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The Finite Element Method for Nonlinear Nonstandard Volterra Integral Equations

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Abstract: In this work, we look at the implementation of the finite element method to a nonlinear (nonstandard) Volterra integral equation. We consider the Galerkin approach, where we choose the weight function in such a way that it takes the form of the approximate solution. We work on a uniform mesh and choose the Lagrange polynomials as basis functions. We consider the error analysis of the method. We look at a specific example to illustrate the implementation of the finite element method. Finally, we consider the estimated rate of convergence.

Keywords: *Volterra integral equations; finite element method; Galerkin approach.*

Mathematics Subject Classification (2010): 45D05, 65R20, 65N30.

1 Introduction

In this paper, we consider the nonlinear Volterra integral equation of the second kind

$$u(x) = \sum_{m=1}^{r} b_m \left(g_m(x) + \int_0^x k_m(x,y)u(y)dy \right)^m, \qquad x \in [0,L], \tag{1}$$

where $r \in \mathbb{N}, r \geq 2$, $b \in \mathbb{R}$, $g : [0,L] \to \mathbb{R}$ and $k : [0,L] \times [0,L] \to \mathbb{R}$ are continuous functions. The unknown function $u(x) \in C[0,L]$.

In essence, (1) is nonstandard in that in its simplest form, it has the structure

$$u = \sum_{m=1}^{r} b_m \left(g_m + W_m u \right)^m, \qquad x \in [0,L],$$

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where W_m is the standard linear Volterra operator, see [9]. Volterra integral equations find applications in many fields of science and engineering, including population dynamics, the spread of epidemics, semi-conductor devices, wave propagation, super-fluidity and traveling wave analysis, see Saveljeva [14], for example. Many authors have used numerical methods to solve Volterra integral equations. The authors in [1, 8, 11, 13] solved linear Volterra integral equations using quadrature rules such as the repeated Simpson's and repeated trapezoidal rule. Other authors, see [3–7], used collocation methods to find approximate solutions for Volterra integral equations.

Sloss and Blyth [15] implemented Corrington's Walsh function method to (1) and also proved the existence and uniqueness of the solution in a Banach space L^2. Malindzisa and Khumalo [9] used the collocation methods and quadrature rules to approximate the solution to (1). They pointed out that the repeated Simpson's rule gives better solutions when a reasonable value of the step size is used. They also provided sufficient conditions for the existence and uniqueness of the solutions for (1) and for the case where $b_1 = 0$ and $r = 2$ in the space $C[0, L]$. A similar study was done by Mamba and Khumalo [10]. They presented convergence analysis for the collocation methods and trapezoidal rule. Benitez and Bolos [2] highlighted that the collocation methods have proven to be appropriate techniques for finding the approximate solutions for nonlinear integral equations because of their accuracy and stability. However, the finite element method has not been used to approximate solutions to (1). In this work, we will consider the implementation of the finite element method to find the approximate solutions for (1). In particular, we will consider the case where $b_1 = 0$ and $r = 2$.

$$u(x) = b\left(g(x) + \int_0^x k(x,y)u(y)dy\right)^2, \tag{2}$$

where $b \in \mathbb{R},\ g : [0, L] \to \mathbb{R}$ and $k : [0, L] \times [0, L] \to \mathbb{R}$ are continuous functions. The unknown function $u(x) \in C[0, L]$. We will also compare the results obtained from the implementation of the finite element method to the ones obtained in [9, 10] using the collocation method and quadrature rules.

2 Well-Posedness of the Problem

The following theorem shows that when $b_1 = 0$ and $r = 2$, the solution of the VIE of the non-standard type (1) exists in the space $C[0, L]$. In this section, we will prove that the solution exists using a method analogous to the one used by Malindzisa and Khumalo in [9]. The uniqueness of the solution for the case where $b_1 = 0$ and $r = 2$ is presented in [9].

Theorem 2.1 *Consider the VIE (2). There exists a solution $u(x)$, where $u \in C[0, L]$, provided that*

$$\begin{aligned} &2K|\,b\,|\big(\|g\|_\infty + KL\big) < 1, \\ &|\,b\,|\big(\|g\|_\infty + KL\big)^2 < L, \end{aligned} \tag{3}$$

where $K = \sup\limits_{[0,1]\times[0,1]} |\,k(x, y)\,|$.

Proof. We use Banach's fixed point theorem. Define $Tz(x) = b\Big(g(x) + \int_0^x k(x,y)z(y)dy\Big)^2$ for each $z \in C[0,L]$. Let $z_1, z_2 \in C[0,L]$. Then

$$\begin{aligned} Tz_2(x) - Tz_1(x) &= b\Big(g(x) + \int_0^x k(x,y)z_2(y)dy\Big)^2 - b\Big(g(x) + \int_0^x k(x,y)z_1(y)dy\Big)^2 \\ &= b\Big[\Big(g(x) + \int_0^x k(x,y)z_2(y)dy\Big)^2 - \Big(g(x) + \int_0^x k(x,y)z_1(y)dy\Big)^2\Big] \\ &= b\Big[\Big(g(x) + \int_0^x k(x,y)z_2(y)dy - g(x) - \int_0^x k(x,y)z_1(y)dy\Big) \\ &\cdot \sum_{i=0}^{1}\Big(g(x) + \int_0^x k(x,y)z_2(y)dy\Big)^i \cdot \Big(g(x) + \int_0^x k(x,y)z_1(y)dy\Big)^{1-i}\Big] \\ & bF(x,z_1,z_2)\int_0^x k(x,y)\big(z_2(y) - z_1(y)\big)dy. \end{aligned} \tag{4}$$

Therefore,

$$\|Tz_2 - Tz_1\|_\infty \le b \sup F(x,z_1,z_2) \cdot K\|z_2 - z_1\|_\infty. \tag{5}$$

Furthermore,

$$\|F\|_\infty \le 2\big(\|g\|_\infty + KL\big).$$

Hence,

$$\|Tz_2 - Tz_1\|_\infty \le 2b\big(\|g\|_\infty + KL\big)K\|z_2 - z_1\|_\infty. \tag{6}$$

Consequently, T is a contraction if

$$2K|\,b\,|\big(\|g\|_\infty + KL\big) < 1. \tag{7}$$

We need to show that $T : C[0,L] \to C[0,L]$. Observe that

$$\Big\|\Big(g(x) + \int_0^x k(x,y)z(y)dy\Big)^2\Big\|_\infty \le \big(\|g\|_\infty + K\|z\|_\infty\big)^2. \tag{8}$$

Therefore

$$\begin{aligned} \|Tz\|_\infty &\le |\,b\,|\big(\|g\|_\infty + K\|z\|_\infty\big)^2 \\ &\le |\,b\,|\big(\|g\|_\infty + KL\big)^2; \end{aligned} \tag{9}$$

thus $T : C[0,L] \to C[0,L]$

$$|\,b\,|\big(\|g\|_\infty + KL\big)^2 < L.$$

Hence T is a contraction and maps $[0,L]$ into itself given (3) holds.

3 Numerical Method

Consider the nonlinear Volterra integral equation (2). Let $\mathcal{I} = \{x_i\}_{i=0}^{n}$ be the partition of $[0, L]$ with a uniform mesh h; that is,

$$0 = x_0 < x_1 < x_2, \ldots x_{n-1} < x_n = L,$$

and let S denote the $(n+1)$-dimensional subspace of $C[0, L]$ spanned by $\{\phi_k\}_{k=0}^{n}$. We seek an approximate solution $u_h \in S$ for $u(x)$ of the form

$$\begin{aligned} u_h(x) =& \alpha_0\phi_0(x) + \alpha_1\phi_1(x) + \alpha_2\phi_2(x) + \cdots + \alpha_n\phi_n(x) \\ =& \sum_{k=0}^{n} \alpha_k\phi_k(x) \approx u(x), \end{aligned} \tag{10}$$

where $\{\alpha_k\}_{k=0}^{n}$ are the coefficients to be determined.

We take S to be a piecewise linear subspace of $C[0, L]$, that is, we choose

$$\phi_k(x) = \begin{cases} \frac{x - x_{k-1}}{h_k}, & \text{for} \quad x_{k-1} \le x \le x_k, \\ \frac{x_{k+1} - x}{h_{k+1}}, & \text{for} \quad x_k \le x \le x_{k+1}, \\ 0, & \text{otherwise} , \end{cases} \tag{11}$$

where $h_k = x_k - x_{k-1}$ is the length of the subinterval (element) k and the derivative of the basis functions is

$$\phi_k'(x) = \begin{cases} \frac{1}{h_k}, & \text{for} \quad x_{k-1} \le x \le x_k, \\ \frac{-1}{h_{k+1}}, & \text{for} \quad x_k \le x \le x_{k+1}, \\ 0, & \text{otherwise} . \end{cases}$$

Each basis function ϕ_k satisfies the following properties, see [12].

1. The interpolation property

$$\phi_k(x_j) = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \ne j. \end{cases}$$

2. The partition of unity

$$\sum_{k=0}^{n} \phi_k(x) = 1, \qquad k, j = 0, 1, 2, \cdots n.$$

Substituting (10) into (2) and rearranging give

$$\sum_{k=0}^{n} \alpha_k\phi_k(x) - b\left(g(x) + \int_0^x k(x, y) \sum_{k=0}^{n} \alpha_k\phi_k(y)dy\right)^2 := R(x), \tag{12}$$

where $R(x)$ represents the residual.

We want to determine the values of α_k such that the residuals over the elements are minimized in a weighted average sense.

$$\int_{x_i}^{x_{i+1}} w(x)R(x)dx = 0, \tag{13}$$

where $w(x)$ is the weight function. Here we use the Galerkin criterion, in which the residual $R(x)$ is orthogonal to the weight function $w(x)$ (see [16]). This follows since $w(x) \in S$ and $R(x)$ is orthogonal to every element $v_h \in S$.

In this study, we chose a weight function which is of the same form as the approximate solution $u_h(x)$ but with arbitrary coefficients. Therefore,

$$w(x) = \beta_0\phi_0(x) + \phi_1(x)\beta_1 + \cdots + \phi_n(x)\beta_n = \sum_{j=0}^{n} \beta_j\phi_j(x). \tag{14}$$

Substituting (12) and (14) in (13) we get the following:

$$\begin{aligned}\int_{x_i}^{x_{i+1}} w(x)R(x)dx &= \int_{x_i}^{x_{i+1}} \sum_{j=0}^{n} \beta_j\phi_j(x) \sum_{k=0}^{n} \alpha_k\phi_k(x)dx \\ &- \int_{x_i}^{x_{i+1}} \sum_{j=0}^{n} \beta_j\phi_j(x)b \left(g(x) + \int_0^x k(x,y) \sum_{k=0}^{n} \alpha_k\phi_k(y)dy \right)^2 dx = 0.\end{aligned}$$

Since the β_j 's are arbitrary, we get the following equation:

$$\begin{aligned}\sum_{j=0}^{n} \beta_j \Bigg[\int_{x_i}^{x_{i+1}} \phi_j(x) \sum_{k=0}^{n} \alpha_k\phi_k(x)dx - \int_{x_i}^{x_{i+1}} \phi_j(x)b\Bigg(g(x)+ \\ \int_0^x k(x,y) \sum_{k=0}^{n} \alpha_k\phi_k(y)dy \Bigg)^2 dx \Bigg] = 0.\end{aligned}$$

Thus

$$\sum_{k=0}^{n} \alpha_k \int_{x_i}^{x_{i+1}} \phi_j(x)\phi_k(x)dx - \int_{x_i}^{x_{i+1}} \phi_j(x)b \left(g(x) + \sum_{k=0}^{n} \alpha_k \int_0^x k(x,y)\phi_k(y)dy \right)^2 dx = 0.$$

Therefore

$$\sum_{k=0}^{n} \alpha_k \int_{x_i}^{x_{i+1}} \phi_k(x)\phi_j(x)dx - \int_{x_i}^{x_{i+1}} \phi_j(x)b \left(g(x) + \sum_{k=0}^{n} \alpha_k \int_0^x k(x,y)\phi_k(y)dy \right)^2 dx = 0, \tag{15}$$

which is a nonlinear system of equations which must be solved for $\{\alpha_k\}_{k=0}^{n}$.

4 Convergence Analysis

We now analyze the error, to start the analysis we first define the norm

$$\begin{aligned}\|e(x)\|_\infty &= \|u(x) - u_h(x)\|_\infty \\ &= \max_{0\le k\le n} \|u(x_k) - u_h(x_k)\| \\ &= \max_{0\le k\le n} \|u(x_k) - \alpha_k\|. \end{aligned} \tag{16}$$

Consider the Volterra integral equation (2). Then substitute (10) into (2), we get

$$u_h(x) = b\left(g(x) + \int_0^x k(x,y)u_h(y)dy\right)^2, \quad x \in [0,L]. \tag{17}$$

The following theorem gives the upper bound of $\|e(x)\|_\infty$.

Theorem 4.1 *Assume that $b \in \mathbb{R}$, $g : [0,L] \to \mathbb{R}$ and $k : [0,L] \times [0,L] \to \mathbb{R}$ are continuous functions. $u(x)$ is the exact solution of (2) and $u_h(x)$ is the approximate solution. Then*

$$\begin{aligned}\|e(x)\|_\infty &= \|u(x) - u_h(x)\|_\infty \\ &\leqslant D_1\frac{h^{q+1}}{4(q+1)} + \left(D_3\frac{h^{q+1}}{4(q+1)}\right)\left(D_5\frac{h^{q+1}}{4(q+1)} + D_7\max_{0\le k\le n} |\,\alpha_k\,|\right),\end{aligned}$$

where D_1, D_3, D_5 and D_7 are appropriately defined constants.

Proof. Note that

$$e(x) = u(x) - u_h(x). \tag{18}$$

Let $x = x_i$ in (18). Therefore,

$$e_i = u(x_i) - u_h(x_i) = b\left(g(x_i) + \int_0^{x_i} k(x_i,y)u(y)dy\right)^2 - b\Bigg(g(x_i) + \int_0^{x_i} k(x_i,y)u_h(y)dy\Bigg)^2 \tag{19}$$

$$\begin{aligned}&= b\Bigg[2g(x_i)\int_0^{x_i} k(x_i,y)\left[R + \Lambda_n(y) - u_h(y)\right]dy + \\ &\left(\int_0^{x_i} k(x_i,y)\left[R + \Lambda_n(y) - u_h(y)\right]dy\right)\cdot\left(\int_0^{x_i} k(x_i,y)\left[R + \Lambda_n(y) - u_h(y)\right]dy + \right. \\ &\left. 2\int_0^{x_i} k(x_i,y)u_h(y)dy\right)\Bigg],\end{aligned} \tag{20}$$

where $u(y)$ is as in (10),

$$\Lambda_n(y) = \sum_{k=0}^{n} u(y_k)\phi_k(y), \tag{21}$$

and $R = u(y) - \Lambda_n(y)$ is the remainder of interpolation corresponding to the finite element and $\Lambda_n(y)$ is the N-order Lagrange finite element solution. By the use of interpolation polynomial error estimation, we have the following:

$$|\,R\,| \leqslant \frac{h^{q+1}}{4(q+1)}\max_{\xi\in[0,L]} |u(\xi)^{q+1}| = \frac{h^{q+1}}{4(q+1)}M, \tag{22}$$

where $M = \max_{\xi\in[0,L]} |u(\xi)^{q+1}|$ and q is the degree of the piecewise Lagrange polynomials.

Substitute equations (21) and (10) into (19), we then get the following equation:

$$e = b\Bigg[\int_0^{x_i} 2g(x_i)k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]ds + \left(\int_0^{x_i} k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]dy\right)\cdot\Bigg(\int_0^{x_i} k(x_i,y)\Bigg[R+ \sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\Bigg]dy + 2\int_0^{x_i} k(x_i,y)\Big(\sum_{k=0}^{n}\alpha_k\phi_k(y)\Big)dy\Bigg)\Bigg]. \tag{23}$$

Then

$$|\,e\,| = \Bigg|b\Bigg[\int_0^{x_i} 2g(x_i)k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]dy + \left(\int_0^{x_i} k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]dy\right)\cdot\Bigg(\int_0^{x_i} k(x_i,y)\Bigg[R+ \sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\Bigg]dy + 2\int_0^{x_i} k(x_i,y)\Big(\sum_{k=0}^{n}\alpha_k\phi_k(y)\Big)dy\Bigg)\Bigg]\Bigg|$$

$$|\,e\,| = |\,b\,|\Bigg|\Bigg[\int_0^{x_i} 2g(x_i)k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]dy + \left(\int_0^{x_i} k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]dy\right)\cdot\Bigg(\int_0^{x_i} k(x_i,y)\Bigg[R+ \sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\Bigg]dy + 2\int_0^{x_i} k(x_i,y)\Big(\sum_{k=0}^{n}\alpha_k\phi_k(y)\Big)dy\Bigg)\Bigg]\Bigg|$$

$$\begin{aligned}|\,e\,| \le |\,b\,|\Bigg[&\Bigg|\int_0^{x_i} 2g(x_i)k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]ds\Bigg| + \Bigg|\left(\int_0^{x_i} k(x_i,y)\left[R+\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right]dy\right)\Bigg|\cdot\Bigg|\Bigg(\int_0^{x_i} k(x_i,y)\Bigg[R+ \sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\Bigg]dy + 2\int_0^{x_i} k(x_i,y)\Big(\sum_{k=0}^{n}\alpha_k\phi_k(y)\Big)dy\Bigg)\Bigg|\Bigg]\\ \le |\,b\,|\Bigg[&\int_0^{x_i}|\,2g(x_i)\,|\,|\,k(x_i,y)\,|\left[|\,R\,|+\left|\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right|\right]dy + \left(\int_0^{x_i}|\,k(x_i,y)\,|\left[|\,R\,|+\left|\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right|\right]dy\right)\cdot\Bigg(\int_0^{x_i}|\,k(x_i,y)\,|\Bigg[|\,R\,|+ \left|\sum_{k=0}^{n}\left(u(y_k)-\alpha_k\right)\phi_k(y)\right|\Bigg]dy + 2\int_0^{x_i}|\,k(x_i,y)\,|\,|\,u_h(y)\,|dy\Bigg)\Bigg].\end{aligned}$$

Now set

$$\delta = [e(y_0), e(y_1) \dots e(y_k), \dots e(y_n)]^T, \qquad \Phi = [\phi_0(y), \phi_1(y), \dots \phi_k(y), \dots, \phi_n(y)]^T,$$

and

$$\Gamma = [\alpha_0, \alpha_1, \dots \alpha_k, \dots, \alpha_n]^T,$$

where $e(y_k) = u(y_k) - u_h(y_k) = u(y_k) - \alpha_k$.

Then

$$|\, \Lambda_n(y) - u_h(y) \,| = \left| \sum_{k=0}^{n} (\alpha(y_k) - \alpha_k)\, \phi_k(y) \right| = |\, \delta \cdot \Phi \,| \leqslant \|\delta\|_\infty \cdot \|\Phi\|_\infty \leqslant C\|\delta\|_\infty. \quad (24)$$

Since $e(y_k)$ for $k = 0, 1, 2, ..., n$ represents the interpolation error, we have

$$e(y_0) = e(y_1) = \cdots = e(y_n) = 0, \quad (25)$$

$$\left| \sum_{k=0}^{n} \alpha_k \phi_k(y) \right| = |\, \Gamma \cdot \Phi \,| \leqslant \|\Gamma\|_\infty \cdot \|\Phi\|_\infty \leqslant C\|\Gamma\|_\infty. \quad (26)$$

Using equation (22), (24), (25) and (26) we then obtain the following:

$$\begin{aligned}
\|e\|_\infty \leqslant\ & D_1 \frac{h^{q+1}}{4(q+1)} + D_2 \|\delta\|_\infty \\
& + \left(D_3 \frac{h^{q+1}}{4(q+1)} + D_4 \|\delta\|_\infty \right) \left(D_5 \frac{h^{q+1}}{4(q+1)} + D_6 \|\delta\|_\infty + D_7 \|\Gamma\|_\infty \right) \\
=\ & D_1 \frac{h^{q+1}}{4(q+1)} + D_2 \max_{0 \le y_k \le n} |\, e(y_k) \,| \\
& + \left(D_3 \frac{h^{q+1}}{4(q+1)} + D_4 \max_{0 \le y_k \le n} |\, e(y_k) \,| \right) \\
& \cdot \left(D_5 \frac{h^{q+1}}{4(q+1)} + D_6 \max_{0 \le y_k \le n} |\, e(y_k) \,| + D_7 \max_{0 \le k \le n} |\, \alpha_k \,| \right) \\
=\ & D_1 \frac{h^{q+1}}{4(q+1)} + \left(D_3 \frac{h^{q+1}}{4(q+1)} \right) \left(D_5 \frac{h^{q+1}}{4(q+1)} + D_7 \max_{0 \le k \le n} |\, \alpha_k \,| \right),
\end{aligned}$$

where $G = \max_{0 \le x_i \le L} |\, g(x_i) \,|$, $K = \max_{\substack{0 \le x_i \le L \\ 0 \le y \le L}} |\, k(x_i, y) \,|$, $L = \max_{0 \le i \le n} |\, x_i \,|$ $D_1 = 2|\, b \,|GKLM$, $D_2 = 2|\, b \,|CGKL$, $D_3 = |\, b \,|KL$, $D_4 = |\, b \,|CKL$, $D_5 = KLM$, $D_6 = CKL$, $D_7 = 2CKL$.

Corollary 4.1 *Let q be the degree of the piece-wise Lagrange polynomials. If $q = 1$, by using the interpolation polynomial error estimation (22) we get*

$$|\, R \,| \leqslant \frac{h^2}{8} M. \quad (27)$$

Therefore

$$\|e(x)\|_\infty = \|u(x) - u_h(x)\|_\infty \leqslant D_1 \frac{h^2}{8} + \left(D_3 \frac{h^2}{8} \right) \left(D_5 \frac{h^2}{8} + D_7 \max_{0 \le k \le n} |\, \alpha_k \,| \right).$$

5 Numerical Computation

We now consider a specific example and implement the finite element method to illustrate the procedure.

Example 5.1 Consider the following nonlinear(nonstandard) Volterra integral equations:

$$u(x) = 2\left(1 + \int_0^x (x-y)u(y)dy\right)^2, \qquad x \in [0,1], \tag{28}$$

whose exact solution is unavailable. To simplify the numerical computation for the above example, we linearize the problem and implement the finite element method. The algebraic equations obtained from the implementation of the finite element method can be written in a vector form as follows:

$$\mathbf{K}\alpha = \mathbf{F}, \tag{29}$$

where $\mathbf{K}$ is the stiffness (coefficient) matrix and $\mathbf{F}$ is the source(force) vector. We now choose the number of elements n. Let $n = 10$, the algebraic equations obtained from the implementation of the finite element method for ten elements in a vector form are written as:

$$\mathbf{A}\alpha = \mathbf{d}, \tag{30}$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{83}{2500} & \frac{499}{30000} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{163}{10000} & \frac{239}{3750} & \frac{529}{30000} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{379}{30000} & \frac{209}{3750} & \frac{233}{10000} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{49}{30000} & \frac{53}{1250} & \frac{1129}{30000} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-207}{10000} & \frac{89}{3750} & \frac{1939}{30000} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1751}{30000} & \frac{-1}{3750} & \frac{1083}{10000} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-3461}{30000} & \frac{-37}{1250} & \frac{5179}{30000} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1957}{10000} & \frac{-241}{3750} & \frac{7849}{30000} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-9101}{30000} & \frac{-391}{3750} & \frac{3793}{10000} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-13271}{30000} & \frac{-187}{1250} & \frac{15889}{30000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-6167}{10000} & \frac{1433}{2500} \end{bmatrix},$$

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \end{bmatrix}; \qquad \mathbf{d} = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{10} \end{bmatrix}.$$

After solving the above system of equations, we consider the graphical representation of the solution for $n = 5$, $n = 10$, $n = 20$ and $n = 40$.

Since the exact solution of (28) is unavailable, the graphical representation in Figure 1 of the approximate solution is found to resemble those obtained in [9]. We now estimate the rate of convergence using the numerical results obtained from implementing the finite element method on (28). To estimate the rate of convergence we use the following:

$$p = \frac{\log \left| \frac{u^{\frac{h}{2}} - u^h}{u^{\frac{h}{4}} - u^{\frac{h}{2}}} \right|}{\log 2},$$

where u^h, $u^{\frac{h}{2}}$ and $u^{\frac{h}{4}}$ are the approximate solution to $u(x)$ for the step size h, $\frac{h}{2}$ and $\frac{h}{4}$, respectively .

Table 1: The rate of convergence for different values of h using (28).

h	$\mathbf{u}_h(x)$	**p**
$\frac{1}{5}$	1.961133515140453	3.044666782640548
$\frac{1}{10}$	1.994338539181266	1.720724066245102
$\frac{1}{20}$	1.998362629972080	1.966888879806644
$\frac{1}{40}$	1.999583524683324	1.999142447697078
$\frac{1}{80}$	1.999895834518747	
$\frac{1}{160}$	1.999973958401418	

The results in Table 1 indicate a second-order rate of convergence. Hence we can conclude that the finite element method is better compared to the one-point collocation methods and quadrature methods presented in [9], since the methods in [9] indicate the first-order rate of convergence.

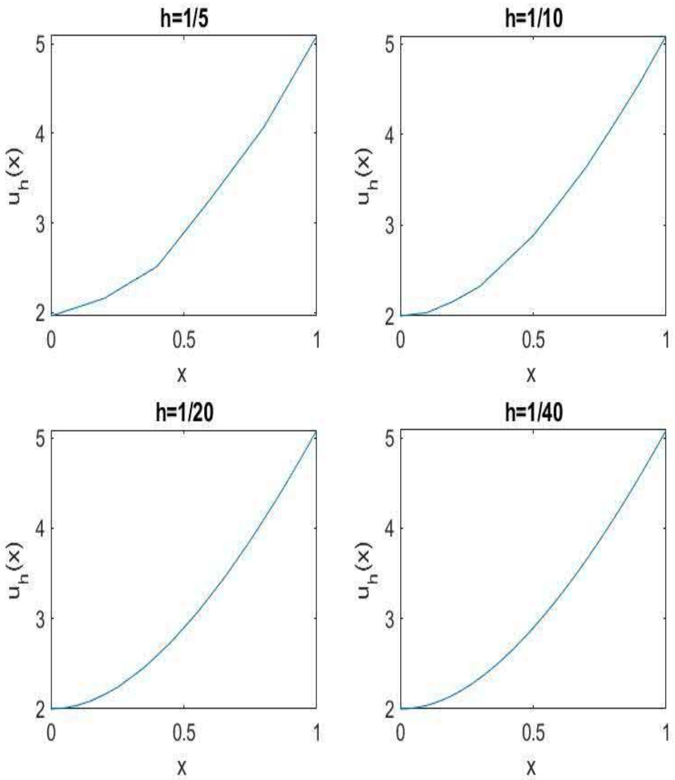

Figure 1: The graphical representation of the solutions for different values of h.

6 Conclusion

In this paper, we solved the nonlinear (nonstandard) Volterra integral equation on a uniform mesh and used the Lagrange polynomials as basis functions together with the Galerkin finite element method, where the weight function is chosen in such a way that it takes the form of the approximate solution but with arbitrary coefficients. We implemented the finite element method to the nonlinear (nonstandard) Volterra integral equations. We proved the error analysis of the approximate solution. We implemented the finite element method to a specific nonlinear (nonstandard) Volterra integral equation. We looked at the graphical representation of the approximate solution for $n = 5$, $n = 10$, $n = 20$ and $n = 40$. Since the exact solution of (28) is unavailable, we compared the graphical representation in Figure 1 with the graphical representation of the approximate solution obtained by using the one-point collocation and quadrature methods in the paper presented by Malindzisa and Khumalo [9], from which we observed that the graphs in Figure 1 are similar to the ones obtained [9]. We also consider the estimate for the rate of convergence for different values of h using (28) from which we obtained the results in Table 1 which indicates a second-order rate of convergence. From the results obtained in Table 1 we conclude that the finite element method is better compared to the one-point collocation methods and quadrature methods presented by Malindzisa and Khumalo in [9] for this example since the methods in [9] indicate the first-order rate of convergence.

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