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Periodic Solutions of a Singular Lagrangian System Related to Dispersion-Managed Fiber Communication Devices

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Abstract: We prove the existence of periodic solutions to a certain singular Lagrangian system that describes the evolution of the optical pulse width and chirp for so-called dispersion-managed solitons.

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1 Introduction and Main Results

In data communication systems like transoceanic transmission along a fiber cable, there is increasing demand to achieve transmission rates as high as possible, mainly to the extensive use of the internet. To do so, a recent approach is to utilize non-linear light-wave communications with suitable periodic amplifications to compensate for loss and dispersive effects. The transmission of such optical signal is described by

$$i\Psi_z - \frac{1}{2}\beta_2(z)\Psi_{tt} + \sigma(z)|\Psi|^2\Psi = iG(z)\Psi, \tag{1}$$

see [6, 8, 9]. Here $\Psi = \Psi(z,t)$ is some complex-valued envelope function of the original electric field, t is time, and z is the longitudinal coordinate of the fiber cable, which should be thought of to be a periodic variable, since both amplification and dispersion repeat periodically. Moreover, $G(z)$ accounts for both loss and amplification in the fiber, whereas $\beta_2(z)$ is related to the dispersion; $\sigma(z)$ is some additional function.

The transformation $\Psi(z,t) = A(z,t)\exp\Big(\int^z G(z')\,dz'\Big)$ removes the term on the right-hand side of (1) to yield the nonlinear Schrödinger equation

$$iA_z + d(z)A_{tt} + c(z)|A|^2A = 0, \tag{2}$$

with coefficient functions $c(z)$ and $d(z)$ being periodic of some period $L > 0$. It is then well-accepted that the central part of the desired pulse-shaped solution to (2) is described to leading order by

$$A(z,t) = \frac{Q(t/T(z))}{\sqrt{T(z)}} \exp\left(i\,\frac{M(z)}{T(z)}\,t^2\right), \tag{3}$$

see the references cited above, and also [2, 4, 5]; the function $Q(x)$ is an input pulse which often is taken as $Q(x) = C_0 \exp(-x^2/2)$, and $M(z)$ resp. $T(z)$ describe the optical pulse width resp. the chirp (time-dependent phase) of the breathing central part of the optical soliton. Most importantly for our purposes, $T(z)$ and $M(z)$ are L-periodic solutions to

$$\frac{dT}{dz} = 4d(z)M, \qquad \frac{dM}{dz} = \frac{d(z)C_1}{T^3} - \frac{c(z)C_2}{T^2}, \tag{4}$$

with fixed constants

$$C_1 = \frac{\int |Q'(x)|^2\,dx}{\int x^2|Q(x)|^2\,dx}, \qquad C_2 = \frac{\int |Q(x)|^4\,dx}{4\int x^2|Q(x)|^2\,dx}.$$

It is hence of fundamental importance for the whole approach to deduce whether or not periodic solutions of (4) do exist. In some of the papers cited above, this problem is studied numerically for the dispersion map $d(z)$ taken as an L-periodic step function,

$$d(z) = \begin{cases} d_+ & : \quad 0 \le z \le L/4,\ \ 3L/4 \le z \le L \\ -d_- & : \quad L/4 < z < 3L/4 \end{cases}, \tag{5}$$

with d_+, $d_- > 0$; the function $c(z)$ was chosen to be constant as is physically reasonable in case the compensation period is much larger than the amplification distance. Taking $d(z)$ as in (5) corresponds to a transmission line consisting of two pieces of fibers with opposite dispersion. Eq. (4), even with dispersion map as in (5), poses interesting mathematical problems, but despite that there is a large mathematical literature on singular Lagrangian problems, cf. e.g. [1, 3] and many others, it does not seem that there are general results that apply to a system as (4), which is Hamiltonian with

$$\mathcal{H}(T,M,z) = 2d(z)M^2 + \frac{d(z)C_1}{2T^2} - \frac{c(z)C_2}{T}.$$

As we are interested in periodic solutions of period L (the "fixed period problem"), it would be natural to consider the action functional $\mathcal{I}$ corresponding to (4) which is here

$$\mathcal{I}(T,M) = \int_0^L \left[T(z)\,\frac{dM}{dz}(z) - \mathcal{H}(T(z),M(z),z)\right]dz$$

for M, T in a suitable function space. A critical point of $\mathcal{I}$ then would provide a solution to (4), but it is not clear how the necessary assumptions on $\mathcal{I}$ can be verified to apply some minimax-argument.

The following theorem is our main result.

Theorem 1.1 *Assume $c(z) = c > 0$ is a constant and $d(z)$ is given by (5). Then (4) has a periodic solution of period L if $d_+ > d_-$.*

The proof of Theorem 1.1 is rather elementary and possible through direct calculation and estimates. Rather than this we would have preferred to give a more functional analytical proof that also works for non-explicit dispersion maps, but such an approach was not clear to us. Nevertheless, the same proof also yields some results for a dispersion map which has the more general form

$$d(z) = \begin{cases} d_+ & : \quad 0 \le z \le L_1, \;\; L - L_1 \le z \le L \\ -d_- & : \quad L_1 < z < L - L_1 \end{cases}, \tag{6}$$

for some $L_1 \in (0, L/2)$; see Theorem 2.4.

Theorem 1.1 discusses the case of a dispersion map with positive average dispersion $\langle d \rangle = \frac{1}{L} \int\limits_0^L d(z)\,dz = \frac{1}{2}(d_+ - d_-)$, cf. [7] for some results in the same direction. Due to numerical observations in [8, 9] there should also exist periodic solutions for the zero-average case $d_+ = d_-$, at least if those values are sufficiently large. If the average dispersion is negative, $d_+ < d_-$, then it will be seen below by means of a symmetry argument that again a periodic solution $T(z)$, $M(z)$ of (4) can be found. However, it is of no practical relevance for the original problem, since it will be negative contrary to what is needed in the ansatz (3). The situation for negative average dispersion currently is rather unclear.

2 Existence of Periodic Solutions

In this section we carry out the proof of Theorem 1.1. First we rewrite (4), introducing $t = z$, $a_+ = 4d_+\sqrt{C_1}$, $a_- = 4d_-\sqrt{C_1}$, $b = cC_2/\sqrt{C_1}$ and $q(t) = T(z)$. Then (4) reads as

$$\ddot{q} = \begin{cases} \frac{a_+^2}{4q^3} - \frac{a_+ b}{q^2} = -V_+'(q) & : \quad 0 \le t \le L/4, \;\; 3L/4 \le t \le L \\ \frac{a_-^2}{4q^3} + \frac{a_- b}{q^2} = -V_-'(q) & : \quad L/4 < t < 3L/4 \end{cases}, \tag{7}$$

where

$$V_+(q) = \frac{a_+^2}{8q^2} - \frac{a_+ b}{q} \quad \text{and} \quad V_-(q) = \frac{a_-^2}{8q^2} + \frac{a_- b}{q}.$$

Throughout we assume $b > 0$, and we also introduce the corresponding energies

$$H_+(q, \dot{q}) = \frac{1}{2}\,\dot{q}^2 + V_+(q) \quad \text{and} \quad H_-(q, \dot{q}) = \frac{1}{2}\,\dot{q}^2 + V_-(q).$$

For the proof of Theorem 1.1, from $d_+ > d_-$ we have the hypothesis

$$a_+ > a_-. \tag{8}$$

It should be noted that the transformation $\bar{q}(t) = -q(L/2 + t)$ changes the rôles of a_+ and a_- in (7). However, since the solution q will be positive under assumption (8), it turns out that for the negative dispersion case $a_+ < a_-$ the function $\bar{q}$ is negative and hence cannot play the rôle of $T(z)$, cf. the corresponding remarks in the introduction.

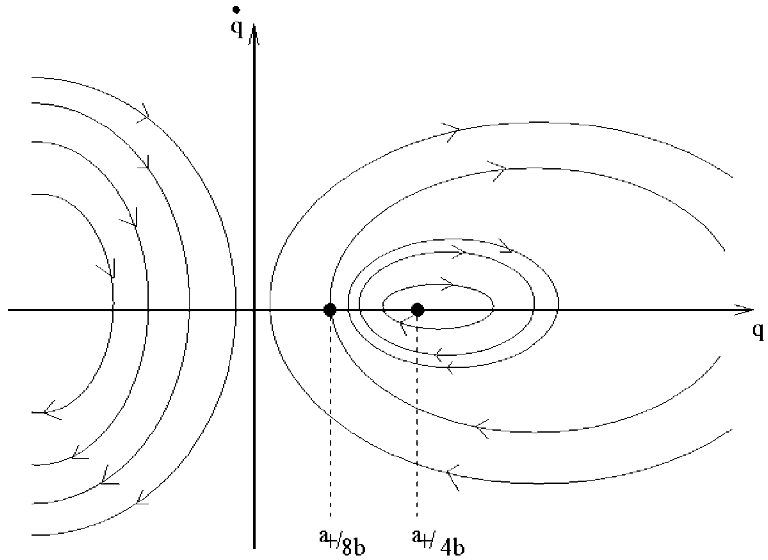

Figure 2.1. Phase portrait of $\ddot{q} = -V_+'(q)$.

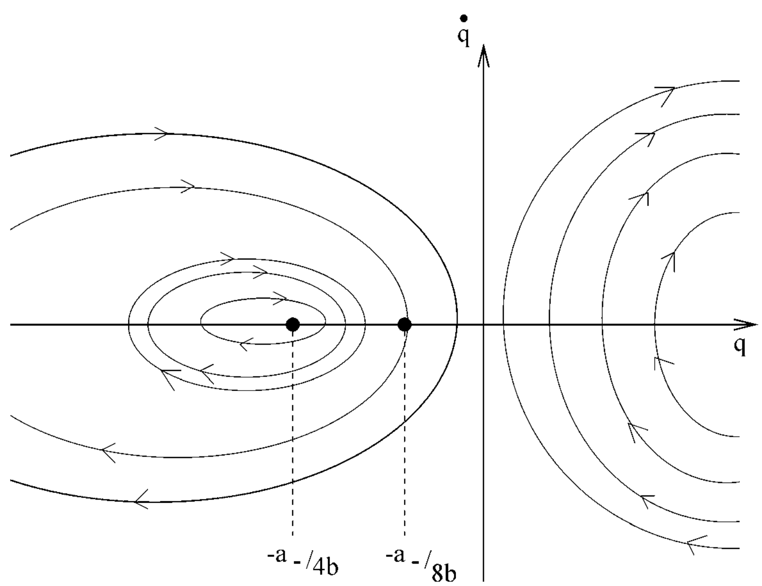

Figure 2.2. Phase portrait of $\ddot{q} = -V_-'(q)$.

To get a clue where to look for periodic solutions of (7), the phase portraits for $\ddot{q} = -V_+'(q)$ resp. for $\ddot{q} = -V_-'(q)$ are given in Figure 2.1 resp. Figure 2.2.

Thus the only possibility to have a periodic solution in $\{q > 0\}$ is to match a periodic orbit from Figure 2.1 to a trajectory from Figure 2.2. The periodic orbits in Figure 2.1 are found to have energies $h_+ \in [-2b^2, 0)$, the value $h_+ = -2b^2$ corresponding to the fixed-point $q = a_+/4b$. The respective periods may then be calculated explicitly as

$$\frac{1}{2}\,T(h_+) = \int\limits_{q_1}^{q_0} \frac{dq}{[2(h_+ - V_+(q))]^{1/2}} = \frac{a_+ b\,\pi}{2\sqrt{2}(-h_+)^{3/2}},$$

where $(q_1, 0)$ and $(q_0, 0)$ with $q_1 \le q_0$ are the intersection points of the orbit of energy h_+ with the axis $\{\dot q = 0\}$.

Let

$$q^* = \begin{cases} a_+/4b & : \quad a_+\pi > 2b^2 L \\ (V_+)^{-1}\Big(-\,(\sqrt{2}a_+ b\pi/L)^{2/3}\Big) & : \quad a_+\pi \le 2b^2 L \end{cases} \tag{9}$$

with $(V_+)^{-1}\Big(-\,(\sqrt{2}a_+ b\pi/L)^{2/3}\Big) \in [a_+/4b, \infty)$; observe $V_+ \colon [a_+/4b, \infty) \to [-2b^2, 0)$ is strictly increasing. We define a map $q_0 \mapsto q_1 \mapsto q_2$ as follows.

(1) For given $q_0 \ge q^*$, determine the energy

$$h_+ = V_+(q_0) = \frac{a_+^2}{8q_0^2} - \frac{a_+ b}{q_0} \in [-2b^2, 0). \tag{10}$$

(2) The point $q_1 \le q_0$ then is defined through

$$\frac{L}{4} = \int\limits_{q_1}^{q_0} \frac{dq}{[2(h_+ - V_+(q))]^{1/2}}. \tag{11}$$

(3) Next, $\dot q_1 \ge 0$ is calculated from

$$h_+ = H_+(q_1, \dot q_1) = \frac{1}{2}\,\dot q_1^2 + \frac{a_+^2}{8q_1^2} - \frac{a_+ b}{q_1}. \tag{12}$$

(4) Then we let

$$h_- = H_-(q_1, \dot q_1) = \frac{1}{2}\,\dot q_1^2 + \frac{a_-^2}{8q_1^2} + \frac{a_- b}{q_1} > 0. \tag{13}$$

(5) Finally, $q_2 > 0$ is defined as the unique intersection point of the orbit with energy h_- of $\ddot q = -V_-'(q)$ with the axis $\{\dot q = 0\}$, i.e., the solution of $h_- = H_-(q_2, 0)$.

Remark 2.1 The map $q_0 \mapsto q_1 \mapsto q_2$ is well-defined, since by definition of q^* in (9) we have $\frac{1}{2}T(h_+) \ge \frac{L}{4}$ for $q_0 \ge q^*$ in both cases, and therefore q_1 exists. Note also that all quantities are determined by q_0, or equivalently, by h_+.

Thus the existence of an L-periodic orbit of (7) is equivalent to finding a zero q_0 of the function

$$F(q_0) = \int\limits_{q_2}^{q_1} \frac{dq}{[2(h_- - V_-(q))]^{1/2}} - \frac{L}{4}. \tag{14}$$

Since F is continuous, the existence of a zero will be a consequence of

$$F(q_0) \to -\frac{L}{4} < 0 \qquad \text{as} \quad q_0 \to q^*, \quad \text{and}$$
$$F(q_0) \to \frac{L}{4}\left(\frac{a_+}{a_-} - 1\right) > 0 \quad \text{as} \quad q_0 \to \infty,$$

cf. (8). The following Lemmas 2.1 and 2.2 verify these assertions, completing the proof of Theorem 1.1. Before going on, we will state some identities that will be used frequently throughout. First, from (12) and (13) we infer

$$h_- = h_+ + \frac{a_-^2 - a_+^2}{8q_1^2} + \frac{(a_- + a_+)b}{q_1}. \tag{15}$$

Next, by direct integration of the right-hand side in (11) we obtain

$$\frac{L}{4} = -\frac{\sqrt{X(q_1)}}{4h_+} + \frac{a_+ b}{2\sqrt{2}(-h_+)^{3/2}}\left[\frac{\pi}{2} + \arcsin\left(\frac{2h_+q_1 + a_+b}{a_+\sqrt{b^2 + \frac{1}{2}h_+}}\right)\right] \tag{16}$$

with $X(q) = 8h_+q^2 + 8a_+bq - a_+^2$; to derive this it is useful to note that $\frac{2h_+q_0 + a_+b}{a_+\sqrt{b^2+\frac{1}{2}h_+}} = -1$ by (10). Similarly, integrating (14) we deduce

$$F(q_0) + \frac{L}{4} = \frac{\sqrt{X(q_1)}}{4h_-} + \frac{a_-b}{2\sqrt{2}h_-^{3/2}} \log\left(\frac{\sqrt{2h_-X(q_1)} + 4h_-q_1 - 2a_-b}{2a_-\sqrt{b^2 + \frac{1}{2}h_-}}\right), \tag{17}$$

utilizing $8h_-q_1^2 - 8a_-bq_1 - a_-^2 = X(q_1)$, cf. (15); the argument of log is ≥ 1, since

$$2a_-\sqrt{b^2 + \frac{1}{2}\,h_-} = 4h_-q_2 - 2a_-b, \tag{18}$$

and $q_1 \geq q_2$.

The right-hand side of (16) contains no q_0, only h_+. It will also be important to have formulae for derivatives w.r. to h_+. To begin with,

$$\frac{dX(q_1)}{dh_+} = 8q_1^2 + 8(a_+b + 2h_+q_1)\left(\frac{dq_1}{dh_+}\right).$$

Through a tedious and lengthy calculation one may then show by differentiating the right-hand side of (16) w.r. to h_+ that

$$\frac{dq_1}{dh_+} = \frac{3L}{16}\,\frac{\sqrt{X(q_1)}}{(-h_+q_1)} - \frac{q_1}{h_+} + \frac{a_+\left(2bq_1 - \frac{1}{2}a_+\right)}{16\left(b^2 + \frac{1}{2}h_+\right)(-h_+q_1)}; \tag{19}$$

this works by inserting formula (16) after differentiation again for the $\arcsin(\dots)$-term. Additionally, we get from (15)

$$\frac{dh_-}{dh_+} = 1 - \frac{a_-^2 - a_+^2}{4q_1^3}\left(\frac{dq_1}{dh_+}\right) - \frac{(a_- + a_+)b}{q_1^2}\left(\frac{dq_1}{dh_+}\right). \tag{20}$$

After this preparation we can proceed to the proof of Lemma 2.1 and Lemma 2.2.

Lemma 2.1 *As* $q_0 \to q^*$ *we have* $F(q_0) \to -\frac{L}{4}$.

Proof We first consider the case $a_+\pi > 2b^2L$, i.e., $q_0 \to a_+/4b$. By definition, $\bar{q}_0 \le q_1 \le q_0$, with $\bar{q}_0$ and q_0 being the two solutions to $h_+ = H_+(q,0)$. Since $h_+ \to -2b^2$ by (10), it follows that $\bar{q}_0 = (-a_+/2h_+)\Big[b - \sqrt{b^2 + \frac{1}{2}h_+}\,\Big] \to a_+/4b$, therefore $q_1 \to a_+/4b$, and hence also $X(q_1) \to 0$ as $q_0 \to a_+/4b$. By (15), $h_- \to 2b^2(a_-/a_+)(2 + a_-/a_+)$, and therefore $h_- = H_-(q_2, 0)$ gives $q_2 \to a_+/4b$ as $q_0 \to a_+/4b$. Consequently, $F(q_0) \to -L/4$ as $q_0 \to a_+/4b$ by (17) and (18).

What concerns the second case $a_+\pi \le 2b^2L$ in (9), we then have $T(h_+^*)/2 = L/4$, with $h_+^* = -\big(\sqrt{2}a_+b\pi/L\big)^{2/3}$, by definition of q^*. As $q_0 \to q^*$ therefore q_1 tends to the smaller solution $\bar{q}^*$ of $h_+^* = H(q,0)$, i.e., we have $X(q_1) \to 0$. According to step (3)–(5) in the above construction of the map, q_2 degenerates to $q_2 \to \bar{q}^*$ as $q_0 \to q^*$. Since $h_- \to a_-^2/8(\bar{q}^*)^2 + a_-b/\bar{q}^* > 0$, we may argue as before to conclude $F(q_0) \to -L/4$ as $q_0 \to q^*$.

It remains to analyze the limiting behaviour of $F(q_0)$ as $q_0 \to \infty$.

Lemma 2.2 *As* $q_0 \to \infty$ *we have* $F(q_0) \to \frac{L}{4}\left(\frac{a_+}{a_-} - 1\right)$.

Proof All limits that are taken in this proof are as $q_0 \to \infty$, or, equivalently, as $h_+ \to 0$. Since both terms on the right-hand side of (16) are non-negative and $h_+ \to 0$, we must also have $X(q_1) \to 0$, whence

$$q_1(h_+q_1 + a_+b) \to \frac{a_+^2}{8}, \qquad h_+q_1 \to -a_+b, \tag{21}$$

and therefore $\frac{2h_+q_1 + a_+b}{a_+\sqrt{b^2 + \frac{1}{2}h_+}} \to -1$. By the de L'Hospital rule we are led to check whether

$$\begin{aligned}\Lambda_1 &= -\frac{a_+b}{3\sqrt{2}}\,\frac{\frac{d}{dh_+}\arcsin(\dots)}{(-h_+)^{1/2}} \\ &= -\frac{a_+b}{3\Big(b^2 + \frac{1}{2}h_+\Big)}\left(\frac{2\Big(b^2 + \frac{1}{2}h_+\Big)\Big[h_+\Big(\frac{dq_1}{dh_+}\Big) + q_1\Big] - \frac{1}{4}(2h_+q_1 + a_+b)}{(-h_+)\sqrt{X(q_1)}}\right)\end{aligned} \tag{22}$$

has a limit as $h_+ \to 0$. Utilizing (19), one arrives after some simplification at

$$\begin{aligned}&2\Big(b^2 + \frac{1}{2}\,h_+\Big)\left[h_+\left(\frac{dq_1}{dh_+}\right) + q_1\right] - \frac{1}{4}\,(2h_+q_1 + a_+b) \\ &\qquad = -\frac{3L}{8}\Big(b^2 + \frac{1}{2}h_+\Big)\,\frac{\sqrt{X(q_1)}}{q_1} - \frac{X(q_1)}{16q_1}.\end{aligned} \tag{23}$$

Inserting (23) into (22) implies by (21), and since $X(q_1) \to 0$, that $\Lambda_1 \to L/8$. Thus de L'Hospital yields from (16),

$$\frac{a_+b}{2\sqrt{2}(-h_+)^{3/2}}\left[\frac{\pi}{2} + \arcsin\left(\frac{2h_+q_1 + a_+b}{a_+\sqrt{b^2 + \frac{1}{2}h_+}}\right)\right] \to \frac{L}{8}, \qquad -\frac{\sqrt{X(q_1)}}{4h_+} \to \frac{L}{8}. \tag{24}$$

By (15) and (21),

$$\frac{h_-}{h_+} \to -\frac{a_-}{a_+}. \tag{25}$$

Thus as a first step towards deriving the limiting behaviour of $F(q_0)$ we conclude from (24) and (25) that

$$\frac{\sqrt{X(q_1)}}{4h_-} = \left(-\frac{\sqrt{X(q_1)}}{4h_+}\right)\left(-\frac{h_+}{h_-}\right) \to \frac{L}{8}\left(\frac{a_+}{a_-}\right). \tag{26}$$

Next we have to analyze the contribution of the second term on the right-hand side of (17). For this, we proceed as before and consider first the quotient

$$\begin{aligned}\Lambda_2 &= \frac{a_-b}{3\sqrt{2}}\,\frac{\frac{d}{dh_+}\log(\dots)}{h_-^{1/2}\left(\frac{dh_-}{dh_+}\right)}\\ &= \frac{a_-b}{3\sqrt{2}\left(b^2+\frac{1}{2}h_-\right)\left[\sqrt{2h_-X(q_1)}+4h_-q_1-2a_-b\right]\left(\frac{dh_-}{dh_+}\right)}\left(\frac{\Lambda_{21}+\Lambda_{22}}{h_-^{1/2}}\right),\end{aligned}$$

where

$$\Lambda_{21} = 4\left(b^2+\frac{1}{2}\,h_-\right)\left[h_-\left(\frac{dq_1}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)q_1\right]-\frac{1}{2}\,(2h_-q_1-a_-b)\left(\frac{dh_-}{dh_+}\right),$$

$$\Lambda_{22} = \frac{\left(b^2+\frac{1}{2}\,h_-\right)}{\sqrt{2h_-X(q_1)}}\left[h_-\left(\frac{dX(q_1)}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)X(q_1)\right]-\frac{1}{4}\,\sqrt{2h_-X(q_1)}\left(\frac{dh_-}{dh_+}\right).$$

By (25) we have $\mathcal{O}(h_+)=\mathcal{O}(h_-)$ as $h_\pm \to 0$, whence we can denote such terms simply by $\mathcal{O}(h)$. Because $\frac{1}{q_1^2}\left(\frac{dq_1}{dh_+}\right) \to \frac{1}{a_+b}$ according to (19), (21) shows $\frac{1}{q_1^3}\,\frac{dq_1}{dh_+} = \mathcal{O}(h)$. In addition, $\sqrt{X(q_1)} = \mathcal{O}(h)$ by (24) and $h_+q_1+a_+b=\mathcal{O}(h)$ by (21). Using this information and the explicit representations (20) of $\frac{dh_-}{dh_+}$ and (19) of $\frac{dq_1}{dh_+}$, it follows after some calculations that

$$\Lambda_{21} = \mathcal{O}(h).$$

Turning our attention to Λ_{22}, we first note $\frac{1}{4}\,\sqrt{2h_-X(q_1)}\left(\frac{dh_-}{dh_+}\right) = \mathcal{O}(h^{3/2})$, since $\frac{dh_-}{dh_+} \to -\frac{a_-}{a_+}$ by (20) and the preceding arguments. Consequently,

$$\begin{aligned}\Lambda_2 = &\frac{a_-b}{6\left[\sqrt{2h_-X(q_1)}+4h_-q_1-2a_-b\right]\left(\frac{dh_-}{dh_+}\right)}\\ &\times\ \frac{\left[h_-\left(\frac{dX(q_1)}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)X(q_1)\right]}{h_-\sqrt{X(q_1)}}+\mathcal{O}(h^{1/2}).\end{aligned} \tag{27}$$

As before, an elementary but quite lengthy calculation yields

$$h_-\left(\frac{dX(q_1)}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)X(q_1) = -\frac{3L}{2}\left[(a_-+2a_+)b+2h_+q_1\right]\frac{\sqrt{X(q_1)}}{q_1}+\mathcal{O}(h^3). \tag{28}$$

As a consequence of $h_-q_1 = (h_+q_1)\left(\frac{h_-}{h_+}\right) \to a_-b$, by inserting (28) into (27) we get $\Lambda_2 \to \frac{L}{8}\left(\frac{a_+}{a_-}\right)$. Thus the rule of de L'Hospital yields

$$\frac{a_-b}{2\sqrt{2}h_-^{3/2}} \log\left(\frac{\sqrt{2h_-X(q_1)} + 4h_-q_1 - 2a_-b}{2a_-\sqrt{b^2 + \frac{1}{2}h_-}}\right) \to \frac{L}{8}\left(\frac{a_+}{a_-}\right). \tag{29}$$

Summarizing (26) and (29), we finally obtain from (17) that $F(q_0) + \frac{L}{4} \to \frac{L}{4}\left(\frac{a_+}{a_-}\right)$.

The method of proof can also be adapted for

$$\ddot{q} = \begin{cases} \frac{a_+^2}{4q^3} - \frac{a_+b}{q^2} & : \quad 0 \le t \le L_1, \;\; L - L_1 \le t \le L \\ \frac{a_-^2}{4q^3} + \frac{a_-b}{q^2} & : \quad L_1 < t < L - L_1 \end{cases}, \tag{30}$$

with $L_1 \in (0, L/2)$, corresponding to the more general dispersion maps (6). We obtain

Theorem 2.1 *For* $2L_1\left(1 + \frac{a_+}{a_-}\right) > L$, *(30) has an* L*-periodic solution.*

Proof We can proceed as before, and in particular we find $F(q_0) + (L/2 - L_1) \to L_1(a_+/a_-)$ as $q_0 \to \infty$. The condition $\lim_{q_0\to\infty} F(q_0) > 0$ then means $2L_1(1+a_+/a_-) > L$.

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