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Sinc-Galerkin Method for Solving Higher Order Fractional Boundary Value Problems

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Abstract: In this work we use the sinc-Galerkin method to solve higher order fractional boundary value problems. We estimate the second order fractional derivative in the Caputo sense. More precisely, we find a numerical solution for

$$g_1(t)D^{\alpha}u(t) + g_2(t)D^{\beta}u(t) + p(t)u^{(4)}(t) + q(t)u(t) = f(t),$$

$$0 < t < 1, \quad 0 < \beta < 1, \quad 1 < \alpha < 2,$$

subject to the boundary conditions $u(0) = 0, \;\; u'(0) = 0, \;\; u(1) = 0, \;\; u'(1) = 0$. Our contribution appears in the estimate of $D^{\alpha}u$ for higher order α. Numerical examples are described to show the accuracy of this attempt where we applied the sinc-Galerkin method for fractional order differential equations with singularities.

Keywords: *higher order fractional boundary value problems; Caputo derivative; sinc-Galerkin method; numerical solution.*

Mathematics Subject Classification (2010): 34K37, 35A35, 45D05, 65M70.

1 Introduction

Boundary value problems come into view in many areas of science, engineering, and economy. One of the physical modelings for boundary value problems is to suppose a finite length elastic beam, which is fixed at one end, and rested on an elastic bearing at the other end. We may add along its length a load to cause deformations, see [1]. In this work we solve a more general model which has mechanical interpretation that involves higher order fractional derivatives.

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Several papers discussed the numerical solution of boundary value problems [2–5]. The existence and uniqueness of solutions of such problems are covered in [6–9]. Recently, fractional boundary value problems have been of interest to many mathematicians and scientists, see [10–13]. The present work is motivated by the desire to obtain numerical solutions to the general higher-order fractional boundary value problem of the form

$$\mathbf{L}u : g_1(t)D^{\alpha}u(t) + g_2(t)D^{\beta}u(t) + p(t)u^{(4)}(t) + q(t)u(t) = f(t), \tag{1}$$

$$0 < t < 1, \quad 0 < \beta < 1, \quad 1 < \alpha < 2,$$

subject to the conditions

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \tag{2}$$

where the notation $y^{(4)}(t)$ stands for the $4th$ derivative of $y(t)$ and $D^{\alpha}u$ is the Caputo fractional derivative.

Definition 1.1 For $f : [a, b] \to \mathbb{R}$ and $n - 1 < \alpha < n$, the left Caputo fractional derivative of order α is

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha+1} \frac{d^n}{d\tau^n} f(\tau)d\tau$$

and the right Riemann-Liouville fractional derivative of order α is

$$D_R^{\alpha}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_t^b (t-\tau)^{n-\alpha+1} f(\tau)d\tau.$$

The relation between the left Caputo and right Riemann-Liouville fractional derivatives is given in the following integration by parts formula:

$$\int_a^b g(t)D^{\alpha}f(t)dt = \int_a^b f(t)D_R^{\alpha}g(t)dt + \sum_{k=0}^{n-1} D_R^{\alpha-n+k}g(t)\frac{d^{n-k-1}}{dt^{n-k-1}}f(t)\Big|_a^b. \tag{3}$$

One of the most common numerical methods to solve differential equations is the sinc-Galerkin method (SGM). In [14], the authors applied the SGM to solve the general case for the linear fourth-order boundary value problems. For more details about the SGM, see [15–17]. In [18], the author applied the sinc-Galerkin method to solve first order fractional differential equations, in our work we generalize the case to provide a good approximation for higher order fractional boundary value problems. One of the important benefits of the sinc-Galerkin method is dealing with the singularities occurred at the boundaries as we will see in the provided examples. For more details about sinc solutions of analytic problems with singularities, see [19].

The outline of this paper is as follows. In Section 2, we introduce basic definitions and results of the sinc-Galerkin method to formulate the discrete system. Section 3 is devoted to the proof of our main result on the discrete system that is obtained by implementing the sinc-Galerkin method to construct a numerical solution. In Section 4, we demonstrate the accuracy of our suggested scheme by presenting two concrete numerical examples, where the exact solutions are explicitly given. The conclusion is drawn in the last section.

2 The Sinc Function and the Quadrature Formula

The quadrature formula is the main recipe for this paper. It is a rule that approximates integral of some class of functions using the sinc function (see [19, 20]).

The *sinc* function is defined on the whole complex plane $\mathbb{C}$ by

$$\operatorname{sinc}(z) \equiv \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{4}$$

For $h > 0$ and $k = 0, \pm 1, \pm 2, ...$, the translated sinc function with evenly spaced nodes is given by

$$S(k,h)(z) \equiv \operatorname{sinc}\left(\frac{z - kh}{h}\right). \tag{5}$$

One of the important results on the sinc function is the orthogonality relation

$$\frac{1}{h}\int_{-\infty}^{\infty} S(k,h)(x)S(j,h)(x)dx = \delta_{kj} := \begin{cases} 1, & k = j, \\ 1, & k \neq j. \end{cases} \tag{6}$$

This implies that for any $f \in B(h)$, where $B(h)$ is the Paley-Wiener space (see [19]), we have

$$f(x) = \sum_{k=-\infty}^{\infty} f(hk)S(k,h)(x). \tag{7}$$

The sinc-Galerkin method is originally designed to solve ODEs on the infinite domain $(-\infty, \infty)$. To solve the boundary value problem on the domain (T_1, T_2), we introduce the conformal mapping ϕ that sends (T_1, T_2) onto $(-\infty, \infty)$:

$$\phi(T_1, T_2; z) = \log(\frac{z - T_1}{T_2 - z}), \tag{8}$$

which maps the eye-shaped domain in the $z-$plane $\mathcal{D}_E$ onto the infinite strip in the $w-$plane, $\mathcal{D}_S$, where

$$\begin{aligned} \mathcal{D}_E &= \{z = x + iy : |arg(\frac{z - T_1}{T_2 - z})| < d \leq \pi/2\}, \\ \mathcal{D}_S &= \{w = u + iv : |v| < d \leq \pi/2\}. \end{aligned}$$

We obtain the basis functions

$$S_k(T_1, T_2; t) := S(k,h) \circ \phi(T_1, T_2; t) = \operatorname{sinc}[\frac{\phi(T_1, T_2; t) - kh}{h}] \tag{9}$$

over the interval $t \in (T_1, T_2)$. We will use $S_k(t)$ for $S_k(0, 1 : t)$.

To discretize our proposed BVP we use the mesh size h which is the mesh size in $\mathcal{D}_E$ for the uniform grids $\{kh\}$, $-\infty < k < \infty$. Using the conformal mapping ϕ, one can obtain the sinc grid points $t_k \in (T_1, T_2)$ under the action of the inverse image of ϕ :

$$t_k = \phi^{-1}(T_1, T_2; kh) = \frac{e^{kh} + T_1}{T_2 + e^{kh}}.$$

Now we define a class of functions in which the quadrature formula is applied (for more details, see [19, 20]).

Definition 2.1 Let $\phi : \mathcal{D}_E \longrightarrow \mathcal{D}_S$ be a conformal mapping of $\mathcal{D}_E$ onto $\mathcal{D}_S$ with inverse ψ. Let $\Gamma = \{\psi(u) \in \mathcal{D}_E : -\infty < u < \infty\} = (T_1, T_2)$. Then $B(\mathcal{D}_E)$ is the class of functions F which are analytic in $B(\mathcal{D}_E)$ and satisfy

$$\int_{\psi(t+L)} |F(z)|dz \longrightarrow 0, \quad t \to \pm\infty,$$

where $L = \{iv : |v| < d \leq \pi/2\}$, and on the boundary of $\mathcal{D}_E$, denoted by $\partial\mathcal{D}_E$, satisfy

$$\mathcal{N}(F) = \int_{\partial\mathcal{D}_E} |F(z)|\, dz < \infty.$$

For the choice of the conformal map $\phi(T_1, T_2; z) = \log(\frac{z-T_1}{T_2-z})$ one can state the quadrature formula as follows.

Theorem 2.1 *Let $F \in B(\mathcal{D}_E)$ and ϕ be defined as in 8 such that*

$$|(T_2 - z)(z - T_1)F(z)| \leq C \exp(-\alpha|\phi(z)|)$$

for some $C > 0$, $\alpha > 0$. Choose $N = M + 1, h = \sqrt{2\pi d/(\alpha M)}$. Then the sinc trapezoidal quadrature formula is

$$\int_0^1 F(z)dz \approx h \sum_{j=-M}^{N} \frac{F(z_j)}{\phi'(z_j)} \tag{10}$$

with the exponential order error $\mathcal{O}(\exp(-\sqrt{2\pi d\alpha M}))$.

3 SGM Approach

In this section, we present the SGM method to discretize the fourth-order fractional differential equation for $1 < \alpha < 2$ and $0 < \beta < 1$,

$$\mathbf{L}u := g_1(t)D^\alpha u(t) + g_2(t)D^\beta u(t) + p(t)u^{(4)}(t) + q(t)u(t) = f(t), \quad 0 < t < 1, \tag{11}$$

subject to the conditions

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = 0. \tag{12}$$

The completeness of the orthogonal system $\{S_k\}$, which is defined in (9), ensures that we can suggest the approximate solution of the form

$$u_m(t) = \sum_{k=-N_1}^{N_2} c_k S_k(t), \quad m = N_1 + N_2 + 1, \tag{13}$$

where c_k are the expansion coefficients which will be determined. These coefficients are determined by the orthogonality property of the basis functions $\{S_k\}_{k=-N_1}^{N_2}$. The orthogonality yields the discrete system

$$< \mathbf{L}u_m - f, S_k >= 0, \quad -N_1 \leq k \leq N_2. \tag{14}$$

One can use the linearity of the inner product to simplify (14) in the form

$$\begin{aligned} < g_1 D^{\alpha} u, S_k > + < g_2 D^{\beta} u, S_k > + < p\ u^{(4)}, S_k > + < q\ u, S_k > \\ - < f, S_k > = 0, \quad -N_1 \le k \le N_2. \end{aligned} \tag{15}$$

As we mentioned in the previous section, we use the weighted inner product

$$< u(t), v(t) > = \int_0^1 u(t) v(t) w(t) dt,$$

where the weight is

$$w(t) = (1-x)^2 x^2. \tag{16}$$

For the general choice of the weight function see [19], where the author suggested

$$w(t) = 1/(\phi'(t))^m$$

for the general case. In the case of higher order derivatives $D^{\alpha} u$ and $\frac{d^m}{dt^m} u$, we suggest the weight

$$w(t) = 1/(\phi'(t))^{N_0},$$

where

$$N_0 \le \frac{\max\{n, m\} + 1}{2} < N_0 + 1.$$

The First Term. We estimate the term $< g_1 D^{\alpha} u, S_k >$ for $1 < \alpha < 2$. We use the integral by parts formula (3) and the boundary conditions $u(0) = u(1) = u'(0) = u'(1) = 0$ to find

$$\begin{aligned} < g_1 D^{\alpha} u, S_k > &= \int_0^1 g_1(t) D^{\alpha}[u(t)] S_k(t) w(t) dt \\ &= \int_0^1 u(t) D_R^{\alpha}[S_k(t) g_1(t) w(t)] dt + \sum_{k=0}^{1} D_R^{\alpha-2+k}[S_k(t) g_1(t) w(t)] \frac{d^{1-k}}{dt^{1-k}} u(t) \Big|_0^1 \\ &= \int_0^1 u(t) D_R^{\alpha}[S_k(t) g_1(t) w(t)] dt \\ &= \int_0^1 \left(u(t) \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_t^1 (\tau - t)^{1-\alpha} S_k(\tau) g_1(t) w(\tau) d\tau \right) dt. \end{aligned}$$

Using the quadrature formula 10, one can write

$$\int_t^1 (\tau - t)^{1-\alpha} S_k(\tau) g_1(t) w(\tau) d\tau \approx h \sum_{r=-N_1}^{N_2} \frac{(\tau_r - t)^{1-\alpha} S_k(\tau_r) g_1(\tau_r) w(\tau_r)}{\phi_t'(\tau_r)},$$

where $h = \pi/\sqrt{N_2}$, and $\tau_k = \phi^{-1}(t, 1; kh)$.

It follows that

$$< g_1 D^{\alpha} u, S_k > \approx \int_0^1 u(t) \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \left(h \sum_{r=-N_1}^{N_2} \frac{(\tau_r - t)^{1-\alpha} S_k(\tau_r) w(\tau_r)}{\phi_t'(\tau_r)} \right) dt.$$

Another use of the quadrature formula (10) for $t_k = \phi^{-1}(0,1;kh)$ yields

$$< g_1 D^\alpha u, S_j >\approx \frac{h^2}{\Gamma(2-\alpha)} \sum_{k=-N_1}^{N_2} \sum_{r=-N_1}^{N_2} \frac{u(t_k)}{\phi'(0,1;t_k)} \frac{d^2}{dt^2} \left(\frac{(\tau_r - t)^{1-\alpha} S_j(\tau_r) g_1(\tau_r) w(\tau_r)}{\phi'(t,1;\tau_r)} \right)\Bigg|_{t=t_k}$$

for $j = -N, ..., N$.

The Second Term. The estimation of the second term is done similarly to that of the first term with the change for $1 < \alpha < 2$ by $0 < \beta < 1$. Hence, the same computations show that

$$< g_2 D^\beta u, S_j >\approx \frac{-h^2}{\Gamma(1-\beta)} \sum_{k=-N_1}^{N_2} \sum_{r=-N_1}^{N_2} \frac{u(t_k)}{\phi'(0,1;t_k)} \frac{d}{dt} \left(\frac{(\tau_r - t)^{1-\beta} S_j(\tau_r) g_2(\tau_r) w(\tau_r)}{\phi'(t,1;\tau_r)} \right)\Bigg|_{t=t_k}$$

for $j = -N, ..., N$.

The Third Term. Now we estimate the term $< pu'''', S_k >$ using the sinc quadrature formula in (10). Using the integration by parts and the fact that $w(0) = w(1) = 0$, one can find that

$$\begin{aligned} < pu'''', S_k > &= -\int_0^1 u'''(x) \frac{d}{dx}[p(x)S_k(x)w(x)]dx \\ &= \int_0^1 u''(x) \frac{d^2}{dx^2}[p(x)S_k(x)w(x)]dx. \end{aligned}$$

Now we use the boundary conditions in (2) and the integration by parts two more times to get

$$< u'''', S_k > = \int_0^1 u(x) \frac{d^4}{dx^4}[p(x)S_k(x)w(x)]dx. \tag{17}$$

If we use the chain rule to write $\frac{d^n}{dx^n}[S_k(x)]$ in terms of $\frac{d^n}{d\phi^n}[S_k]$ and $\phi^{(n)}(0,1;x)$, and if we introduce the notation

$$S_k^{(n)}(x) := \frac{d^n}{d\phi^n}[S_k(x)], \ n = 0,1,2,3,4$$

and simplify the calculations as in [19, 20], then we can find the following result:

$$< pu'''', S_k >= \sum_{j=0}^{4} \Big(\int_0^1 u(x) S_k^{(j)}(x) \eta_j(x) dx \Big), \tag{18}$$

where

$$\begin{aligned}
\eta_0(x) = &\ 48(-1+2x)P'(x) + 12p''(x) + (-1+x)x \\
&\times \Big(72p''(x) + 8(-1+2x)p^{(3)}(x) + (-1+x)xp^{(4)}(x)\Big) + 24p(x), \\
\eta_1(x) = &\ \frac{18(-1+x)x(-1+2x)p''(x) + 4(-1+x)^2x^2p^{(3)}(x))}{(-1+x)x} + \\
&\ \frac{-8\left(1-9x+9x^2\right)p'(x)}{(-1+x)x} - \frac{2(-1+2x)\left(-1-6x+6x^2\right)p(x)}{(-1+x)^2x^2}, \\
\eta_2(x) = &\ \frac{\left(-1-12x+12x^2\right)p(x)}{(-1+x)^2x^2} + \\
&\ \frac{-12\left(-2+13x-15x^2+2x^3\right)p'(x) + 6(-1+x)^2x^2p''(x)}{(-1+x)^2x^2}, \\
\eta_3(x) = &\ \frac{-2(-1+2x)p(x) - 4(-1+x)xp'(x)}{(-1+x)^2x^2}, \\
\eta_4(x) = &\ \frac{p(x)}{(-1+x)^2x^2}.
\end{aligned} \tag{19}$$

Applying the sinc quadrature formula (10) to the right-hand side of (18), we obtain

$$< pu'''', S_k > \approx h \sum_{j=-N_1}^{N_2} \sum_{i=0}^{4} \frac{u(x_j)}{\phi'(0,1;x_j)h^i} \delta_{jk}^{(i)} \eta_i(x_j), \tag{20}$$

where $x_k = \phi^{-1}(0,1;kh)$, and $\delta_{jk}^{(i)}$ are given by

$$\begin{aligned}
\delta_{jk}^{(0)} &= [S(j,h)\circ\phi(x)]\Big|_{x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \\
\delta_{jk}^{(1)} &= h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]\Big|_{x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{(k-j)}, & j \neq k, \end{cases} \\
\delta_{jk}^{(2)} &= h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]\Big|_{x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k, \end{cases} \\
\delta_{jk}^{(3)} &= h^3\frac{d^3}{d\phi^3}[S(j,h)\circ\phi(x)]\Big|_{x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{(k-j)^3}[6-\pi^2(k-j)^2], & j \neq k, \end{cases} \\
\delta_{jk}^{(4)} &= h^4\frac{d^4}{d\phi^4}[S(j,h)\circ\phi(x)]\Big|_{x_k} = \begin{cases} \frac{\pi^4}{5}, & j = k, \\ \frac{-4(-1)^{k-j}}{(k-j)^4}[6-\pi^2(k-j)^2], & j \neq k. \end{cases}
\end{aligned} \tag{21}$$

The Last Two Terms. Finally, use (6) and (7). One can write

$$
\begin{aligned}
< qu-f, S_k > &= \int_0^1 (q(x)u(x)-f(x))\, S_k(x)w(x)dx \\
&= \int_0^1 \left(\frac{q(x)u(x)-f(x)}{\phi'(0,1;x)} w(x)\right) S_k(x)\phi'(0,1;x)dx \\
&= \int_0^1 \left(\sum_{i=-\infty}^{\infty} \frac{q(t_i)u(t_i)-f(t_i)}{\phi'(0,1;t_i)} w(t_i)S_i(x)\right) S_k(x)\phi'(0,1;x)dx \\
&\approx \sum_{i=-N_1}^{N_2} \frac{q(t_i)u(t_i)-f(t_i)}{\phi'(0,1;t_i)} w(t_i) \int_0^1 S_i(x)S_k(x)\phi'(0,1;x)dx \\
&= \sum_{i=-N_1}^{N_2} \frac{q(t_i)u(t_i)-f(t_i)}{\phi'(0,1;t_i)} w(t_i) \int_{-\infty}^{\infty} \mathrm{sinc}(\frac{t-ih}{h})\mathrm{sinc}(\frac{t-ih}{h})dt \\
&= \sum_{i=-N_1}^{N_2} \frac{q(t_i)u(t_i)-f(t_i)}{\phi'(0,1;t_i)} w(t_i)h\delta_{ki} \\
&= h\frac{q(t_k)u(t_k)-f(t_k)}{\phi'(0,1;t_k)} w(t_k).
\end{aligned}
$$

Replacing each term in equation (15) by its approximation, we proved the following theorem.

Theorem 3.1 *Let $\phi(T_1,T_2;x)$, $S_j(x)$,$\eta_j(x)$, $w(x)$, and $\delta_{jk}^{(i)}$ be defined as in (8), (9), (19), (16), and (21), respectively. Discretize $(0,1)$ by $\{t_j\}$, where $t_j=\phi^{-1}(0,1;jh)$ and $(t,1)$ by $\tau_j=\phi^{-1}(t,1;jh)$ for all $0<t<1$. We can discretize the BVP*

$$g_1(x)D^{\alpha}u(t)+g_2(x)D^{\beta}u(t)+p(t)u^{(4)}(t)+q(t)u(t)=f(t),\ 0<t<1,\ 1<\alpha<2, 0<\beta<1,$$

subject to the boundary conditions $u(0)=0$, $u'(0)=0$, $u(1)=0$, $u'(1)=0$, by the system

$$\frac{h^2}{\Gamma(2-\alpha)} \sum_{j=-N_1}^{N_2}\sum_{i=-N_1}^{N_2} \frac{u(t_j)}{\phi'(0,1;t_j)} \frac{d^2}{dt^2}\left(\frac{(\tau_i-t)^{1-\alpha}S_k(\tau_i)g_1(\tau_i)w(\tau_i)}{\phi'(t,1;\tau_i)}\right)\Bigg|_{t=t_j} +$$

$$\frac{-h}{\Gamma(1-\beta)} \sum_{j=-N_1}^{N_2}\sum_{i=-N_1}^{N_2} \frac{u(t_j)}{\phi'(0,1;t_j)} \frac{d}{dt}\left(\frac{(\tau_i-t)^{-\beta}S_k(\tau_i)g_2(\tau_i)w(\tau_i)}{\phi'(t,1;\tau_i)}\right)\Bigg|_{t=t_j} +$$

$$h\sum_{j=-N_1}^{N_2}\sum_{i=0}^{4} \frac{u(t_j)}{\phi'(0,1;t_j)h^i}\delta_{jk}^{(i)}\eta_i(t_j) + h\frac{q(t_k)u(t_k)-f(t_k)}{\phi'(0,1;t_k)}w(t_k)=0,$$

for $k=-N_1,...,0,...,N_2$.

4 Numerical Applications

Example 4.1 Consider the following BVP:

$$x^{3/2}D^{1.5}u(x)+u^{(4)}(x)+\frac{1}{1+x^2}u(x)=f(x) \tag{22}$$

with the boundary conditions

$$u(0)=0,\ \ u'(0)=0,\ \ u(1)=0,\ \ u'(1)=0, \tag{23}$$

where

$$f(x)=24\left(6-20x+15x^2\right)+\frac{(-1+x)^4x^2}{1+x^2}+\frac{4x^2\left(105-840x+2016x^2-1920x^3+640x^4\right)}{105\sqrt{\pi}}.$$

The exact solution for this problem is $u(x)=(x-1)^4x^2$.

For the choice of $w(x)=(1-x)^2x^2$, we have

$$\begin{aligned}\eta_0(x)&=24,\\ \eta_1(x)&=\frac{-2-4x\left(2-9x+6x^2\right)}{(-1+x)^2x^2},\\ \eta_2(x)&=\frac{-1+12(-1+x)x}{(-1+x)^2x^2},\\ \eta_3(x)&=-\frac{2}{(-1+x)^2}+\frac{2}{x^2},\\ \eta_4(x)&=\frac{1}{(-1+x)^2x^2}.\end{aligned}$$

Use $N_1=N_2=30$, and $h=\frac{\pi}{\sqrt{30}}$. Then, the discrete system is given by the resulting system

$$\sum_{j=--30}^{30}\sum_{i=0}^{4}\frac{c_j}{h^i\phi'(0,1;x_j)}\delta_{jk}^{(i)}\eta_i(x_j)+\frac{h}{\Gamma(1/2)}\sum_{j=-30}^{30}\sum_{i=-30}^{30}\frac{c_j}{\phi'(0,1;x_j)}\times$$
$$\frac{d^2}{dt^2}\left(\frac{(\tau_i)^{3/2}(\tau_i-t)^{-1/2}S_k(\tau_i)w(\tau_i)}{\phi'(t,1;\tau_i)}\right)\Bigg|_{t=t_j}+\frac{c_kw(x_k)}{(1+x_k^2)\phi'(0,1;x_k)}=\frac{f(x_k)w(x_k)}{\phi'(0,1;x_k)}$$

for $k=-30,-29,...,29,30$. Using Mathematica, one can solve this system and find that

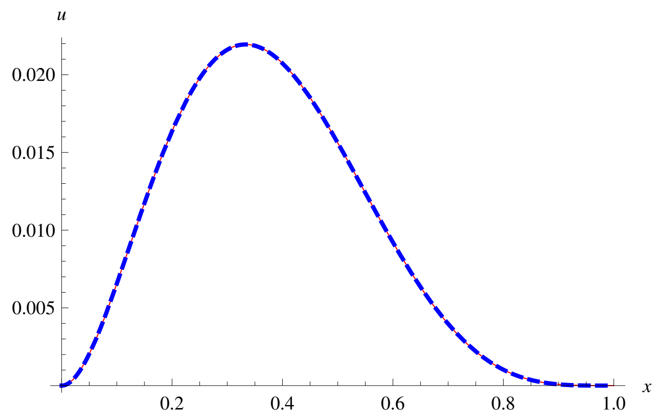

Figure 1: The exact solution is the red color and the approximate solution is the dashed blue color for Example 4.1 when $N_1 = N_2 = 40$ and $h = \pi/\sqrt{60}$.

$c_{-30} = 5.99655 \times 10^{-14}$	$c_{-10} = 0.00001$	$c_{10} = 1.48656 \times 10^{-8}$
$c_{-29} = 1.042706 \times 10^{-13}$	$c_{-9} = 0.0000317$	$c_{11} = 5.5547 \times 10^{-9}$
$c_{-28} = 2.8087 \times 10^{-13}$	$c_{-8} = 0.000097$	$c_{12} = 4.61723 \times 10^{-9}$
$c_{-27} = 4.09977 \times 10^{-13}$	$c_{-7} = 0.000292374$	$c_{13} = 1.82193 \times 10^{-9}$
$c_{-26} = 9.09015 \times 10^{-13}$	$c_{-6} = 0.000848463$	$c_{14} = 1.4434 \times 10^{-9}$
$c_{-25} = 1.15627 \times 10^{-12}$	$c_{-5} = 0.00231724$	$c_{15} = 5.9222 \times 10^{-10}$
$c_{-24} = 2.21194 \times 10^{-12}$	$c_{-4} = 0.00571294$	$c_{16} = 4.5229 \times 10^{-10}$
$c_{-23} = 1.3661 \times 10^{-12}$	$c_{-3} = 0.01192$	$c_{17} = 1.91280 \times 10^{-10}$
$c_{-22} = 3.9494 \times 10^{-13}$	$c_{-2} = 0.0192758$	$c_{18} = 1.420322 \times 10^{-10}$
$c_{-21} = 1.91666 \times 10^{-11}$	$c_{-1} = 0.0217367$	$c_{19} = 6.149521 \times 10^{-11}$
$c_{-20} = 7.5221 \times 10^{-11}$	$c_{0} = 0.0156253$	$c_{20} = 4.4668 \times 10^{-11}$
$c_{-19} = 2.9383 \times 10^{-10}$	$c_{1} = 0.00690247$	$c_{21} = 1.96808 \times 10^{-11}$
$c_{-18} = 9.7171 \times 10^{-10}$	$c_{2} = 0.001944$	$c_{22} = 1.40445 \times 10^{-11}$
$c_{-17} = 3.2402 \times 10^{-9}$	$c_{3} = 0.000382$	$c_{23} = 6.2537 \times 10^{-12}$
$c_{-16} = 1.0346 \times 10^{-8}$	$c_{4} = 0.0000584$	$c_{24} = 4.3941 \times 10^{-12}$
$c_{-15} = 3.3137 \times 10^{-8}$	$c_{5} = 7.59754 \times 10^{-6}$	$c_{25} = 1.9531 \times 10^{-12}$
$c_{-14} = 1.04718 \times 10^{-7}$	$c_{6} = 1.01161 \times 10^{-6}$	$c_{26} = 1.34788 \times 10^{-12}$
$c_{-13} = 3.31083 \times 10^{-7}$	$c_{7} = 1.41935 \times 10^{-7}$	$c_{27} = 5.7864 \times 10^{-13}$
$c_{-12} = 1.04126 \times 10^{-6}$	$c_{8} = 5.67820 \times 10^{-8}$	$c_{28} = 3.84874 \times 10^{-13}$
$c_{-11} = 3.269197 \times 10^{-6}$	$c_{9} = 1.7549 \times 10^{-8}$	$c_{29} = 1.4007 \times 10^{-13}$
		$c_{30} = 8.0230 \times 10^{-14}$

Now the numerical solution is $u(x) = \sum_{i=-30}^{30} c_i S_k(x)$. Figure 1, Table 1, and Table 2 show the accuracy of the SGM in Example 4.1.

Example 4.2 Consider the following BVP:

$$xD^{1.5}u(x) + D^{0.5}u(x) + u^{(4)}(x) + \frac{1}{x}u(x) = f(x) \tag{24}$$

with the boundary conditions

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \tag{25}$$

x_k	Exact value	Approximate Value	Absolute Error
0.000103361	$1.0679078727 \times 10^{-8}$	$1.034606291 \times 10^{-8}$	$3.33015812 \times 10^{-10}$
0.000325432	$1.057680767 \times 10^{-7}$	$1.047189702 \times 10^{-7}$	$1.04910649 \times 10^{-9}$
0.00102413	$1.044557458 \times 10^{-6}$	$1.041265549 \times 10^{-6}$	$3.291908932 \times 10^{-9}$
0.00321811	0.0000102236	0.0000102134	$1.022024664 \times 10^{-8}$
0.0100649	0.0000972841	0.0000972533	$3.081035295 \times 10^{-8}$
0.0310251	0.000848548	0.000848463	$8.53263208 \times 10^{-8}$
0.0915966	0.00571312	0.00571294	$1.826834860 \times 10^{-7}$
0.241011	0.0192759	0.0192758	$1.500778530 \times 10^{-7}$
0.5	0.015625	0.0156253	$3.16788086 \times 10^{-7}$
0.758989	0.00194364	0.00194426	$6.24294102 \times 10^{-7}$
0.908403	0.0000580864	0.0000584573	$3.70957577 \times 10^{-7}$
0.968975	8.6991642×10^{-7}	$1.011614594 \times 10^{-6}$	$1.416981697 \times 10^{-7}$
0.989935	$1.005642443 \times 10^{-8}$	$5.67820105 \times 10^{-8}$	$4.672558615 \times 10^{-8}$
0.996782	$1.065625857 \times 10^{-10}$	$1.486565724 \times 10^{-8}$	$1.475909466 \times 10^{-8}$
0.998976	$1.097828970 \times 10^{-12}$	$4.61723057 \times 10^{-9}$	$4.616132746 \times 10^{-9}$
0.999675	$1.12087543 \times 10^{-14}$	$1.443447221 \times 10^{-9}$	$1.443436012 \times 10^{-9}$
0.999897	$1.14113473 \times 10^{-16}$	$4.522918397 \times 10^{-10}$	$4.52291725 \times 10^{-10}$

Table 1: Numerical values for Example 4.1 when $N_1 = N_2 = 30$ and $h = \pi/\sqrt{30}$.

x_k	Exact value	Approximate Value	Absolute Error
0.000888742	$7.87057693 \times 10^{-7}$	$7.8614223458 \times 10^{-7}$	$9.15458562 \times 10^{-10}$
0.00179253	$3.19018596 \times 10^{-6}$	$3.1883409387 \times 10^{-6}$	$1.84503042 \times 10^{-9}$
0.00361208	0.0000128597	0.0000128559	$3.71143282 \times 10^{-9}$
0.00726518	0.0000512656	0.0000512581	$7.438059772 \times 10^{-9}$
0.0145589	0.000199884	0.000199869	$1.479564583 \times 10^{-8}$
0.0289613	0.000745729	0.0007457	$2.89922931 \times 10^{-8}$
0.0567902	0.00255258	0.00255252	$5.51198146 \times 10^{-8}$
0.108375	0.00742317	0.00742307	$9.86259941 \times 10^{-8}$
0.19703	0.0161384	0.0161383	$1.562575722 \times 10^{-7}$
0.331262	0.0219466	0.0219464	$1.948937535 \times 10^{-7}$
0.5	0.015625	0.0156248	$1.615772327 \times 10^{-7}$
0.668738	0.00538517	0.00538509	$8.17869846 \times 10^{-8}$
0.80297	0.000971686	0.000971658	$2.849247698 \times 10^{-8}$
0.891625	0.00010967	0.000109661	8.2069107×10^{-9}
0.94321	$9.25355745 \times 10^{-6}$	$9.2513881662 \times 10^{-6}$	$2.16928432 \times 10^{-9}$
0.971039	$6.63350097 \times 10^{-7}$	$6.6280111914 \times 10^{-7}$	$5.4897826 \times 10^{-10}$
0.985441	$4.36286973 \times 10^{-8}$	$4.3493511523 \times 10^{-8}$	$1.35185823 \times 10^{-10}$
0.992735	$2.745695415 \times 10^{-9}$	$2.7132500384 \times 10^{-9}$	$3.2445377 \times 10^{-11}$
0.996388	$1.690005952 \times 10^{-10}$	$1.6151162 \times 10^{-10}$	$7.4889679 \times 10^{-12}$

Table 2: Numerical values for Example 4.1 when $N_1 = N_2 = 40$ and $h = \pi/\sqrt{60}$.

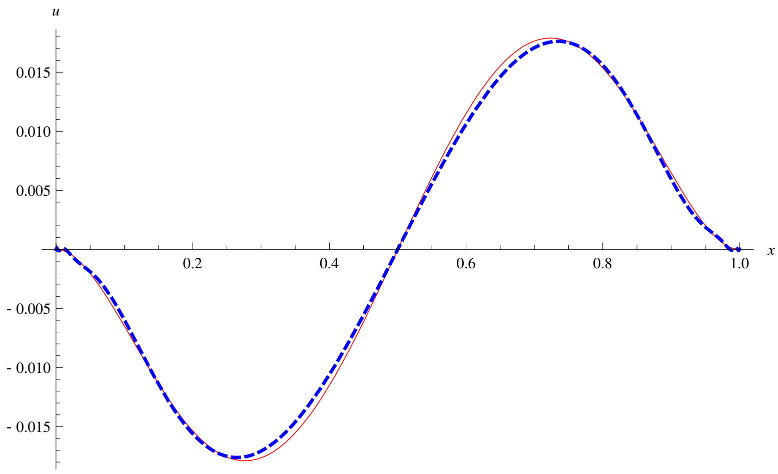

Figure 2: The exact solution is the red color and the approximate solution is the dashed blue color for Example 4.2 when $N_1 = N_2 = 10$ and $h = \frac{\pi}{\sqrt{30}}$.

where

$$f(x) = (-1+2x)(24+(1-x)^2x+96) + \frac{4x^{3/2}}{315\sqrt{\pi}}\left(-525+3528x-6480x^2+3520x^3\right).$$

The exact solution for this problem is $u(x) = x^2(-1+4x-5x^2+2x^3)$. In this example we consider a fractional differential equation with singularity at $x = 0$ to show the power of SGM with this kind of problems. We consider a different family of parameters, the first $N_1 = N_2 = 10$, and $h = \pi/\sqrt{10}$. Next, we apply the SGM when $N_1 = N_2 = 40$, and $h = \pi/\sqrt{20}$. It is clear from Figures 2 and 3, and Table 3 that we can obtain a high quality approximation for a good choice of the parameters.

5 Conclusion

The sinc-Galerkin method is established for the higher order fractional boundary value problems in this paper. The suggested method utilizes the properties of fractional derivatives in order to solve the higher order BVP. The numerical scheme is computationally captivating. We demonstrated our results by tables and figures, it is proved that the convergence rate of the sinc method is of $\mathcal{O}(\exp(-\kappa\sqrt{N})$ with some $\kappa > 0$, where N is the number of nodes or bases used in the method. We provided a fractional differential equation with singularities, the method shows the best response to this example.

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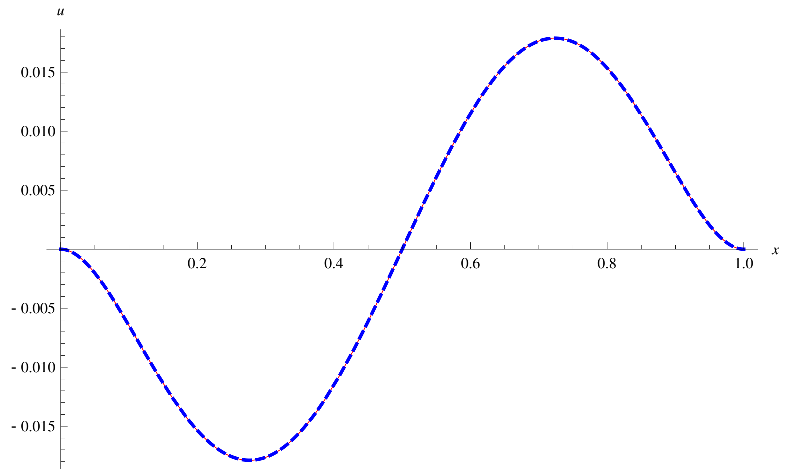

Figure 3: The exact solution is the red color and the approximate solution is the dashed blue color for Example 4.2 when $N_1 = N_2 = 40$ and $h = \frac{\pi}{2\sqrt{30}}$.

x_k	Exact value	Approximate Value	Absolute Error
0.00691383	−0.0000464904	−0.0000460076	$4.828512978699654 \times 10^{-7}$
0.0113114	−0.000122241	−0.00012146	$7.809294063951228 \times 10^{-7}$
0.0184542	−0.000315995	−0.000314746	$1.2489442106647592 \times 10^{-6}$
0.0299707	−0.000794545	−0.000792583	$1.962081982870246 \times 10^{-6}$
0.0483203	−0.0019103	−0.00190731	$2.995110557358835 \times 10^{-6}$
0.0770126	−0.00427437	−0.00427001	$4.365622987672832 \times 10^{-6}$
0.120583	−0.00853317	−0.00852726	$5.90892890370287 \times 10^{-6}$
0.183893	−0.0142393	−0.0142322	$7.096707035069219 \times 10^{-6}$
0.270229	−0.0178716	−0.0178646	$6.9694864587993566 \times 10^{-6}$
0.37831	−0.0134626	−0.013458	$4.585623157854837 \times 10^{-6}$
0.5	0.	$1.57961907 \times 10^{-7}$	$1.5796190761125167 \times 10^{-7}$
0.62169	0.0134626	0.0134582	$4.371289509910378 \times 10^{-6}$
0.729771	0.0178716	0.0178646	$6.9222409211575076 \times 10^{-6}$
0.816107	0.0142393	0.0142322	$7.127970986300913 \times 10^{-6}$
0.879417	0.00853317	0.00852723	$5.94465157464398 \times 10^{-6}$
0.922987	0.00427437	0.00426998	$4.386834537174061 \times 10^{-6}$
0.95168	0.0019103	0.0019073	$3.0051297730975847 \times 10^{-6}$
0.970029	0.000794545	0.000792579	$1.966324959320467 \times 10^{-6}$
0.981546	0.000315995	0.000314744	$1.2506400348780188 \times 10^{-6}$
0.988689	0.000122241	0.000121459	$7.815852781527434 \times 10^{-7}$
0.993086	0.0000464904	0.0000460073	$4.83099931514434 \times 10^{-7}$

Table 3: Numerical values for Example 4.2 when $N_1 = N_2 = 40$ and $h = \frac{\pi}{2\sqrt{30}}$.

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