(t) \nabla \frac{\partial \psi}{\partial x_i} + \frac{\partial}{\partial x_i}(h(t)\nabla z)) p_i dt \right] dx \\
&+ \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dt dx.
\end{aligned} \tag{30}
$$

Using (22), we obtain

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} \int_\Omega \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega \left[\int_0^T \frac{\partial \psi}{\partial x_i} (\frac{\partial p_i}{\partial t} - \Delta^2 p_i - Q(t) \nabla p_i) dt + \frac{\partial}{\partial x_i}(h(t)\nabla z) p_i dt \right] dx \\
&+ \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dt dx. \\
&= \sum_{i=1}^{n} \int_\Omega \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega \int_0^T (h(t)\nabla z) \frac{\partial p_i}{\partial x_i} dt dt + \int_\Omega \int_0^T \varepsilon h Q_\varepsilon dt dx \\
&= \int_\Omega \int_0^T h(t) \left[\nabla z \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega \sum_{i=1}^{n} \frac{\partial p_i}{\partial x_i} + \varepsilon h Q_\varepsilon \right] dt dx.
\end{aligned} \tag{31}
$$

We deduce the characterization

$$
Q_\varepsilon(t) = \frac{-1}{\varepsilon} (\widetilde{\chi}_\omega \nabla z)(\widetilde{\chi}_\omega Div(p)). \tag{32}
$$

5 Numerical Approach

We choose the following one dimensional equation:

$$
\begin{cases}
\frac{\partial z}{\partial t} + \frac{\partial^2 z}{\partial x^2} = Q(t) \frac{\partial z}{\partial x}, & [0,1], \\
z(x,0) = z_0(x), & [0,1], \\
z = 0, & \text{at } x = 0, 1.
\end{cases} \tag{33}
$$

Such transport equation may describe the concentration of a contaminant in a convective-diffusive problem, see [6]. The optimal control Q_n is calculated by choosing $\varepsilon = \frac{1}{n}$ and

$$
\begin{cases}
Q_{n+1}(t) = -n(\widetilde{\chi}_\omega \nabla z_n)(\widetilde{\chi}_\omega Div(p_n)), \\
Q_0 = 0,
\end{cases} \tag{34}
$$

where p_n is the output of

$$\begin{cases} \dfrac{\partial p_n(x,t)}{\partial t} = \dfrac{\partial^2 p_n(x,t)}{\partial x^2} + Q_n(t)\dfrac{\partial p_n}{\partial x}(x,t), & [0,1], \\ p_n(x,T) = (\dfrac{\partial z_Q(T)}{\partial x} - \widetilde{\chi}_\omega^* z^d(x)), & [0,1], \\ p = 0, \text{ at } x = 0,1. \end{cases} \tag{35}$$

The optimal control (34) is a bounded sequence deduced from Theorem 4.1, which allows us to establish the following algorithm:

Step 1 : Initializing of the considered problem

Time T.
Desired function z^d.
Error ε.
Subregion ω.

Step 2 : While $\| Q_{n+1} - Q_n \| \leq \varepsilon$

Find z_n and $\dfrac{\partial z_n}{\partial x}(T)$ solution of (33).
Find $p_n(t)$ solution of (35).
Find the control Q_{n+1} by (34).

Step 3 : The solution of the problem (20) is Q_n verifying $\| Q_{n+1} - Q_n \| \leq \varepsilon$.

6 Open Problems

The coupled systems are a very important class of bilinear systems. One of the important applications of such systems are the predator-prey models, which are a couple of nonlinear differential equations used to describe the interaction between two species, one as a predator and the other as a prey. Consider the following important regional control problem of the predator-prey system:

$$\min_{Q \in L^2(0,T;L^2(\Omega))} J_\varepsilon(Q). \tag{36}$$

The cost J_ε is defined for $\varepsilon > 0$ by

$$J_\varepsilon(Q) = \frac{1}{2}\left\|\widetilde{\chi}_\omega y(T) - \widetilde{\chi}_\omega z(T)\right\|^2_{L^2(\omega)} + \frac{\varepsilon}{2}\left\|Q(x,t)\right\|^2_{L^2(0,T;L^2(\Omega))} \tag{37}$$

and constrained by the model

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta^2 y - Q(x,t)\dfrac{\partial z}{\partial x}, & \Theta, \\ \dfrac{\partial z}{\partial t} = \Delta^2 z - Q(x,t)\dfrac{\partial y}{\partial x}, & \Theta, \\ y(x,0) = y_0(x), z(x,0) = z_0(x), & \Omega. \\ y = z = 0, & \Sigma. \end{cases} \tag{38}$$

This important problem of regional optimal control is still under consideration.

7 Conclusion

This work proposes a solution for the gradient optimal control problem governed by an infinite dimensional bilinear system. The approach gives a respond to many open nonlinear problems, for example, the control problems governed by bilinear coupled systems.

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