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Generalized Monotone Method for Nonlinear Caputo Fractional Impulsive Differential Equations

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Abstract: Generalized monotone method is a useful technique to prove the existence of coupled minimal and maximal solutions when the nonlinear function is the sum of an increasing and decreasing functions. In this work, we develop generalized monotone method for Caputo fractional impulsive differential equations with initial conditions, using coupled lower and upper solutions of Type 1. For that purpose we develop comparison results for Caputo fractional impulsive differential equation. Further, under uniqueness assumption, we prove the existence of the unique solution of the nonlinear Caputo fractional impulsive differential equation with initial conditions.

Keywords: *nonlinear Caputo fractional differential equations; impulsive differential equations; generalized monotone method.*

Mathematics Subject Classification (2010): 34A08, 34A37.

1 Introduction

In the past few decades, the impulsive equations have exhibited more advantages in the mathematical models of physical and biological models. See [2, 3, 6–8, 14, 23] for details. These equations can describe more naturally and more closed to the real world problems. See [9, 12, 15]. In the past four decades, the study of fractional differential equations has gained lots of importance due to its applications. See [1, 4, 5, 10, 11, 13, 25, 26]. In fact, the dynamic equations with fractional derivative have represented as better and economical models in various branches of science and engineering. See [12, 13, 15–17].

In this work, we develop generalized monotone method combined with coupled lower and upper solutions for nonlinear Caputo fractional impulsive differential equations with initial conditions. In general, explicit solution for nonlinear problems is rarely possible.

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In addition, explicit solution even for linear Caputo fractional differential equations with variable coefficients with or without impulses and initial conditions is not trivial either. However, explicit solution of the solution and/or representation form of the solution of linear Caputo fractional impulsive differential equation with initial condition is possible. See [22] for more details. In addition, in [22], the uniqueness of the solution of the linear Caputo fractional impulsive differential equation has been proved by developing a comparison result.

We apply generalized monotone method, Laplace transform and some properties in the main result. See [6, 18–21, 24, 27, 28] for more details. In [22], we have obtained explicit solutions for the linear Caputo fractional impulsive differential equations with initial condition. In addition, we have also developed a comparison result in [22] relative to coupled lower and upper solutions.

In the present work, we have also developed linear comparison results as an auxiliary result which is useful in our main result. We have developed monotone sequences $\{v_n\}$ and $\{w_n\}$ which are piece-wise left continuous using the coupled lower and upper solutions, when the nonlinear function is the sum of non-decreasing and non-increasing functions. We have established the monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of the nonlinear problem. Furthermore, under uniqueness assumptions on the nonlinear terms, we prove that the coupled minimal and maximal solutions reduce to the unique solutions of the nonlinear problem.

2 Preliminary Results

In this section, we introduce some known definitions and results, which are needed for the main results. First, we recall some basic definitions.

Definition 2.1 The Riemann-Liouville fractional integral of $u(t)$ of order q is defined by

$$D_t^{-q} u = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds, \tag{1}$$

where $0 < q \leq 1$.

Definition 2.2 The Caputo (left) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as:

$$^cD_t^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s) \mathrm{d}s. \tag{2}$$

Definition 2.3 The Riemann-Liouville (left-sided) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} u(s) ds, \ t > 0. \tag{3}$$

The relation between Caputo derivative and Riemann-Liouville derivative of a function $f(t)$ is given by

$$^cD^q u(t) = D^q (u(t) - u(0)).$$

This relation will be useful for results relative to differential inequalities.

Next we define the Mittag-Leffler function which is useful in computing the solution of the linear fractional differential equations.

Definition 2.4 The two parameter Mittag-Leffler function is defined as

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+r)}. \tag{4}$$

If $r = q$, the relation (4) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma q(k+1)}. \tag{5}$$

If $r = 1$, the Mittag-leffler function is defined as

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma q(k+1)}. \tag{6}$$

See [5, 10, 13, 16] for more details.

In our next definition we assume $p = 1-q$, when $0 < q < 1$, $J = (0, T]$ and $J_0 = [0, T]$.

Definition 2.5 A function $\phi(t) \in C(J, \mathbb{R})$ is a C_p continuous function on J if $t^{1-q}\phi(t) \in C(J_0, R)$. The set of C_p continuous functions is denoted by $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$, we call the function $t^{1-q}\phi(t)$ the continuous extension of $\phi(t)$.

Next, we introduce some theorems and lemmas which are useful to our main results.

Lemma 2.1 *Let $J = [0, T]$, $m \in C_p(J, \mathbb{R})$ be such that for some $t^0 \in J$, we have $m(t^0) = 0$ and $m(t) \leq 0$ for $t \in [0, t^0]$, then (Riemann-Liouville fractional derivative) $D^q m(t^0) \geq 0$.*

See [4, 5] for the details of the proof.

Lemma 2.2 *Let $J = [0, T]$, such that $0 < t_1 < t_2 < ... < t_{N-1} < t_{N-1} = T$, and m be piece-wise left continuous on each $(t_i, t_{i+1}]$. Suppose there exists a $t^0 \in J$, such that $m(t^0) = 0$ and $m(t) \leq 0$ for $t \in [0, t^0]$, then $D^q m(t^0) \geq 0$.*

See [4, 21] for the details of the proof.

Remark: The above result is also true with Caputo derivative in place of Riemann-Liouville derivative. The proof can be easily obtained by applying the relation between the Caputo derivative and the Riemann-Liouville derivative, which is ${}^cD^q m(t) = D^q \left(m(t) - m(0)\right)$.

Consider the linear Caputo fractional differential equation

$${}^cD^q u = \lambda u + f(t), \qquad u(0) = u_0. \tag{7}$$

Then the solution of (7) is given by

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s) \mathrm{d}s. \tag{8}$$

Consider the nonlinear Caputo fractional impulsive differential equations with initial condition

$$\begin{cases} {}^cD^q u(t) = \lambda u(t) + \sum\limits_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)u(t_i) \\ \qquad + \sum\limits_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)u(t_i) + f(t,u(t)) + g(t,u(t)), \\ u(o) = u_0, \end{cases} \tag{9}$$

where $t \in [0,T]$, and $0 < t_1 < t_2 < \cdots < t_N = T$. Also, $\chi(t-t_i)$ is the Heaviside unit step function which is left continuous,

$$\chi(t-t_i) = \begin{cases} 1, & if\ t > t_i, \\ 0, & if\ t \le t_i. \end{cases} \tag{10}$$

Furthermore, we assume that $\lambda \neq 0$, and for each $1 \le i \le N$, $c_i\chi(t-t_i)s_i(t-t_i) \ge 0$ and $b_i\chi(t-t_i)r_i(t-t_i) \le 0$. The function $f(t,u)$ is nondecreasing in u and $g(t,u)$ is nonincreasing in u. In addition, $s_i(t-t_i)$ and $r_i(t-t_i)$ are continuous on each interval $[t_i, t_{i+1}]$ for $i = 1,\dots,N-1$. Therefore, they are bounded on each interval.

Next we define the coupled lower and upper solutions of natural type as well of Type 1. See [9] for other types of coupled lower and upper solutions.

Definition 2.6 *If $u : C[0,T] \to \mathbb{R}$ which is piecewise left continuous at t_i, $i = 1,2,\dots,N$, such that $0 < t_1 \le t_2 \le \cdots \le t_N = T$, and whose Caputo derivative of order q exists on $[0,T]$. Then we denote $f \in PC^q[[0,T],\mathbb{R}]$.*

Definition 2.7 We say that v, w are $PC^q[[0,T],\mathbb{R}]$ piecewise left continuous on (t_i, t_{i+1}) for $i = 1,\dots,N-1$. Then we say v and w are coupled lower and upper solutions of natural type of (9) if they satisfy the inequalities:

$$\begin{aligned} {}^cD^q v(t) &\le \lambda v(t) + \sum_{i=1}^{N} a_i\chi(t-t_i)s_i(t-t_i)v(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)v(t_i) \\ &\quad + f(t,v) + g(t,v), \\ v(0) &\le u_0, \end{aligned} \tag{11}$$

$$\begin{aligned} {}^cD^q w(t) &\ge \lambda w(t) + \sum_{i=1}^{N} a_i\chi(t-t_i)s_i(t-t_i)w(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)w(t_i) \\ &\quad + f(t,w) + g(t,w), \\ w(0) &\ge u_0. \end{aligned} \tag{12}$$

Definition 2.8 We say that v, w are $PC^q[[0,T],\mathbb{R}]$ piecewise left continuous on (t_i, t_{i+1}) for $i = 1,\dots,N-1$. Then we say v and w are coupled lower and upper solutions of type 1 if they satisfy the inequalities:

$$\begin{aligned} {}^cD^q v(t) &\le \lambda v(t) + \sum_{i=1}^{N} a_i\chi(t-t_i)s_i(t-t_i)v(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)w(t_i) \\ &\quad + f(t,v) + g(t,w) \\ v(0) &\le u_0, \end{aligned} \tag{13}$$

$$
\begin{aligned}
{}^cD^q w(t) &\geq \lambda w(t) + \sum_{i=1}^{N} a_i \chi(t-t_i) s_i(t-t_i) w(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) v(t_i) \\
&\quad + f(t,w) + g(t,v), \\
w(0) &\geq u_0.
\end{aligned} \tag{14}
$$

Theorem 2.1 *If $\lambda \neq 0$, $v(t)$ and $w(t)$ are coupled lower and upper solutions of type 1 of the nonlinear Caputo impulsive fractional differential equation (9), where $f(t,u)$ and $g(t,u)$ satisfy the one-sided Lipschitz condition in u, of the following form with $u_1 \geq u_2$*

$$f(t,u_1) - f(t,u_2) \leq L_1(u_1 - u_2), \tag{15}$$

$$g(t,u_1) - g(t,u_2) \geq -L_2(u_1 - u_2), \tag{16}$$

where $L_1 \geq 0$ and $L_2 \geq 0$. Then $v(0) \leq w(0)$ implies that $v(t) \leq w(t)$, $\forall t \geq J = [0.T]$.

See [22] for the details of the proof.

3 Auxiliary Results

In this section, we prove a comparison theorem which will be used to prove the generalized monotone method in the main result.

Theorem 3.1 *If the functions $P(t)$ and $Q(t)$ are $PC^q[[0,T],\mathbb{R}]$ such that satisfy the following inequalities:*

$${}^cD^q P \leq \lambda P + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) P(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) Q(t_i), \tag{17}$$

$${}^cD^q Q \geq \lambda Q + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) Q(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) P(t_i), \tag{18}$$

where $\lambda \geq 0$, and for each $1 \leq i \leq N$, $c_i\chi(t-t_i)s_i(t-t_i) \geq 0$ and $b_i\chi(t-t_i)r_i(t-t_i) \leq 0$, then the initial condition $P(0) \leq 0$ and $Q(0) \geq 0$ implies $P(t) \leq 0$ and $Q(t) \geq 0$ for all $t \in [0,T]$.

Proof. We prove by the method of mathematical induction. For $t \in [0, t_1)$

$${}^cD^q P(t) \leq \lambda P(t), \qquad {}^cD^q Q(t) \geq \lambda Q(t). \tag{19}$$

Then we can get

$$P(t) \leq P(0) E_{q,1}(\lambda t^q) \leq 0, \qquad Q(t) \geq Q(0) E_{q,1}(\lambda t^q) \geq 0. \tag{20}$$

For $t = t_1$, we have

$$P(t_1) \leq P(0) E_{q,1}(\lambda t_1^q) \leq 0, \qquad Q(t_1) \geq Q(0) E_{q,1}(\lambda t_1^q) \geq 0. \tag{21}$$

Assume the result is true for $t \in [t_{k-1}, t_k)$, for $0 \leq k \leq N-1$, which yields $P(t_k) \leq 0$ and $Q(t_k) \geq 0$ for all $0 \leq k \leq N-1$. Then, for $t \in [t_k, t_{k+1})$,

$${}^cD^q P(t) \leq \lambda P(t) + \sum_{i=1}^{k} c_i \chi(t-t_i) s_i(t_{k+1}-t_i) P(t_i) + \sum_{i=1}^{k} b_i \chi(t-t_i) r_i(t_{k+1}-t_i) Q(t_i). \tag{22}$$

With the result of $P(t_k) \leq 0$ and $Q(t_k) \geq 0$ for all $0 \leq k \leq N-1$, we can get ${}^cD^qP \leq \lambda P$. Therefore we have $P(t) \leq P(0)E_{q,1}(\lambda t^q) \leq 0$ on $[0, t_{k+1})$. Then for $t = t_{k+1}$, we have $P(t_{k+1}) \leq 0$. Similarly, we have the results for $Q(t)$,

$$Q(t) \geq Q(0)E_{q,1}(\lambda t^q) \geq 0.$$

Then $Q(t_{k+1}) \geq 0$. Since it is true for $i = 1$, therefore, by induction, for all t_i, $0 \leq i \leq N$, $P(t_i) \leq 0$ and $Q(t_i) \geq 0$. Then we have $P(t) \leq 0$ and $Q(t) \geq 0$ for all $0 \leq t \leq t_N$, which completes the proof.

Lemma 3.1 *If the functions $P(t)$ and $Q(t)$ are $PC^q[[0,T],\mathbb{R}]$ such that to satisfy the following inequalities:*

$$ {}^cD^qP \leq \lambda P + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)P(t_i), \tag{23}$$

$$ {}^cD^qQ \geq \lambda Q + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i). \tag{24}$$

where $\lambda \neq 0$, $\sum\limits_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i) \geq 0$, then the initial condition $P(0) \leq 0$ and $Q(0) \geq 0$ implies $P(t) \leq 0$ and $Q(t) \geq 0$ for all $t \in [0,T]$.

Proof. This is a special case of Theorem 3.1 with $b_i = 0$ for all $i = 1, 2, \cdots, N$. Therefore the proof is almost the same as the one in Theorem 3.1.

4 Main Result

In this section, we consider the nonlinear Caputo impulsive differential equation of the form (9), which has application in science and biology. Since it is rarely possible to compute the solution of the nonlinear problem with or without impulses and with integer derivatives or fractional derivatives, hence we develop generalized monotone method together with coupled lower and upper solution. See [9, 18] for more details.

The method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (9) on the sector defined by coupled lower and upper solutions. Furthermore, if the nonlinear functions satisfy uniqueness condition, then the coupled minimal and maximal solutions coincide to be the unique solution of (9).

Note that the generalized monotone method is a more appropriate method to prove the existence of the nonlinear Caputo fractional impulsive differential equations when the nonlinear function is the sum of nondecreasing and nonincreasing functions.

In order to prove our main results, we need the existence and uniqueness of solution of two linear systems of Caputo fractional impulsive differential equations with initial condition. This is precisely the next result.

Theorem 4.1 *Let v_0, w_0 be coupled lower and upper solutions of (9) of type 1, such that $v_0(t) \leq w_0(t)$ on $t \in [0,T]$. Suppose η and μ are any two functions such that*

$v_0 \le \eta \le \mu \le w_0$ on $[0,T]$, then the solution of the following linear Caputo fractional impulsive differential equations:

$$\begin{cases} {}^cD^q p = \lambda p + f(t,\eta) + g(t,\mu) + \sum\limits_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)p(t_i) + \sum\limits_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)q(t_i), \\ p(0) = u_0, \end{cases} \tag{25}$$

$$\begin{cases} {}^cD^q q = \lambda q + f(t,\mu) + g(t,\eta) + \sum\limits_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)q(t_i) + \sum\limits_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)p(t_i), \\ q(0) = u_0, \end{cases} \tag{26}$$

exists and it is unique.

Proof. Since $\mu(t)$ and $\eta(t)$ are known functions of t, it is easy to see that $f(t,\mu)$, $f(t,\eta)$, $g(t,\mu)$ and $g(t,\eta)$ become functions of t and let us denote

$$f(t,\eta) + g(t,\mu) = F(t), \qquad f(t,\mu) + g(t,\eta) = G(t). \tag{27}$$

Then the equations (25) and (26) become linear system of Caputo fractional impulsive differential equations, namely

$$\begin{aligned} &{}^cD^q p = \lambda p + F(t) + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)p(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)q(t_i), \\ &p(0) = u_0, \end{aligned} \tag{28}$$

$$\begin{aligned} &{}^cD^q q = \lambda q + G(t) + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)p(t_i), \\ &q(0) = 0. \end{aligned} \tag{29}$$

Applying the Laplace transformation, the solution of the $p(t)$ and $q(t)$ are given by

$$\begin{cases} p = u_0E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)F(s)\mathrm{d}s \\ \qquad + \sum\limits_{i=1}^{N} c_i\chi(t-t_i)S_i(t-t_i)p(t_i) + \sum\limits_{i=1}^{N} b_i\chi(t-t_i)R_i(t-t_i)q(t_i), \\ q = u_0E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)G(s)\mathrm{d}s \\ \qquad + \sum\limits_{i=1}^{N} c_i\chi(t-t_i)S_i(t-t_i)q(t_i) + \sum\limits_{i=1}^{N} b_i\chi(t-t_i)R_i(t-t_i)p(t_i), \end{cases} \tag{30}$$

where $S_i(t-t_i) = \mathfrak{L}^{-1}\left(\frac{\mathfrak{L}(s_i)}{s^q-\lambda}\right)$ and $R_i(t-t_i) = \mathfrak{L}^{-1}\left(\frac{\mathfrak{L}(r_i)}{s^q-\lambda}\right)$, for $i = 1, 2, \cdots, N$. $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ are the Laplace transformation and the inverse Laplace transformation, respectively. Then for $t \in [0, t_1)$, the equations (28) and (29) reduce to

$${}^cD^q p = \lambda p + F(t), \qquad {}^cD^q q = \lambda q + G(t). \tag{31}$$

Use the result of (8). The solution $p(t)$ and $q(t)$ can be given by

$$\begin{cases} p = u_0E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)F(s)\mathrm{d}s, \\ q = u_0E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)G(s)\mathrm{d}s. \end{cases} \tag{32}$$

For $t = t_1$, we get

$$\begin{cases} p(t_1) = u_0 E_{q,1}(\lambda t_1^q) + \int_0^{t_1} (t_1 - s)^{q-1} E_{q,q}(\lambda (t_1 - s)^q) F(s) \mathrm{d}s, \\ q(t_1) = u_0 E_{q,1}(\lambda t_1^q) + \int_0^{t_1} (t_1 - s)^{q-1} E_{q,q}(\lambda (t_1 - s)^q) G(s) \mathrm{d}s. \end{cases} \tag{33}$$

For $t \in [t_1, t_2)$, the equations (28) and (29) reduce to

$$^cD^q p = \lambda p + F(t) + c_1 s_1(t - t_1) p(t_1) + b_1 r_1(t - t_1) q(t_1), \tag{34}$$

$$^cD^q q = \lambda q + G(t) + c_1 s_1(t - t_1) q(t_1) + b_1 r_1(t - t_1) p(t_1). \tag{35}$$

The solution can be given as

$$\begin{cases} p = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t - s)^{q-1} E_{q,q}(\lambda (t - s)^q) F(s) \mathrm{d}s \\ \qquad + c_1 S_1(t - t_1) p(t_1) + b_1 R_1(t - t_1) q(t_1), \\ q = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t - s)^{q-1} E_{q,q}(\lambda (t - s)^q) G(s) \mathrm{d}s \\ \qquad + c_1 S_1(t - t_1) q(t_1) + b_1 R_1(t - t_1) p(t_1). \end{cases} \tag{36}$$

After substituting $p(t_1)$ and $q(t_1)$, the equation (36) reduces to

$$\begin{cases} p = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t - s)^{q-1} E_{q,q}(\lambda (t - s)^q) F(s) \mathrm{d}s \\ \qquad + c_1 S_1(t - t_1) \left(u_0 E_{q,1}(\lambda t_1^q) + \int_0^{t_1} (t_1 - s)^{q-1} E_{q,q}(\lambda (t_1 - s)^q) F(s) \mathrm{d}s \right) \\ \qquad + b_1 R_1(t - t_1) \left(u_0 E_{q,1}(\lambda t_1^q) + \int_0^{t_1} (t_1 - s)^{q-1} E_{q,q}(\lambda (t_1 - s)^q) G(s) \mathrm{d}s \right), \\ q = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t - s)^{q-1} E_{q,q}(\lambda (t - s)^q) G(s) \mathrm{d}s \\ \qquad + c_1 S_1(t - t_1) \left(u_0 E_{q,1}(\lambda t_1^q) + \int_0^{t_1} (t_1 - s)^{q-1} E_{q,q}(\lambda (t_1 - s)^q) G(s) \mathrm{d}s \right) \\ \qquad + b_1 R_1(t - t_1) \left(u_0 E_{q,1}(\lambda t_1^q) + \int_0^{t_1} (t_1 - s)^{q-1} E_{q,q}(\lambda (t_1 - s)^q) F(s) \mathrm{d}s \right), \end{cases} \tag{37}$$

where $S_1(t - t_1) = \mathfrak{L}^{-1}\left(\frac{\mathfrak{L}(s_1)}{s^q - \lambda}\right)$ and $R_1(t - t_1) = \mathfrak{L}^{-1}\left(\frac{\mathfrak{L}(r_1)}{s^q - \lambda}\right)$. Then we can find the value of $p(t_2)$ and $q(t_2)$ by substituting t_2 into the equation (37). Then after another iteration, we can get the solution for $t \in [t_2, t_3)$. If we continue the above process, we can obtain a closed form of solution of (25)-(26) for all $t \in [0, T]$.

In order to prove the uniqueness of the solution of the equations (25) and (26), let (p_1, q_1) and (p_2, q_2) be two solutions. Then let $m = p_1 - p_2$ and $n = q_1 - q_2$. Then,

$$\begin{aligned} &^cD^q m = \lambda m + \sum_{i=1}^N c_i \chi(t - t_i) s_i(t - t_i) m(t_i) + \sum_{i=1}^N b_i \chi(t - t_i) r_i(t - t_i) n(t_i), \\ &m(0) = 0, \end{aligned} \tag{38}$$

$$\begin{aligned} &^cD^q n = \lambda n + \sum_{i=1}^N c_i \chi(t - t_i) s_i(t - t_i) n(t_i) + \sum_{i=1}^N b_i \chi(t - t_i) r_i(t - t_i) m(t_i), \\ &n(0) = 0. \end{aligned} \tag{39}$$

Then by applying Theorem 3.1, we can get that $m \equiv 0$ and $n \equiv 0$ for all $t \in [0, T]$, which means $p_1 \equiv p_2$ and $q_1 \equiv q_2$ for all $t \in [0, T]$. Hence the solution of the system (25)-(26) is unique. This concludes the proof.

In the next result, we construct the sequences v_n and w_n, which are monotonically increasing and decreasing sequences. The sequences v_n and w_n are the solution of the following linear system of Caputo fractional impulsive differential equation. They are defined as

$$\begin{aligned} {}^cD^q v_n &= \lambda v_n + f(t, v_{n-1}) + g(t, w_{n-1}) \\ &\quad + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) v_n(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) w_n(t_i), \qquad (40) \\ v_n(0) &= u_0, \end{aligned}$$

$$\begin{aligned} {}^cD^q w_n &= \lambda w_n + f(t, w_{n-1}) + g(t, v_{n-1}) \\ &\quad + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) w_n(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) v_n(t_i), \qquad (41) \\ w_n(0) &= u_0. \end{aligned}$$

Here v_0 and w_0 are coupled lower and upper solutions of Type 1 of the problem (9).

In order to prove our first next main result, we need the following sector Ω, defined as

$$\Omega = [(t, u) : v_0(t) \le u \le w_0(t), t \in [0, T]], \qquad (42)$$

where v_0 and w_0 are coupled lower and upper solution of suitable type of equation (9)

Theorem 4.2 *Assume*
(A_1). v_0 and w_0 are coupled lower and upper solutions of type 1 of the equation (9), such that $v_0 \le w_0$ on $[0, T]$;
(A_2). $f(t, u)$ and $g(t, u)$ are nondecreasing and nonincreasing, respectively, on Ω.
Then the sequences $\{v_n\}$ and $\{w_n\}$ defined by (40)-(41) are well defined and satisfy the following results:
(i). $\{v_n\}$ and $\{w_n\}$ satisfy the inequality

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le w_n \le w_{n-1} \le \cdots \le w_1 \le w_0, \qquad \forall t \in [0, T]. \qquad (43)$$

(ii). If u is any solution of equation (9) such that $v_0 \le u \le w_0$, then the sequences $\{v_n\}$ and $\{w_n\}$ converge uniformly and monotonically to the coupled minimal and maximal solutions $v(t)$ and $w(t)$, respectively, such that $v(t) \le u \le w(t)$.
(iii). Furthermore, if $f(t, u)$ and $g(t, u)$ satisfy the one-sided Lipschitz condition of the form

$$f(t, u_1) - f(t, u_2) \le L_1(u_1 - u_2), \qquad g(t, u_1) - g(t, u_2) \ge L_2(u_1 - u_2), \qquad (44)$$

where $u_1 \ge u_2$, $L_1 \ge 0$ and $L_2 \ge 0$, $\forall t \in [0, T]$, then we have $v(t) = w(t) = u(t)$ being the unique solution of (9) on $[0, T]$.

Proof. We know that $v_0 \le w_0$. Then from Theorem 4.1, it is easy to see that $v_1(t)$ and $w_1(t)$ exist and are unique as well as $v_n(t)$ and $w_n(t)$ for each $n \ge 1$. In order to prove that v_n and w_n are monotonically non-decreasing and non-increasing respectively and $v_n \le w_n$ for all $n \ge 1$, we use the method of mathematical induction. Initially, we prove $v_0 \le v_1$ and $w_1 \le w_0$. Assume $P(t) = v_0(t) - v_1(t)$ and $Q(t) = w_0(t) - w_1(t)$. Then we have

$$P(0) \le u_0 - u_0 = 0 \qquad Q(0) \ge u_0 - u_0 = 0, \qquad (45)$$

and

$$\begin{aligned}
{}^cD^qP &= {}^cD^q(v_0 - v_1) = {}^cD^qv_0 - {}^cD^qv_1 \\
&\leq \lambda P + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)P(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)Q(t_i), \\
{}^cD^qQ &= {}^cD^q(w_0 - w_1) = {}^cD^qw_0 - {}^cD^qw_1 \\
&\geq \lambda Q + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)P(t_i).
\end{aligned} \tag{46}$$

Using Theorem 3.1, we have $P(t) \leq 0$ and $Q(t) \geq 0$. This proves $v_0 \leq v_1$ and $w_1 \leq w_0$ for all $t \in [0, T]$.

Assume that $v_n \leq v_{n+1}$ and $w_{n+1} \leq w_n$ are true for $n = k$, $k \geq 0$. Therefore, $v_k \leq v_{k+1}$ and $w_{k+1} \leq w_k$ for all $t \in [0, T]$. Then let $n = k+1$, let $P(t) = v_{k+1} - v_{k+2}$ and $Q(t) = w_{k+1} - w_{k+2}$. Therefore $P(0) = Q(0) = 0$.

With the assumption (A_2) on f and g, we can get

$$\begin{aligned}
{}^cD^qP &= \lambda P + f(t, v_k) - f(t, v_{k+1}) + g(t, w_k) - g(t, w_{k+1}) \\
&\quad + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)P(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)Q(t_i) \\
&\leq \lambda P + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)P(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)Q(t_i).
\end{aligned} \tag{47}$$

Similarly, for $Q(t)$ we can get

$$\begin{aligned}
{}^cD^qQ &= \lambda Q + f(t, w_k) - f(t, w_{k+1}) + g(t, v_k) - g(t, v_{k+1}) \\
&\quad + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)P(t_i) \\
&\geq \lambda Q + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)P(t_i).
\end{aligned} \tag{48}$$

Using Theorem 3.1, we have $P(t) \leq 0$ and $Q(t) \geq 0$. This proves $v_{k+1} \leq v_{k+2}$ and $w_{k+2} \leq w_{k+1}$ for all $0 \leq t \leq t_N$. Certainly, it is true for $k = 1$, hence, by induction, we have the result

$$v_0 \leq v_1 \leq \cdots \leq v_{n-1} \leq v_n, \qquad w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq w_0. \tag{49}$$

Next we prove that $v_n \leq w_n$ on $t \in [0, T]$ for all $n \geq 1$. We prove it using the method of mathematical induction.

Let $p(t) = v_1(t) - w_1(t)$, then $p(0) = v_1(0) - w_1(0) = u_0 - u_0 = 0$. Using the assumption (A_2) on f and g, we can get

$$\begin{aligned} {}^cD^q p &= \lambda p + f(t, v_0) - f(t, w_0) + g(t, w_0) - g(t, v_0) \\ &\quad + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) p(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i)(-p(t_i)) \\ &\le \lambda p + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) p(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i)(-p(t_i)) \\ &\le \lambda p + \sum_{i=1}^{N} \big(c_i \chi(t-t_i) s_i(t-t_i) - b_i \chi(t-t_i) r_i(t-t_i)\big) p(t_i). \end{aligned} \tag{50}$$

By Lemma 3.1, we have $p(t) \le 0$. Therefore, $v_1 \le w_1$ for all $t \in [0, T]$.

Assume the result $v_n \le w_n$ is true for $n = k$, which is $v_k \le w_k$ for all $t \in [0, T]$. For $n = k + 1$, we let $p(t) = v_{k+1}(t) - w_{k+1}(t)$, then $p(0) = u_0 - u_0 = 0$. With the assumption (A_2), we have

$$\begin{aligned} {}^cD^q p &= \lambda p + f(t, v_k) - f(t, w_k) + g(t, w_k) - g(t, v_k) \\ &\quad + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) p(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i)(-p(t_i)) \\ &\le \lambda p + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) p(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i)(-p(t_i)) \\ &\le \lambda p + \sum_{i=1}^{N} \big(c_i \chi(t-t_i) s_i(t-t_i) - b_i \chi(t-t_i) r_i(t-t_i)\big) p(t_i). \end{aligned} \tag{51}$$

Using the result of Lemma 3.1, we have $p(t) \le 0$. Therefore, $v_{k+1} \le w_{k+1}$ for all $t \in [0, T]$. Since it is true for $k = 1$, therefore, by induction, we have the conclusion $v_n \le w_n$ is true for every $n \ge 1$. Since we have already assumed that $v_0 \le w_0$, we can obtain the inequality

$$v_0 \le v_1 \le \cdots \le v_{n-1} \le v_n \le w_n \le w_{n-1} \le \cdots \le w_1 \le w_0. \tag{52}$$

In the next result, we will show that $v_0 \le u \le w_0$ implies $v_n \le u \le w_n$ for all $n \ge 1$. We prove by the method of mathematical induction. For $n = 1$, let

$$P(t) = v_1(t) - u(t), \qquad Q(t) = u(t) - w_1(t). \tag{53}$$

The initial condition is $P(0) = Q(0) = u_0 - u_0 = 0$.
Then with the assumption (A_2), we have

$$\begin{aligned} {}^cD^q P &= \lambda P + f(t, v_0) - f(t, u) + g(t, w_0) - g(t, u) \\ &\quad + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) P(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) Q(t_i) \\ &\le \lambda P + \sum_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) P(t_i) + \sum_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) Q(t_i). \end{aligned} \tag{54}$$

Similarly, for $Q(t)$ we have

$$\begin{aligned}
{}^cD^qQ &= \lambda Q + f(t,u) - f(t,w_0) + g(t,u) - g(t,v_0) \\
&\quad + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)P(t_i) \\
&\le \lambda Q + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)P(t_i).
\end{aligned} \tag{55}$$

Then, by Theorem 3.1, we can get $P(t) \le 0$ and $Q(t) \le 0$. Therefore, $v_1 \le u \le w_1$ for all $t \in [0,T]$.

Assume the result $v_n \le w_n$ is true for $n = k$, then we have $v_k \le u \le w_k$. Then for $n = k+1$, let

$$P(t) = v_{k+1}(t) - u(t), \qquad Q(t) = u(t) - w_{k+1}(t). \tag{56}$$

The initial condition is $P(0) = Q(0) = u_0 - u_0 = 0$.

Using the assumption (A_2), we can get

$$\begin{aligned}
{}^cD^qP &= \lambda P + f(t,v_k) - f(t,u) + g(t,w_k) - g(t,u) \\
&\quad + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)P(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)Q(t_i) \\
&\le \lambda P + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)P(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)Q(t_i).
\end{aligned} \tag{57}$$

Similarly, for $Q(t)$ we have

$$\begin{aligned}
{}^cD^qQ &= \lambda Q + f(t,u) - f(t,w_k) + g(t,u) - g(t,v_k) \\
&\quad + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)P(t_i) \\
&\le \lambda Q + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)Q(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)P(t_i).
\end{aligned} \tag{58}$$

Then, by Theorem 3.1, we can get $P(t) \le 0$ and $Q(t) \le 0$. Therefore, $v_{k+1} \le u \le w_{k+1}$ for all $t \in [0,T]$. Since the result is true for $k = 1$, then by induction, we have $v_n(t) \le u(t) \le w_n(t)$ for all $n \ge 0$ and $t \in [0,T]$,

If we consider the result above and the result (i) we proved, we can have

$$v_0 \le v_1 \le \cdots \le v_n \le u \le w_n \le w_{n-1} \le \cdots \le w_1 \le w_0. \tag{59}$$

For the next result, we will prove that the sequences $\{v_n\}$ and $\{w_n\}$ are uniformly bounded and equicontinuous.

Since $v_0(t)$ and $w_0(t)$ are continuous on each interval $[t_k, t_{k+1}]$, we can get they are bounded on the whole interval $[0,T]$. Then assume $|v_0(t)| \le M_v$ and $|w_0(t)| \le M_w$. Then for every n and $t \in [0,T]$, by monotonicity we have

$$0 \le v_n - v_0 \le w_0 - v_0. \tag{60}$$

We take the absolute value to obtain

$$|v_n| \le |v_n - v_0| + |v_0| \le |w_0 - v_0| + |v_0| \le |w_0| + |v_0| + |v_0| \le M_w + 2M_v. \tag{61}$$

Therefore, there exists some positive constant M which is independent of t or N, such that $|v_n| \le M$.

Similarly,

$$|v_n| \le |w_n - w_0| + |w_0| \le |v_0 - w_0| + |w_0| \le |v_0| + |w_0| + |w_0| \le M_v + 2M_w. \tag{62}$$

Therefore, there exists some positive constant M' which is independent of t or N such that $|w_n| \le M'$. Furthermore, M and M' do not depend on n or t. Then the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded on the interval $[0, T]$.

In order to prove the equicontinuity, we use the integral representation of $v_n(t)$,

$$\begin{aligned} v_n(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \bigg(& f(s, v_{n-1}(s)) + g(s, w_{n-1}(s)) + \lambda v_n(s) \\ & + \sum_{i=1}^{k} c_i s_i(t-t_i) v_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) w_n(t_i) \bigg) \mathrm{d}s. \end{aligned} \tag{63}$$

Then for any $k = 0, 1, \ldots, N-1$, let $t^1 \in [t_k, t_{k+1}]$, $t^2 \in [t_k, t_{k+1}]$. Without losing the generalization, we assume that $t^1 > t^2$ and $\left|t^1 - t^2\right| < \delta$, where M is some positive constant. Since $s_i(t-t_i)$, $r_i(t-t_i)$ and $f(t, u(t))$, $g(t, u(t))$ are continuous in t on the interval $[t_i, t_{i+1}]$, we can let $|c_i s_i(t-t_i)| \le C_s$, $|b_i r_i(t-t_i)| \le C_r$ and $|f(t, u(t))| \le M_f$, $|g(t, u(t))| \le M_g$. Based on the uniformly boundedness, we have $|v_n| \le M_v$ and $|w_n| \le M_w$, then we have

$$\begin{aligned} \left|v_n(t^1) - v_n(t^2)\right| = \bigg| & \frac{1}{\Gamma(q)} \int_0^{t^1} (t^1-s)^{q-1} \bigg(f(s, v_{n-1}(s)) + g(s, w_{n-1}(s)) + \lambda v_n(s) \\ & + \sum_{i=1}^{k} c_i s_i(t^1-t_i) v_n(t_i) + \sum_{i=1}^{k} b_i r_i(t^1-t_i) w_n(t_i) \bigg) \mathrm{d}s \\ & - \frac{1}{\Gamma(q)} \int_0^{t^2} (t^2-s)^{q-1} \bigg(f(s, v_{n-1}(s)) + g(s, w_{n-1}(s)) + \lambda v_n(s) \\ & + \sum_{i=1}^{k} c_i s_i(t^2-t_i) v_n(t_i) + \sum_{i=1}^{k} b_i r_i(t^2-t_i) w_n(t_i) \bigg) \mathrm{d}s \bigg|. \end{aligned} \tag{64}$$

For any $t \in [t_k, t_{k+1}]$ we have

$$\begin{aligned} & \left| f(t, v_{n-1}(t)) + g(t, w_{n-1}(t)) + \lambda v_n(t) + \sum_{i=1}^{k} c_i s_i(t-t_i) v_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) w_n(t_i) \right| \\ & \le M_f + M_g + \sum_{i=1}^{k} C_s M_v + \sum_{i=1}^{k} b_i M_w. \end{aligned} \tag{65}$$

We let $M = M_f + M_g + \sum_{i=1}^{k} C_s M_v + \sum_{i=1}^{k} b_i M_w$, then for any $t \in [t_k, t_{k+1}]$,

$$\left| f(t, v_{n-1}(t)) + g(t, w_{n-1}(t)) + \lambda v_n(t) + \sum_{i=1}^{k} c_i s_i(t-t_i) v_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) w_n(t_i) \right| \leq M. \tag{66}$$

Therefore, we have

$$\begin{aligned} \left| v_n(t^1) - v_n(t^2) \right| &\leq \frac{M}{\Gamma(q)} \int_0^{t^2} \left| (t^1 - s)^{q-1} - (t^2 - s)^{q-1} \right| \mathrm{d}s + \frac{M}{\Gamma(q)} \int_{t^2}^{t^1} \left| (t^1 - s)^{q-1} \right| \mathrm{d}s \\ &\leq \frac{M}{\Gamma(q+1)} (t^1 - t^2)^q + \frac{M}{\Gamma(q+1)} (t^1 - t^2)^q = \frac{2M}{\Gamma(q+1)} \left| t^1 - t^2 \right|^q < \epsilon. \end{aligned} \tag{67}$$

Providing $\left| t^1 - t^2 \right| \leq \delta = \left(\frac{\epsilon \Gamma(q+1)}{2M} \right)^{\frac{1}{q}}$, we can have that v_n is equicontinuous.

Similarly, for w_n we have

$$\begin{aligned} &\left| f(t, w_{n-1}(t)) + g(t, v_{n-1}(t)) + \lambda w_n(t) + \sum_{i=1}^{k} c_i s_i(t-t_i) w_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) v_n(t_i) \right| \\ &\leq M_f + M_g + \sum_{i=1}^{k} C_s M_w + \sum_{i=1}^{k} b_i M_v. \end{aligned} \tag{68}$$

Let $M' = M_f + M_g + \sum_{i=1}^{k} C_s M_w + \sum_{i=1}^{k} b_i M_v$, then for any $t \in [t_k, t_{k+1}]$,

$$\left| f(t, w_{n-1}(t)) + g(t, v_{n-1}(t)) + \lambda w_n(t) + \sum_{i=1}^{k} c_i s_i(t-t_i) w_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) v_n(t_i) \right| \leq M'. \tag{69}$$

Therefore, we have

$$\begin{aligned} \left| w_n(t^1) - w_n(t^2) \right| &\leq \frac{M'}{\Gamma(q)} \int_0^{t^2} \left| (t^1 - s)^{q-1} - (t^2 - s)^{q-1} \right| \mathrm{d}s + \frac{M'}{\Gamma(q)} \int_{t^2}^{t^1} \left| (t^1 - s)^{q-1} \right| \mathrm{d}s \\ &\leq \frac{M'}{\Gamma(q+1)} (t^1 - t^2)^q + \frac{M'}{\Gamma(q+1)} (t^1 - t^2)^q = \frac{2M'}{\Gamma(q+1)} \left| t^1 - t^2 \right|^q < \epsilon. \end{aligned} \tag{70}$$

We provide $\left| t^1 - t^2 \right| \leq \delta = \left(\frac{\epsilon \Gamma(q+1)}{2M'} \right)^{\frac{1}{q}}$, then w_n is equicontinuous. Therefore, if we take the minimum of these two, $\delta = \min \left(\left(\frac{\epsilon \Gamma(q+1)}{2M} \right)^{\frac{1}{q}}, \left(\frac{\epsilon \Gamma(q+1)}{2M'} \right)^{\frac{1}{q}} \right)$, then can obtain that $\{v_n(t)\}$ and $\{w_n(t)\}$ are equicontinuous on the interval $[t_k, t_{k+1}]$. Since $k = 0, 1, \ldots, N-1$ was arbitrary, we proved that $\{v_n(t)\}$ and $\{w_n(t)\}$ are equicontinuous on the interval $[0, t_N = T]$.

Since we have proved that $\{v_n(t)\}$ and $\{w_n(t)\}$ are equicontinuous and uniformly bounded on the interval $[0, T]$, by Ascoli-Arzela's theorem, there exist subsequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$, which converge uniformly to $v(t)$ and $w(t)$, respectively, on $[0, T]$. Because of the monotonicity of the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ we have shown, we can get that the entire sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ converge uniformly and monotonically to $v(t)$ and $w(t)$, respectively.

For the next step, we will prove that $v(t)$ and $w(t)$ we have above are the minimal and maximal solutions of the problem (9). Furthermore, we want to show that they are equivalent to the solution of the equation (9).
We use the integral representation.

$$\begin{aligned} v_n(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(& f(s, v_{n-1}(s)) + g(s, w_{n-1}(s)) + \lambda v_n(s) \\ & + \sum_{i=1}^{k} c_i s_i(t-t_i) v_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) w_n(t_i) \Big) \mathrm{d}s. \end{aligned} \tag{71}$$

Then, we take the limit of n on both sides. Since $\{v_n\}$ converges uniformly, we have

$$\begin{aligned} \lim_{n\to\infty} v_n = \lim_{n\to\infty} \Big(u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(& f(s, v_{n-1}(s)) + g(s, w_{n-1}(s)) + \lambda v_n(s) \\ & + \sum_{i=1}^{k} c_i s_i(t-t_i) v_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) w_n(t_i) \Big) \mathrm{d}s. \end{aligned} \tag{72}$$

Then,

$$\begin{aligned} v(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(& f(s, v(s)) + g(s, w(s)) + \lambda v(s) \\ & + \sum_{i=1}^{k} c_i s_i(t-t_i) v(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) w(t_i) \Big) \mathrm{d}s. \end{aligned} \tag{73}$$

Similarly, for w_n we have

$$\begin{aligned} w_n(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(& f(s, w_{n-1}(s)) + g(s, v_{n-1}(s)) + \lambda w_n(s) \\ & + \sum_{i=1}^{k} c_i s_i(t-t_i) w_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) v_n(t_i) \Big) \mathrm{d}s. \end{aligned} \tag{74}$$

After taking the limits of n on both sides, we can get

$$\begin{aligned} \lim_{n\to\infty} w_n = \lim_{n\to\infty} \Big(u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(& f(s, w_{n-1}(s)) + g(s, v_{n-1}(s)) + \lambda w_n(s) \\ & + \sum_{i=1}^{k} c_i s_i(t-t_i) w_n(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) v_n(t_i) \Big) \mathrm{d}s \Big). \end{aligned} \tag{75}$$

Then,

$$\begin{aligned} w(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(& f(s, w(s)) + g(s, v(s)) + \lambda w(s) \\ & + \sum_{i=1}^{k} c_i s_i(t-t_i) w(t_i) + \sum_{i=1}^{k} b_i r_i(t-t_i) v(t_i) \Big) \mathrm{d}s. \end{aligned} \tag{76}$$

Now we can get that $v(t)$ and $w(t)$ satisfy the equation (9). Therefore, $v(t)$ and $w(t)$ are coupled minimal and maximal solutions of equation (9). Thus, we have already shown that $v_n \leq u \leq w_n$. Taking the limits of n we can get $\lim_{n\to\infty} v_n \leq \lim_{n\to\infty} u \leq \lim_{n\to\infty} w_n$. Then we can obtain $v \leq u \leq w$.

In the last result, we will show that if f and g satisfy the one-sided Lipschitz condition, then the coupled minimal and maximal solutions are equivalent to the solution u of the equation (9).

Let $m(t) = w(t) - v(t)$, then $m(0) = w(0) - v(0) = u_0 - u_0 = 0$, and we can get

$$\begin{aligned} {}^cD^q m(t) &= {}^cD^q(w(t) - v(t) = {}^cD^q w(t) - {}^cD^q v(t) \\ &= \lambda(w(t) - v(t)) + [f(t, w(t)) - f(t, v(t))] + [g(t, v(t)) - g(t, w(t))] \\ &+ \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)(w(t_i) - v(t_i)) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)(v(t_i) - w(t_i)). \end{aligned} \tag{77}$$

Let $\Lambda = \lambda + L_1 + L_2$, we can get

$$ {}^cD^q m(t) \leq \Lambda m(t) + \sum_{i=1}^{N} c_i\chi(t-t_i)s_i(t-t_i)m(t_i) - \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)m(t_i). \tag{78}$$

Then, by using the Laplace transformation, we can get

$$ m(t) \leq m(0)E_{q,1}(\Lambda t^q) + \sum_{i=1}^{N-1} c_iS_i(t-t_i)m(t_i) - \sum_{i=1}^{N-1} b_iR_i(t-t_i)m(t_i). \tag{79}$$

We know that $m(0) = 0$, then according to the result of Theorem 3.1, we have $m(t) \leq 0$, $\forall t \in [0, t_N]$. By definition of $m(t)$ we can get $\forall t \in [0, T]$, $w(t) \leq v(t)$. Since we have proved the monotonicity $v(t) \leq u(t) \leq w(t)$, we can get that $\forall t \in [0, T]$, $v(t) = u(t) = w(t)$, which concludes the proof.

Theorem 4.3 *Assume*
(A_1). v_0 and w_0 are coupled lower and upper solutions of natural type of the equation (9), such that $v_0 \leq u \leq w_0$ on $[0, T]$;
(A_2). $f(t, u)$ and $g(t, u)$ are nondecreasing and nonincreasing, respectively, on Ω.
Then the sequences v_n and w_n defined by (40)-(41) are well defined and satisfy the following results:
(i). For all $n \geq 1$, on $[0, T]$ we have

$$ v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq w_0, \tag{80}$$

provided $v_0 \leq v_1$ and $w_1 \leq w_0$.
(ii). The sequences v_n and w_n converge uniformly and monotonically to the coupled minimal and maximal solutions $v(t)$ and $w(t)$, respectively. Furthermore, if u is any solution of equation (9), then $v(t) \leq u \leq w(t)$.
(iii). Furthermore, if $f(t, u)$ and $g(t, u)$ satisfy the one-sided Lipschitz condition, which is for any $u_1 \geq u_2$, we have

$$ f(t, u_1) - f(t, u_2) \leq L_1(u_1 - u_2), \qquad g(t, u_1) - g(t, u_2) \geq L_2(u_1 - u_2), \tag{81}$$

where $L_1 \geq 0$ and $L_2 \geq 0$, then $\forall t \in [0, T]$, we have $v(t) = u(t) = w(t)$, the uniqueness of (9) holds on $[0, T]$.

Proof. The proof follows the same lines as the proof of Theorem 4.2 except in the first part, instead of proving $v_0 \leq v_1$ and $w_1 \leq w_0$, we have this result provided. The rest of the proof is the same.

5 Conclusion

We generalized the monotone method and use the method to prove that for the nonlinear Caputo fractional impulsive differential equation (9), under certain conditions, the coupled lower and upper solutions of both the natural type and type 1 converge to the exact solution of the problem. Therefore, in the future work, the monotone method will be significantly useful to approximate the solution of the problem. In the numerical results, we will discuss another method which converges faster than this method.

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