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Oscillation Criteria for Delay Equations with Several Non-Monotone Arguments

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Abstract: Consider the first-order linear differential equation with several retarded arguments $x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0,\ t \ge t_0$, where the functions $p_i,\ \tau_i \in C\left([t_0,\infty), \mathbb{R}^+\right)$, for every $i = 1, 2, \dots, m,\ \tau_i(t) \le t$ for $t \ge t_0$ and $\lim_{t\to\infty} \tau_i(t) = \infty$. In this paper we review the most interesting sufficient conditions under which all solutions oscillate. An example illustrating the results is given.

Keywords: *oscillation; retarded; differential equations; non-monotone arguments.*

Mathematics Subject Classification (2010): Primary 34K11, Secondary 34K06.

1 Introduction

Consider the first-order linear differential equation with several non-monotone retarded arguments

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0,\ t \ge t_0, \tag{1.1}$$

where the functions $p_i,\ \tau_i \in C([t_0,\infty), \mathbb{R}^+)$, for every $i = 1, 2, \dots, m$, (here $\mathbb{R}^+ = [0,\infty)$), $\tau_i(t) \le t$ for $t \ge t_0$ and $\lim_{t\to\infty} \tau_i(t) = \infty$.

Let $T_0 \in [t_0, +\infty)$, $\tau(t) = \min\{\tau_i(t) : i = 1, \dots, m\}$ and $\tau_{-1}(t) = \sup\{s : \tau(s) \le t\}$. By a solution of the equation (1.1) we understand a function $x \in C([T_0, +\infty), \mathbb{R})$, continuously differentiable on $[\tau_{-1}(T_0), +\infty]$ and that satisfies (1.1) for $t \ge \tau_{-1}(T_0)$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *non-oscillatory.*

For the general theory the reader is referred to [9, 11, 12, 17].

The oscillatory behavior of functional differential equations has been the subject of many investigations. See, for example, [1–20] and the references cited therein.

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2 Oscillation Conditions for Eq. (1.1)

Concerning the differential equation (1.1) with several non-monotone arguments the following related oscillation results have been recently published.

Assume that there exist non-decreasing functions $\sigma_i \in C\left([t_0,\infty),\mathbb{R}^+\right)$ such that

$$\tau_i(t) \le \sigma_i(t) \le t,\ \ i = 1,2,\dots,m. \tag{2.1}$$

In 2015, Infante, Kopladatze and Stavroulakis [14] proved that if

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} \sum_{i=1}^{m} p_i(\xi) \exp\left(\int_{\tau_i(\xi)}^{\xi} \sum_{i=1}^{m} p_i(u)\, du \right) d\xi \right) ds \right]^{1/m} > \frac{1}{m^m}, \tag{2.2}$$

then all solutions of Eq. (1.1) oscillate.

Also, in 2015, Kopladatze [15] improved the above condition as follows. Let there exist some $k \in \mathbb{N}$ such that

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(m \int_{\tau_i(s)}^{\sigma_i(t)} \left(\prod_{\ell=1}^{m} p_\ell(\xi) \right)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right) ds \right]^{\frac{1}{m}}$$

$$> \frac{1}{m^m} \left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right], \tag{2.3}$$

where

$$\psi_1(t) = 0,\ \ \psi_i(t) = \exp\left(\sum_{j=1}^{m} \int_{\tau_j(t)}^{t} \left(\prod_{\ell=1}^{m} p_\ell(s) \right)^{\frac{1}{m}} \psi_{i-1}(s)\, ds \right),\ \ i = 2,3,\dots,$$

$$0 < \alpha_i := \liminf_{t\to\infty} \int_{\sigma_i(t)}^{t} p_i(s)\, ds < \frac{1}{e},\ \ i = 1,2,\dots,m, \tag{2.4}$$

and

$$c_i(\alpha_i) = \frac{1 - \alpha_i - \sqrt{1 - 2\alpha_i - \alpha_i^2}}{2},\ \ i = 1,2,\dots,m, \tag{2.5}$$

then all solutions of Eq. (1.1) oscillate.

In 2016, Bravermen, Chatzarakis and Stavroulakis [7] obtained the following iterative sufficient oscillation conditions

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(t), \tau_i(u))\, du > 1, \tag{2.6}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i\left(u\right) a_r\left(h\left(t\right), \tau_i\left(u\right)\right) du > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.7}$$

or

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i\left(u\right) a_r\left(h\left(t\right), \tau_i\left(u\right)\right) du > \frac{1}{e}, \tag{2.8}$$

where

$$h\left(t\right) = \max_{1\le i\le m} h_i\left(t\right) \text{ and } h_i\left(t\right) = \sup_{t_0\le s\le t} \tau_i\left(s\right),\ i = 1, 2, \ldots, m,$$

$$0 < \alpha := \liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i\left(s\right) ds \le \frac{1}{e} \tag{2.9}$$

and $a_1\left(t,s\right) = \exp^{\left(\int_s^t \sum_{i=1}^m p_i(u)du\right)}$, $a_{r+1}\left(t,s\right) = \exp^{\left(\int_s^t \sum_{i=1}^m p_i(u)a_r(u,\tau_i(u))du\right)}$, $r \in \mathbb{N}$.

Also, in 2016, Akca, Chatzarakis and Stavroulakis [1] improved that result replacing condition (2.6) by the iterative condition

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i\left(u\right) a_r\left(h\left(u\right), \tau_i\left(u\right)\right) du > \frac{1+\ln\lambda_0}{\lambda_0}, \tag{2.10}$$

where λ_0 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$, $0 < \alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i\left(s\right) ds \le \frac{1}{e}$ and $\tau\left(t\right) = \max_{1\le i\le m} \tau_i\left(t\right)$.

In 2018, Attia et al. [3] established the following oscillation conditions.

Assume that $0 < \rho := \liminf_{t\to\infty} \int_{g(t)}^t \sum_{k=1}^n p_k(s)ds \le \frac{1}{\mathrm{e}}$, and

$$\limsup_{t\to\infty} \left(\int_{g(t)}^t Q(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^t \sum_{i=1}^n p_i(s)ds} \right) > 1,$$

where

$$Q(t) = \sum_{k=1}^n \sum_{i=1}^n p_i(t) \int_{\tau_i(t)}^t p_k(s) \mathrm{e}^{\int_{g_k(t)}^t \sum_{i=1}^n p_i(s)ds + (\lambda(\rho)-\epsilon)\int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^n p_\ell(u)du} ds,$$

$\epsilon \in (0,\ \lambda(\rho))$, or

$$\limsup_{t\to\infty} \left(\int_{g(t)}^t Q_1(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^t \sum_{i=1}^n p_i(s)ds} \right) > 1,$$

where

$$Q_1(t) = \sum_{k=1}^n \sum_{i=1}^n p_i(t) \int_{\tau_i(t)}^t p_k(s) \mathrm{e}^{\int_{g_k(t)}^t \sum_{i=1}^n p_i(s)ds + \int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^n (\lambda(q_\ell)-\epsilon_\ell) p_\ell(u)du} ds,$$

$\epsilon_\ell \in (0,\ \lambda(q_\ell))$, and $q_\ell = \liminf_{t\to\infty}\ \int_{\tau_\ell(t)}^t p_\ell(s)ds,\quad \ell = 1,2,...,m$, or

$$\limsup_{t\to\infty}\left(\prod_{j=1}^{n}\left(\prod_{k=1}^{n}\int_{g_j(t)}^{t}R_k(s)ds\right)^{\frac{1}{n}}+\frac{\prod_{k=1}^{n}c(\beta_k)}{n^n}\mathrm{e}^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n}p_\ell(s)ds}\right)>\frac{1}{n^n},$$

where

$$R_k(s)=\mathrm{e}^{\int_{g_k(s)}^{s}\sum_{i=1}^{n}p_i(u)du}\sum_{i=1}^{n}p_i(s)\int_{\tau_i(s)}^{s}p_k(u)\mathrm{e}^{(\lambda(\rho)-\epsilon)\int_{\tau_k(u)}^{g_k(s)}\sum_{\ell=1}^{n}p_\ell(v)dv}du,$$

$\epsilon \in (0,\ \lambda(\rho))$, and $0 < \beta_k := \liminf_{t\to\infty}\ \int_{\sigma_i(t)}^t p_i(s)ds \le \frac{1}{\mathrm{e}}$. Then Eq. (1.1) is oscillatory.

Recently Bereketoglu et al. [4] established the following conditions.

Assume that there exist non-decreasing functions $\sigma_i \in C([t_0,\infty),\mathbb{R}^+)$ such that (2.1) is satisfied and for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)}P_k(u)\,du\right)ds\right)\right]^{1/m}>\frac{1}{m^m},\tag{2.11}$$

or

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)}P_k(u)\,du\right)ds\right)\right]^{1/m}>\frac{1}{m^m}\left[1-\prod_{i=1}^{m}c_i(\alpha_i)\right],\tag{2.12}$$

where

$$P_k(t)=\sum_{j=1}^{m}p_j(t)\left\{1+m\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{t}P_{k-1}(u)\,du\right)ds\right]^{1/m}\right\},$$

with $P_0(t) = m\left[\prod_{\ell=1}^{m}p_\ell(t)\right]^{1/m}$, α_i is given by (2.4) and $c_i(\alpha_i)$ by (2.5). Then all solutions of Eq.(1.1) oscillate.

Very recently Moremedi, Jafari and Stavroulakis [19] further improved the above conditions as follows.

Assume that there exist non-decreasing functions $\sigma_i \in C([t_0,\infty),\mathbb{R}^+)$ such that (2.1) is satisfied and for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)}P_k(u)\,du\right)ds\right)\right]^{1/m}>\frac{1}{m^m},\tag{2.13}$$

or

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m}\left[1-\prod_{i=1}^{m} c_i(\alpha_i)\right], \tag{2.14}$$

where

$$P_k(t) = P(t)\left[1+\int_{\sigma_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)\,d\xi\right)du\right)ds\right] \tag{2.15}$$

with $P_0(t) = P(t) = \sum_{i=1}^{m} p_i(t)$, α_i is given by (2.4) and $c_i(\alpha_i)$ by (2.5). Then all solutions of Eq.(1.1) oscillate.

Remark 2.1 It is clear that the left-hand sides of both conditions (2.11), (2.12) and (2.13), (2.14) are identically the same and also the right-hand side of (2.12) and (2.14) reduces to (2.11) and (2.13) when $c_i(\alpha_i) = 0$. Thus, it seems that the above conditions (2.14) and (2.12) are exactly the same as conditions (2.13) and (2.11), when $c_i(\alpha_i) = 0$. One may notice, however, that the condition (2.4) is required in (2.14) and (2.12) but not in (2.13) and (2.11).

In the case of monotone arguments we have the following.

Let τ_i be non-decreasing functions and for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\tau_j(t)}^{t}\left(p_i(s)\exp\left(\int_{\tau_i(s)}^{\tau_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m} \tag{2.16}$$

or

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\tau_j(t)}^{t}\left(p_i(s)\exp\left(\int_{\tau_i(s)}^{\tau_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m}\left[1-\prod_{i=1}^{m} c_i(\alpha_i)\right], \tag{2.17}$$

where

$$P_k(t) = P(t)\left[1+\int_{\tau_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)d\xi\right)du\right)ds\right] \tag{2.18}$$

with $P_0(t) = P(t) = \sum_{j=1}^{m} p_j(t), \alpha_i$ is given by (2.4), and $c_i(\alpha_i)$ by (2.5). Then all solutions of (1.1) oscillate.

At this point it should be mentioned that in the case of monotone arguments several oscillation conditions involving the liminf were established.

In 1982, Ladas and Stavroulakis [16] considered the differential equation with several constant delays of the form

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0, \tag{2.19}$$

where τ_i, $i = 1, 2, ..., m$ are positive constants and $p_i(t)$, $i = 1, 2, ..., m$ are positive and continuous functions, and established the following oscillation conditions involving liminf. (See also [1, 2]). Consider the differential equations (2.19) and assume that

$$\liminf_{t\to\infty} \int_{t-(\tau_i/2)}^{t} p(s)ds > 0, \quad i = 1, 2, ..., m. \tag{2.20}$$

Then each one of the following conditions

$$\liminf_{t\to\infty} \int_{t-\tau_i}^{t} p_i(s)ds > \frac{1}{e}, \text{ for some } i,\ i = 1, 2, ..., m, \tag{2.21}$$

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{e}, \text{where } \tau = \min[\tau_1, ..., \tau_m\}, \tag{2.22}$$

$$\left[\prod_{i=1}^{m}\left(\sum_{j=1}^{m} \liminf_{t\to\infty} \int_{t-\tau_j}^{t} p_i(s)ds\right)\right]^{\frac{1}{n}} > \frac{1}{e} \tag{2.23}$$

or

$$\frac{1}{m}\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_i(s)ds\right) + \frac{2}{m}\sum_{i<j,i,j=1}^{m}\left[\left(\liminf_{t\to\infty}\int_{t-\tau_j}^{t} p_i(s)ds\right) \times \left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_j(s)ds\right)\right]^{1/2} > \frac{1}{e} \tag{2.24}$$

implies that every solution of (2.19) oscillates.

Later in 1996, Li [18] proved that the same conclusion holds if

$$\liminf_{t\to\infty} \sum_{i=1}^{m} \int_{t-\tau_i}^{t} p_i(s)ds > \frac{1}{e}. \tag{2.25}$$

In 1984, Hunt and Yorke [13] considered the differential equation with variable delays of the form

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t - \tau_i(t)) = 0, \tag{2.26}$$

where τ_i are continuous and positive valued on $[0, \infty)$ and proved the following. If there is a uniform upper bound τ_0 on the $\tau_i^{,}\ s$ and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\tau_i(t) > \frac{1}{e}, \tag{2.27}$$

then all solutions of Eq. (2.26) oscillate.

In 2003, Grammatikopoulos, Koplatadze and Stavroulakis [10] also studied Eq.(1.1) in the case that $\tau_i(t)$ $(i = 1, 2, ..., m)$ are nondecreasing, and established the following result. Assume that

$$\int_0^\infty \mid p_i(t) - p_j(t) \mid dt < \infty \quad (i, j = 1, 2, ..., m) \tag{2.28}$$

$$\liminf_{t\to\infty}\int_{\tau_i(t)}^{t} p_i(s)ds > 0 \quad (i = 1, 2, ..., m) \tag{2.29}$$

and

$$\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\int_{\tau_i(t)}^{t} p_i(s)ds\right) > \frac{1}{e}. \tag{2.30}$$

Then all solutions of Eq. (1.1) oscillate.

3 Corollaries and Examples

In the case $m = 1$, Eq.(1.1) reduces to the equation

$$x'(t) + p(t)\,x(\tau(t)) = 0. \tag{3.1}$$

In 2012, Braverman and Karpuz [6] derived the following sufficient oscillation condition for Eq.(3.1):

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\} ds > 1, \tag{3.2}$$

while in 2014, Stavroulakis [20] improved the above condition as follows:

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\} ds > 1 - \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right), \tag{3.3}$$

where $\sigma(t) := \sup_{s\le t}\tau(s)$.

From the above conditions (2.13) and (2.14) the following corollary is immediate. It is clear that the conditions in this corollary essentially improve the conditions (3.2) and (3.3).

Corollary 3.1 *Assume that there exists a non-decreasing function $\sigma(t)$ such that $\tau(t) \le \sigma(t) \le t$ and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\,du\right) ds > 1 \tag{3.4}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\,du\right) ds > 1 - c(\alpha), \tag{3.5}$$

where

$$P_k(t) = p(t)\left[1 + \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\,d\xi\right) du\right) ds\right], P_0(t) = p(t),$$

$$0 < \alpha := \liminf_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\,ds \ \le \frac{1}{e}, \tag{3.6}$$

and $c(\alpha) = \dfrac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}$. *Then all solutions of Eq.(3.1) oscillate.*

The following example is given to illustrate the results. Note that in this example (cf. [4, 6, 14, 19]) the interval of the values that p can take is increased.

Example 3.1 Consider the equation

$$x'(t) + px(\tau(t)) = 0,\ t \geq 0,\ p > 0, \tag{3.7}$$

with the retarded argument

$$\tau(t) = \begin{cases} t-1, & t \in [3n,\ 3n+1], \\ -3t + (12n+3), & t \in [3n+1,\ 3n+2], \\ 5t - (12n+13), & t \in [3n+2,\ 3n+3]. \end{cases}$$

For this equation, as in [6, 14], one may choose the function

$$\sigma(t) = \begin{cases} t-1, & t \in [3n,\ 3n+1], \\ -3n, & t \in [3n+1,\ 3n+2.6], \\ 5t - (12n+13), & t \in [3n+2.6,\ 3n+3]. \end{cases}$$

If we choose $t_n = 3n+3$, (cf. [[6], Example 1] and [[14], Example 4.2]), then for $k = 1$, the condition (3.4) of Corollary 1 reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du \right) ds \geq \lim_{n\to\infty} \int_{3n+2}^{3n+3} p \exp\left(\int_{5s-(12n+13)}^{3n+2} P_1(u)\,du \right) ds,$$

where

$$\begin{aligned} P_1(t) &= p\left[1 + \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{t} p \exp\left(\int_{\tau(u)}^{u} p\,d\xi \right) du \right) ds \right] \\ &\geq p\left[1 + \int_{3n+2}^{3n+3} p \exp\left(\int_{5s-(12n+13)}^{3n+3} p \exp(p)\,du \right) ds \right] \\ &= p\left[1 + \left(\frac{e^{6pe^p} - e^{pe^p}}{5} \right) e^{-p} \right]. \end{aligned}$$

Therefore

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du \right) ds \geq \frac{p}{5P_1}\left(e^{5P_1} - 1\right),$$

where $P_1 = p\left[1 + \left(\frac{e^{6pe^p} - e^{pe^p}}{5}\right) e^{-p}\right]$. For $p = 0.251$, $P_1 \approx 0.4676$, and so $\frac{p}{5P_1}\left(e^{5P_1} - 1\right) \approx 1.0052 > 1$. Therefore all solutions of Eq.(3.7) oscillate.

Observe, however, that when we consider the conditions stated in [1, 6, 7, 14, 15, 20] and [4] for the above equation (3.7), we obtain the following.

1. Observe that, for $t_n = 3n+3$,

$$\int_{\sigma(3n+3)}^{3n+3} p\exp\left\{\int_{\tau(s)}^{\sigma(3n+3)} pd\xi\right\} ds = \int_{3n+2}^{3n+3} p\exp\left\{\int_{5s-(12n+13))}^{3n+2} pd\xi\right\} ds = \frac{e^{5p}-1}{5}$$

and condition (3.2) reduces to $\frac{e^{5p}-1}{5} > 1$. But, for $p = 0.251$ $\frac{e^{5p}-1}{5} \approx 0.50157 < 1$, therefore the condition (3.2) is not satisfied.

2. Similarly, in the condition (3.3), $\mathfrak{a} = \liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} p(s)\, ds = \lim_{n\to\infty} \int\limits_{3n+2}^{3n+3} pds =$ p and $c(\mathfrak{a}) = c(p) = \dfrac{1-p-\sqrt{1-2p-p^2}}{2}$. And, as before, (3.3) reduces to $\frac{e^{5p}-1}{5} >$ $1 - \dfrac{1-p-\sqrt{1-2p-p^2}}{2}$. Taking $p = 0.251$, the left-hand side of (3.3) is equal to 0.50157, while the right-hand side is 0.95527. Therefore this condition is not satisfied.

3. The condition (2.2) reduces to

$$\limsup_{t\to+\infty} \int\limits_{\sigma(t)}^{t} p\exp\left(\int\limits_{\tau(s)}^{\sigma(t)} p\exp\left(\int\limits_{\tau(\xi)}^{\xi} pdu\right) d\xi\right) ds > 1, \tag{3.8}$$

and, as in [[14], Example 4.2], the choice of $t_n = 3n+3$ leads to the inequality

$$\frac{\left(e^{5pe^p}-1\right)}{5e^p} > 1. \tag{3.9}$$

Observe, however, that for $p = 0.251$,

$$\frac{\left(e^{5pe^p}-1\right)}{5e^p} \approx 0.62524 < 1.$$

Therefore the condition (3.9) is not satisfied.

4. The condition (2.3), for $k = 2$, reduces to

$$\limsup_{t\to\infty} \int\limits_{\sigma(t)}^{t} p\exp\left(\int\limits_{\tau(s)}^{\sigma(t)} p\ \psi_2(\xi)\, d\xi\right) ds > 1 - c(\alpha), \tag{3.10}$$

where $\psi_2(\xi) = 1$, and for $t_n = 3n+3$, as before, it leads to

$$\frac{e^{5p}-1}{5} > 1 - \frac{1-p-\sqrt{1-2p-p^2}}{2}.$$

For $p = 0.251$, we have

$$\frac{e^{5p}-1}{5} \approx 0.50157,$$

while the right-hand side

$$1-c(p)\approx 0.95527.$$

Therefore the condition (3.10) is not satisfied.

5. The condition (2.6) for $r=1$ reduces to

$$\limsup_{t\to\infty}\int_{h(t)}^{t} pa_1(h(t),\tau(s))\,ds>1, \tag{3.11}$$

where

$$h(t)=\sigma(t)\ \text{ and }\ a_1(t,s)=\exp\left(\int_s^t pdu\right).$$

That is, to the condition

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right)ds>1, \tag{3.12}$$

and, as before, for $t_n=3n+3$ and $p=0.251$, we have

$$\frac{e^{5p}-1}{5}\approx 0.50157<1. \tag{3.10}$$

Therefore the condition (3.11) is not satisfied.

6. Similarly, condition (2.10) for $r=1$ reduces to

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right)ds>\frac{1+\ln\lambda_0}{\lambda_0}, \tag{3.14}$$

where λ_0 is the smaller root of the equation $\lambda=e^{p\lambda}$. As before, for $t_n=3n+3$ and $p=0.251$, we have

$$\frac{e^{5p}-1}{5}\approx 0.50157,$$

while

$$\frac{1+\ln\lambda_0}{\lambda_0}\approx 0.94893.$$

Therefore the condition (3.14) is not satisfied.

7. For $k=1$, condition (2.11) reduces to

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du\right)ds>1. \tag{3.15}$$

If we choose $t_n = 3n+3$,

$$\begin{aligned} P_1(t) &= p\left\{1+\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{t} pdu\right)ds\right\} = p\left\{1+\int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+3} pdu\right)ds\right\} \\ &= p\left(1+\frac{e^{6p}-e^{p}}{5}\right). \end{aligned}$$

And, as before, (3.15) reduces to

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right) > 1.$$

For $p = 0.251$ we find $P_1 \approx 0.412812$ and so

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right) \approx 0.836386 < 1.$$

Therefore the condition (3.15) is not satisfied.

We conclude, therefore, that for $p = 0.251$ no one of the conditions (3.2), (3.3), (2.2), (2.3) for $k = 2$, (2.6) and (2.10) for $r = 1$, and (2.11) is satisfied.

It should be also pointed out that not only for this value of $p = 0.251$ but for all values of $p > 0.251$, especially for all values of $p \in [0.251, 0.358]$, (cf. [[14], Example 4.2]),

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right) > 1$$

and therefore all solutions of (3.7) oscillate. Observe, however, that for $p = 0.358$

$$\frac{e^{5p}-1}{5} \approx 0.99789 < 1,$$

also for $p = 0.3$

$$\frac{\left(e^{5pe^{p}}-1\right)}{5e^{p}} \approx 0.974101 < 1,$$

$$\frac{e^{5p}-1}{5} \approx 0.696337 < 0.912993 \approx \frac{1+\ln\lambda_0}{\lambda_0},$$

and for $p = 0.263$, $P_1 \approx 0.44944$ and so

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right) \approx 0.99024 < 1.$$

Therefore for all values of $p \in [0.251, 0.358]$ the conditions of Corollary 1 are satisfied and so all solutions to Eq.(3.7) oscillate, while no one of the above mentioned conditions is satisfied for these values of $p \in [0.251, 0.358]$.

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