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Stability of Stationary Motions of Mehcanical Syatems with a Rigid Body as the Basic Element

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Abstract: The paper presents a brief description of the problem on the permanent rotations of a rigid body from its statement moment up to factual completion. Stability theory of stationary motions connected with this problem is stated. Their interconnection is shown and the closest generalizations have been considered.

Keywords: *Rigid body systems; stationary motion; stability.*

Mathematics Subject Classification (2000): 70E05, 70E15, 70K20.

1 Introduction

The problem of the permanent rotations of a rigid body with a fixed point in the gravity force field occupies an important place in analytical mechanics and different applications. In rigid body dynamics a complete investigation of the permanent rotations has been made by Staude [1]. This remarkable paper by Staude practically closed this problem unfortunately because specialists on rigid body dynamics lost interest in further investigation of the permanent rotations for many years. However, this problem attracted the attention of the experts in the stability theory in connection with the study of stability of stationary motions of mechanical systems and has played an important role in the development of this problem. Stability problems on the stationary motions of mechanical systems and on the permanent rotations of a rigid body are closely connected, their interinfluence defining their joint development in many respects. The formation of these problems was connected with the Routh theorem [2] and Majevskii criterion [3]. Their systematic investigation started with the appearance of the Chetaev method [4] and Rumyantsev's paper [5] having provided a suitable mathematical apparatus and having defined the direction of research. The introduction in the research domain of the problem of gyrostat motion [6, 7], other new objects [8] and force fields [9] raised the interest in the problem and defined the period of its intensive development. At this time KAM-theory had an essential influence on it and led to the extension of the use of Hamiltonian mechanics methods and the raising of the meaning of necessary conditions. The use of Kolmogorov's idea [10] permits us to characterize stability domains in the phase and parameter space as the domains of the fulfilment of the necessary conditions from which only some subdomains of smaller dimension can be excluded. On this base it is possible to say about practical end of the problems for which the investigation of necessary stability conditions is fulfilled, what is really done for many problems. The registration of this approach signifies the end of an intensive development period of the permanent rotations stability problem. The modern stage is characterized by the study of new objects such as multibody systems with different kinds of joints, a body on a string and others; by the search for new effects and by the movement of interest from stability theory into attractor theory, chaos and other modern topics of dynamical systems theory.

In the presented paper the state of stability theory of stationary motions of mechanical systems, the stability problem of permanent rotations of a rigid body and its generalizations are described. The presentation is in the main based on the results obtained by the Donetsk school of mechanics where these problems were studied the most widely and completely.

2 Objects and Motions

The problem of a motion of a rigid body with a fixed point in a gravity force field occupies the central place in rigid body dynamics. For its study different forms of the motion equations are offered, from which we choose the best-known Euler-Poisson equations

$$A_1\dot{\omega}_1 = (A_2 - A_3)\omega_2\omega_3 + \Gamma(e_2\nu_3 - e_3\nu_2) \quad (123), \tag{1}$$

$$\dot{\nu}_1 = \nu_2\omega_3 - \nu_3\omega_2 \quad (123), \tag{2}$$

where $\omega_2,\, \omega_2,\, \omega_3;\; \nu_1,\, \nu_2,\, \nu_3;\; e_1,\, e_2,\, e_3$ are, respectively, projections on the moving axes of an angular velocity, a vertical unit vector and a unit vector leading from the fixed point in the direction of the mass center of a body; $A_1,\, A_2,\, A_3$ are principal moments of inertia; Γ is the product of the body weight and the distance from the fixed point to the mass center; (123) is a symbol of cyclic index permutation.

Equations (1) and (2) allow the integrals

$$\begin{gathered} A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2 - 2\Gamma(e_1\nu_1 + e_2\nu_2 + e_3\nu_3) = h, \\ A_1\omega_1\nu_1 + A_2\omega_2\nu_2 + A_3\omega_3\nu_3 = k, \\ \nu_1^2 + \nu_2^2 + \nu_3^2 = 1. \end{gathered} \tag{3}$$

Gyrostat. The necessity of accounting for the influence of interior masses motions on the Earth's motion led Volterra [11] to the creation of a new mechanical object named a gyrostat. At the present time by the term gyrostat we understand a rigid body having cavities with liquid performing a potential motion [8] or a body carrying fly-wheels rotating in a definite way [7]: on inertia or with constant relative velocity. Let's write the motion equations and integrals of a gyrostat with fixed point in a gravity force field

$$A_1\dot{\omega}_1 = (A_2 - A_3)\omega_2\omega_3 + \lambda_2\omega_3 - \lambda_3\omega_2 + \Gamma(e_2\nu_3 - e_3\nu_2),$$
$$\dot{\nu}_1 = \nu_2\omega_3 - \nu_3\omega_2 \quad (123), \tag{4}$$

$$A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2 - 2\Gamma(e_1\nu_1 + e_2\nu_2 + e_3\nu_3) = h,$$
$$(A_1\omega_1 + \lambda_1)\nu_1 + (A_2\omega_2 + \lambda_2)\nu_2 + (A_3\omega_3 + \lambda_3)\nu_3 = k, \tag{5}$$
$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

Here in addition to the notations introduced under λ_1, λ_2, λ_3 the projections of gyrostatic moment vector on the moving axes are designated.

A Rigid Body with Vortex Filling. A great number of papers are devoted to the study of the motion of a body with an ellipsoidal cavity completely filled by an ideal uniform incompressible liquid performing the uniform vortex motion. The motion equations of the body-liquid system have the form [8, 12]

$$\dot{\Omega}_1 = (1 - \varepsilon_3)\omega_3\Omega_2 - (1 + \varepsilon_2)\omega_2\Omega_3 + (\varepsilon_2 + \varepsilon_3)\Omega_2\Omega_3,$$
$$\frac{d}{dt}(a_1\omega_1 + b_1\Omega_1) = (a_2\omega_2 + b_2\Omega_2)\omega_3 - (a_3\omega_3 + b_3\Omega_3)\omega_2 + \Gamma(e_2\nu_3 - e_3\nu_2), \tag{6}$$
$$\dot{\nu}_1 = \nu_2\omega_3 - \nu_3\omega_2 \quad (123).$$

Here in addition the designations are introduced: Ω_1, Ω_2, Ω_3 are the projections of the vortex vector on the moving axes; a_1, a_2, a_3 are changed inertia moments of the body-liquid system; Γ is the product of the body-liquid system weight and the distance from the mass center to the fixed point divided by $2c^2M/s$; $c^2 = c_1^2 + c_2^2 + c_3^2$; c_1, c_2, c_3 are semiaxes of the cavity-ellipsoid; M is the liquid mass in the cavity;

$$\varepsilon = \frac{c_3^2 - c_2^2}{c_3^2 + c_2^2}, \qquad b_1 = \frac{2c_2^2c_3^2}{c^2(c_2^2 + c_3^2)} \quad (123).$$

Equations (6) allow the integrals

$$\sum_{i=1}^{3}(a_i\omega_i^2 + b_i\Omega_i^2 - 2\Gamma e_i\nu_i) = h, \qquad \sum_{i=1}^{3}(a_i\omega_i + b_i\Omega_i)\nu_i = k, \tag{7}$$
$$\frac{\Omega_1^2}{c_1^2} + \frac{\Omega_2^2}{c_2^2} + \frac{\Omega_3^2}{c_3^2} = m, \qquad \nu_1^2 + \nu_2^2 + \nu_2^2 = 1.$$

Multibody System. The equations of motion of the system of rigid bodies can be obtained in different forms depending on the choice of coordinate systems and main variables. The number of possible forms of equations is increasing because the bodies can be composed in groups in different ways. A form of such equations that are transparent and accessible for further investigation are required. The equations satisfying these

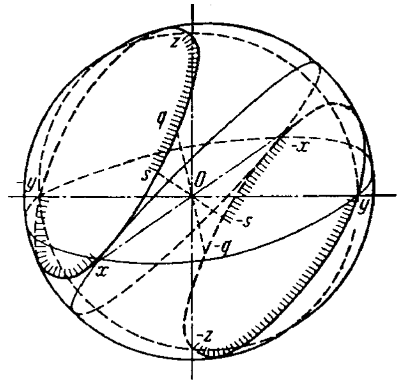

Figure 2.1. Intersection of the Staude cone with the unit sphere.

requirements are offered in [13] for the system of n gyrostats

$$\frac{d}{dt}(A^n\omega^n + \lambda^n) + m^n c^n \times \left[\sum_{l=1}^{n-1}\frac{d}{dt}(\omega^l \times s^l) - g\nu\right] = 0,$$

$$\frac{d}{dt}(A^k\omega^k + \lambda^k) + m^k c^k \times \left[\sum_{l=1}^{k-1}\frac{d}{dt}(\omega^l \times s^l) - g\nu\right] + s^k \tag{8}$$

$$\times \sum_{q=k+1}^{n} m^q \left[\frac{d}{dt}(w^q \times c^q) + \sum_{l=1}^{q-1}\frac{d}{dt}(\omega^l \times s^l) - g\nu\right] = 0 \quad (k = 2, 3, \ldots, n-1),$$

$$\frac{d}{dt}(A^1\omega^1 + \lambda^1) + s^1 \times \sum_{q=2}^{n} m^q \left[\frac{d}{dt}(\omega^q \times c^q) + \sum_{l=1}^{q-1}\frac{d}{dt}(\omega^l \times s^l) - g\nu\right] = m^1 c^1 \times g\nu,$$

$$\dot{\nu} = \nu \times \omega.$$

Carrier-bodies S_0^k, S_0^{k-1} of gyrostats S^k, S^{k-1} $(k > 1)$ have one generic point O^k; where ω^k is the absolute angular velocity of the body S_0^k; vector s^k leads from O^k to the mass center of gyrostat S^k; m^k is the mass of S^k; A^k is the inertia tensor of gyrostat S^k at point O^k; λ^k is the gyrostatic moment of this gyrostat.

The development of equations (8), and their further transformation have played the essential role in the amplification of interest in multibody dynamics and have promoted significant progress in obtaining the exact solutions; their number has more than doubled since the time of their publication.

Permanent Rotations. In rigid body dynamics the most studied stationary motions are permanent rotations which are characterized by the property that the angular velocity vector is constant and leads along the vertical. This provides us with the possibility of representing permanent rotation in the moving (connected with the body) space. The real motion is obtained by the coincidence of the permanent rotation axis with the vertical and rotating the body around the vertical with angular velocity obtained. Under the common values of parameters the set of permanent rotations is one-dimensional and consists of three parts of the curve, lying at the Staude cone (Figure 2.1)

$$(A_2 - A_3)e_1\omega_2\omega_3 + (A_3 - A_1)e_2\omega_3\omega_1 + (A_1 - A_2)e_3\omega_1\omega_2 = 0. \tag{9}$$

Note that in the general situation permanent rotations about principal axes are impossible. They can appear under some conditions on parameters. So they exist in the Lagrange, Kovalevskaya, Hess cases and in the Euler case permanent rotations are admissible only around three principal axes. Note more the singular case: in the Lagrange case the permanent rotations set consists of the principal axis and some surface.

On the whole a similar but more complicated picture exists for a gyrostat [14]. In principle, the picture is different for a body with vortex filling, for which the permanent rotations set forms a solid angle [12]. For systems of the rigid bodies, on the whole, permanent rotations of the Lagrange gyroscopes around principal axes have been considered [12].

3 Stability of Stationary Motions

Two main approaches to the investigation of stationary motions stability have been created. The first is based on the Routh-Lyapunov theorem and Chetaev method, the second – on the Arnold-Moser theorem extended to stationary motions. To form these theorems it is convenient to use Hamilton variables $q_i,\ p_i\ \ (i = 1,\dots,n)$ for the description the motion of a conservative mechanical system with n degrees of freedom. In the presence of cyclic coordinates $q_\alpha\ \ (\alpha = k+1,\dots,n)$ the Hamilton function has the form $H(q_1,\dots,q_k,\, p_1,\dots,p_n)$ and the equations of motion have $n-k$ cyclic integrals $p_\alpha = c_\alpha = \text{const}\ \ (\alpha = k+1,\dots,n)$. The function $H(q_1,\dots,q_k,\, p_1,\dots,p_k,\, c_{k+1},\, c_n)$ defines the system with k degrees of freedom which is called the reduced one.

The stationary motions of the mechanical system are called such motions for which positional coordinates and impulses $q_i,\ p_i\ \ (i = 1,\dots,k)$ and cyclic impulses $p_\alpha\ \ (\alpha = k+1,\dots,n)$ preserve constant values $q_i = q_i^0,\ p_i = p_i^0,\ p_\alpha = c_\alpha$. Constants c_α are arbitrary, values $q_i^0,\, p_i^0$ are obtained from the equations

$$\frac{\partial H}{\partial q_i} = 0, \qquad \frac{\partial H}{\partial p_i} = 0, \qquad i = 1,\dots,k$$

and determine the equilibrium position of the reduced system.

Under stability of stationary motions we understand the stability of these motions with respect to the values $q_i,\ p_i,\ c_\alpha\ \ (i = 1,\dots,k;\ \alpha = k+1,\dots,n)$. The effective tool of investigation of stationary motions stability is the Routh theorem [2] (with the Lyapunov addition [15]) reducing the question about stationary motions stability to the analysis of the extremum of potential energy of the reduced system.

Theorem 3.1 *If potential energy of the reduced system has the minimum both under the given values $p_\alpha = c_\alpha$ responding to the stationary motion considered and under any close to the given values $p_\alpha = c_\alpha + \eta_\alpha$ and also the values q_i inverting it in the minimum are continuous functions of the variables p_α, then stationary motion is stable.*

Different generalizations of Theorem 3.1, its connection with the Chetaev method and application to the investigation of the permanent rotations stability have been considered in the papers [16, 17], where in particular it was pointed out that for the establishment of stationary motions stability it was sufficient to establish by the Lyapunov method the stability of equilibrium position of the reduced system, moreover it was convenient to construct the Lyapunov function by the Chetaev method from the motion integrals.

For Hamiltonian systems in the formulation of Theorem 3.1 the Hamilton function of the reduced system is used instead of the potential energy. Theorem 3.1 does not solve the question about stationary motions stability if the Hamiltonian of the reduced system is not a function of fixed sign in the equilibrium position. For Hamiltonian systems with two-dimensional reduced system, the stability of stationary motions can be obtained with the help of the following theorem [18] extending the known Arnold-Moser theorem [19, 20] to the case of stationary motions.

Theorem 3.2 *Let the Hamiltonian* $H(q_1, q_2, p_1, p_2, \ldots, p_{2+m})$ *be an analytical function of the coordinates and impulses at the point p with the coordinates*

$$q_1 = q_2 = 0, \quad p_1 = p_2 = 0, \quad p_{2+i} = c_i, \quad i = 1, \ldots, m \tag{10}$$

defining the stationary motion considered. The Hamiltonian of the reduced system at this point satisfies the following conditions:

1. *Eigenvalues of the linearized reduced system are pure imaginary* $\pm i\alpha_1, \ \pm i\alpha_2$.
2. *For all integers* $k_1, \ k_2$ *satisfying the condition* $|k_1| + |k_2| \leq 4$ *the inequality* $k_1\alpha_1 + k_2\alpha_2 \neq 0$ *is fulfilled.*
3. $D = -(\beta_{11}\alpha_2^2 - 2\beta_{12}\alpha_1\alpha_2 + \beta_{22}\alpha_1^2) \neq 0$, *where* $\beta_{\nu\mu}$ *are the coefficients of the fourth order form for the Hamiltonian, transformed into the form*

$$H = \sum_{\nu=1}^{2} \frac{\alpha_\nu}{2} R_\nu + \sum_{\nu,\mu=1}^{2} \frac{\beta_{\nu\mu}}{4} R_\nu R_\mu + O_5, \qquad R_\nu = \xi_\nu^2 + \eta_\nu^2$$

(O_5 *is a power series containing the terms of order not less then five).*

Then stationary motion (10) is stable.

Condition 1 of this theorem is fulfilled in the domain of fulfilment of necessary stability conditions. In the common situation nonfulfilment of conditions 2 and 3 leads to the exclusion of some sets of lesser dimension from this domain. In the rest domain, which differs little from the domain of the fulfilment of necessary stability conditions, stationary motions are stable. Therefore in the nonsingular case (conditions 2 and 3 don't take identities) it is practically sufficient to study only the necessary stability conditions. For the analysis of singular cases it is possible to apply the theorems on stability of the equilibrium position under the presence of resonances [21] and the vanishing of discriminant D [22] extended to stationary motions.

4 Permanent Rotations of a Rigid Body

One of the first problems solved on the stability of permanent rotations is the problem of the stability of permanent rotations of the Lagrange gyroscope around its principal axis. The conditions of their stability are known as the Majevskii criterion [3] and were obtained during research into projectile motion. Following attempts [23, 24] didn't bring any serious progress in this problem and only the appearance of the Chetaev method gave the possibility of its systematic investigation which was begun in Rumyantsev's paper [5]. Sufficient stability conditions of permanent rotations were obtained in his paper by the construction of the Lyapunov function in the form of the bundle of the integrals of perturbed motion. With their help the stability domains were found both in the common case of mass distribution and in particular cases when the mass center belonged to one of the principal planes or to the principal axis and also when the ellipsoid of inertia was the ellipsoid of rotation. Subsequent investigations can be divided into three groups: the study of permanent rotations in integrable cases; analysis of the permanent rotations around principal axes; research on the general case (permanent rotations around principal axes are impossible). The interest in integrable cases is caused by the fact that the presence of additional integrals permits us to obtain the necessary and sufficient stability conditions by the Lyapunov functions method. Two rest directions are connected with the motions which are of most interest from the theoretical and applied point of views. Let's examine them in more detail.

Permanent Rotations about Principal Axes. Let the mass center belong to the principal axis, then the body can rotate about this axis permanently with arbitrary velocity. The stability of these motions is studied with respect to the variables ω_1, ω_2, ω_3, ν_1, ν_2, ν_3 under the use of equations (1) or with respect to the Euler angles θ, φ and all generalized impulses p_θ, p_φ, p_ψ under the use of Hamiltonian equations. Using the Chetaev method, Rumyantsev [5, 25] obtained sufficient stability conditions of the considered permanent rotations which are equivalent to the conditions of the property of having fixed sign the square part of Hamiltonian H_2. The following investigation of this problem was fulfilled with the help of Theorem 3.2 in paper [26]. Let's look at its main result.

A body motion is described by Hamilton equations in the Euler angles introduced in the usual way. For Hamiltonian to have no singularities on the considered motion we place the mass center on the first principal axis. The following solution corresponds to the permanent rotations studied

$$\theta^0 = \varphi^0 = \frac{\pi}{2}, \quad \psi^0 = \omega_0 t + \psi_0, \quad p_\theta^0 = p_\varphi^0 = 0, \quad p_\psi^0 = A_1\omega,$$

where ω_0 is the angular velocity of the permanent rotation. Introducing the perturbations

$$\theta = \frac{\pi}{2} + y_1', \quad \varphi = \frac{\pi}{2} + y_2', \quad p_\theta = x_1', \quad p_\varphi = x_2'$$

and going over to the dimensionless variables, we obtain the following presentation for the Hamiltonian H

$$H = H_2 + H_4 + O_5, \tag{11}$$

$$2H_2 = ax_1^2 + bx_2^2 + (\omega^2 - e)y_1^2 + [(a-1)\omega^2 - e]y_2^2 + 2(a-1)\omega x_1 y_2 + 2\omega x_2 y_1,$$

$$\begin{aligned} 2H_4 = {} & (1-a)x_1^2 y_2^2 + x_2^2 y_1^2 + \frac{8\omega^2 + e}{12}\, y_1^4 + \frac{4\omega^2(1-a) + e}{12}\, y_2^4 \\ & + \frac{2\omega^2(a-1) + e}{2}\, y_1^2 y_2^2 + \frac{4\omega(1-a)}{3}\, x_1 y_2^3 + \omega(a-1)x_1 y_1^2 y_2 \\ & + \frac{5}{3}\,\omega x_2 y_1^3 + 2\omega(a-1)x_2 y_1 y_2^2 + 2(a-1)x_1 x_2 y_1 y_2, \end{aligned}$$

where

$$e = \begin{cases} 1 & \text{for} \quad \Gamma < 0, \\ -1 & \text{for} \quad \Gamma > 0 \end{cases}.$$

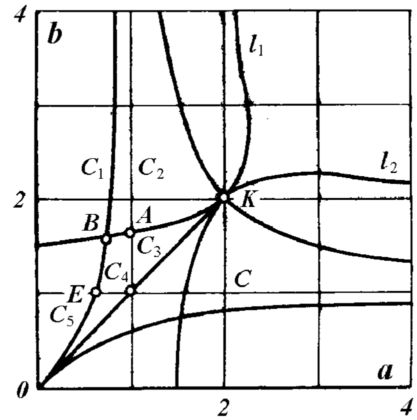

Figure 4.1. Stability domain in the space Oab.

Condition 1 of Theorem 3.2 is fulfilled in the domain G of necessary stability conditions fulfilment which were obtained and analyzed in detail in paper [27]. Condition 2 is reduced to one inequality which is broken under the condition defining the resonance of the third order

$$9\omega^4 + 2(41b - 59)\omega^2 + (9b - 1)(b - 9) = 0. \tag{12}$$

To check the third condition Hamiltonian (11) is transformed to the normal form up to terms of the fourth order inclusively and discriminant D is calculated which has rather simple expression for values $a = 1,\ e = 1$

$$\begin{aligned} D_1 = (b-1)^2\omega^8 + 2(b-1)(b^2+2b-5)\omega^6 + (b-1)(b^3+13b^2 \\ - 41b + 7)\omega^4 + 8(b^4 - 5b^3 + 5b^2 + b + 2)\omega^2 + 4b(b-1)^2. \end{aligned} \tag{13}$$

A conclusion about the stability of permanent rotations is obtained by the application of Theorem 3.2. For the descriptive representation of the obtained results there has been introduced an extended parametric space being the direct product of the space of mechanical system parameters and cyclic constants space, in this case the plane $Ob\omega$. The equation (12) and $D_1 = 0$ determine in the plane $Ob\omega$ resonance curve s_1 and discriminant curve s_2 (Figure 4.1). The theorem is true.

Theorem 4.1 *Let a rigid body having equal inertia moments about two first axes* $(a = 1)$ *be rotated permanently about the first axes carrying the mass center situated higher than fixed point* $(e = 1)$*. Then in extended parametric space-plane $Ob\omega$-stability domain represents domain G_1 of fulfilment of necessary stability conditions from which curves s_1, s_2 are excluded (Figure 4.1).*

Returning to the common case we note from the formulas (12) and (13) that in the space $Oab\omega$ conditions 2 and 3 of Theorem 3.2 are not fulfilled on resonance and discriminant surfaces S_1 and S_2; that is the following result occurs.

Theorem 4.2 *Let a rigid body be rotated permanently around the principal axis carrying the mass center. Then in the extended parametric space $Oab\omega$ stability domain represents the domain G from which surfaces S_1 and S_2 are excluded.*

It should be noted that the permanent rotations corresponding to resonance curve s_1 have been studied in paper [28]. They are found to be stable and they should not be excluded from the domain G_1.

Permanent Rotations about Staude Cone Axes. The problem of searching for the stability conditions of the permanent rotations under arbitrary mass distribution seemed at first hopelessly difficult [23]. In 1920 R.Grammel obtained the necessary stability conditions which he couldn't analyze because of their complexity and was forced "to invert" the statement of the problem. Sufficient conditions were obtained by Rumyantsev [5] by the construction of the Lyapunov function in the form of the bundle of integrals (3):

$$\mu\nu_1^2 + \Gamma\,\frac{e_1}{\nu_1} > 0, \qquad \Gamma\,\frac{e_1 e_2}{\nu_1\nu_2} + \mu\left(\frac{e_1}{\nu_1}\,\nu_2^2 + \frac{e_2}{\nu_2}\,\nu_1^2\right) > 0, \tag{14}$$
$$\mu\left(\frac{e_2e_3}{\nu_2\nu_3}\,\nu_1^2 + \frac{e_1e_3}{\nu_1\nu_3}\,\nu_2^2 + \frac{e_1e_2}{\nu_1\nu_2}\,\nu_3^2\right) + \Gamma\frac{e_1e_2e_3}{\nu_1\nu_2\nu_3} > 0.$$

Here the variables ν_1, ν_2, ν_3 satisfy equation (9), where μ is an arbitrary constant.

In paper [5] only preliminary analysis of conditions (14) is fulfilled. On its basis some stability domains of permanent rotations at Staude cones are noted.

The application of Theorem 3.2 to the analysis of these rotations is made difficult by the awkwardness of the calculations. Therefore it is natural to do their analysis by computer. Such research was fulfilled for gyrostats [29] and is described in the subsection below.

A Body in the Newtonian Gravity Force Field. Where there is a significant distance between a body and the attracting center in many cases it is assumed only taking account of the forces containing linear terms of the expansions on the degrees of value inverse to this distance. The force field obtained in this way is called the Newtonian field. Motion equations and integrals of a body with a fixed point have the form [30]

$$A_1\dot\omega_1 = (A_2 - A_3)\omega_2\omega_3 + \Gamma(e_2\nu_3 - e_3\nu_2) - \mu(A_2 - A_3)\nu_2\nu_3,$$
$$\dot\nu_1 = \nu_2\omega_3 - \nu_3\omega_2 \qquad (123), \tag{15}$$
$$A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2 - 2\Gamma(e_1\nu_1 + e_2\nu_2 + e_3\nu_3)$$
$$+\,\mu(A_1\nu_1^2 + A_2\nu_2^2 + A_3\nu_3^2) = h,$$
$$A_1\omega_1\nu_1 + A_2\omega_2\nu_2 + A_3\omega_3\nu_3 = k, \tag{16}$$
$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

Here μ is a constant characterizing the force field.

Stability of stationary motions of this system was studied by Kuz'min [9]. He established that as for the case of constant gravity stationary motions are permanent rotations about "vertical" the axes of which belong to the Staude cone. Note the convenient parametrization of the permanent rotations introduced in this paper

$$\nu_1 = \frac{\Gamma e_1}{\Omega(\rho - A_1)} \qquad (123), \qquad \Omega^2 = \Gamma^2\sum_{i=1}^{3}\frac{e_i^2}{(\rho - A_i)^2}, \tag{17}$$

where $\Omega = \omega^2 - \mu$, and ω is the angular velocity of permanent rotation. Stability conditions are obtained in accordance with the Routh-Lyapunov theorem as the conditions of the property of having fixed sign second variation of the first integral (16) under conservation the rest integrals on the perturbed motions

$$\Omega T > 0, \tag{18}$$

$$4\omega^2 \Omega T + \Omega^2 IS > 0. \tag{19}$$

Here

$$T = \sum_{(123)} (\rho - A_1)(A_2 - A_3)\nu_2^2 \nu_3^2,$$

$$S = \sum_{(123)} (\rho - A_2)(\rho - A_3)\nu_1^2,$$

$$I = \sum_{(123)} A_1 \nu_1^2,$$

where symbol $\sum\limits_{(123)}$ means the summation of three terms obtained from the one shown under the sum symbol by the cyclic permutation of indexes.

Condition (19) excluding the boundary is not only one of the sufficient conditions but the necessary one. In addition, setting $\Omega = \omega^2$ in inequalities (18) and (19) we obtain from them the stability conditions for constant gravity softening Rumyantsev conditions (14).

5 Permanent Rotations of a Gyrostat

The investigation of permanent rotations stability of a gyrostat was begun by Volterra [11] who considered in detail the permanent rotations of a gyrostat on inertia. Rumyantsev [6] analyzed these rotations by the Lyapunov method. In this paper the sufficient stability conditions of permanent rotations of a heavy gyrostat around the principal axis with arbitrary angular velocity are also obtained. The case when a gyrostat can rotate around the principal axis only with some fixed velocity was studied by Anchev [31]. The investigation of the permanent rotations around the principal axis was continued with the help of Theorem 3.2 in paper [32]. Here is its main result.

Rotations Around the Principal Axis. Under the assumption that the mass center of a gyrostat belongs to the first principal axis and the vector λ of gyrostatic moment is directed along the same axis the gyrostat can rotate permanently around the first axis with angular velocity ω and this rotation is defined by the following values of the variables

$$p_\theta = p_\varphi = 0, \quad p_\psi = \frac{\omega}{a_1} + \lambda, \quad \theta = \varphi = \frac{\pi}{2}, \quad \psi = \omega t + \psi_0, \tag{20}$$

where a_1 is the first component of tensor A^{-1}.

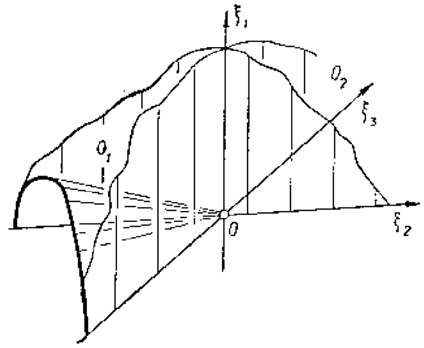

Figure 5.1. The domain of the fulfilment of the necessary stability conditions in the space $O\xi_1\xi_2\xi_3$.

In dimensionless variables the Hamilton function of perturbed motion has the form

$$
\begin{aligned}
H &= H_2 + H_4 + \dots, \\
2H_2 &= ax_1^2 + bx_2^2 + (\omega^2 + \omega\lambda - e)y_1^2 + \big[(\omega+\lambda)\big(a(\omega+\lambda) - \omega\big) - e\big]y_2^2 \\
&\quad + 2\big(a(\omega+\lambda) - \omega\big)x_1y_2 + 2\omega x_2y_1, \\
24H_4 &= (3\lambda^2 + 11\lambda\omega + 8\omega^2 + e)y_1^4 + \big[(\omega+\lambda)(4\omega + 3\lambda - 4a(\omega+\lambda)) + e\big]y_2^4 \\
&\quad + 6\big[(\omega+\lambda)(-2\omega - \lambda + 2a(\omega+\lambda)) + e\big]y_1^2y_2^2 + 12(1-a)x_1^2y_2^2 + 12x_2^2y_1^2 \\
&\quad + 4(4\omega + 3\lambda - 4a(\omega+\lambda))x_1y_2^3 + 4(5\omega + 3\lambda)x_2y_1^3 + 12(a-1)(\omega+\lambda)x_1y_1^2y_2 \\
&\quad + 12(-2\omega - \lambda + 2a(\omega+\lambda))x_2y_1y_2^2 + 24(a-1)x_1x_2y_1y_2.
\end{aligned} \tag{21}
$$

To obtain the necessary stability conditions we write the characteristic equation for the linearized system with function H_2

$$
\begin{gathered}
\mu^4 + \xi\mu^2 + \xi_2\xi_3 = 0, \\
\xi_1 = ab(\omega+\lambda)^2 - (a+b)(\omega+\lambda)\omega + 2\omega^2 - e(a+b), \\
\xi_2 = \omega^2(a-1) + a\omega\lambda - ae, \quad \xi_3 = \omega^2(b-1) + b\omega - be.
\end{gathered}
$$

It is convenient to represent the domain D of fulfilment of the necessary stability conditions in the space of parameters ξ_1, ξ_2, ξ_3 (Figure 5.1). In the subdomain G_2 the square form H_2 is of the fixed sign and corresponding permanent rotations are stable. To study the stability of permanent rotations corresponding to the subdomain G_1 Theorem 3.2 is applied. We obtain the expressions for resonance relations and the discriminant by applying the corresponding normalization transformation. Note that under $\lambda = 0$, $a = 1$ the discriminant D has the form (13), i.e. $D \not\equiv 0$. Therefore the equality $D(a,b,\lambda,\omega) = 0$ selects in the space $Oab\lambda\omega$ some manifolds just as resonance relations. Permanent rotations corresponding to the manifolds selected are excluded from consideration. As for the remaining permanent rotations in the domain G_1 on the basis of Theorem 3.2 we conclude that these rotations are stable on Lyapunov.

Comparing these conclusions with the result obtained above for a rigid body we can state that the presence of a rotor in a body renders the stabilization effect on the carrying body motion under the corresponding choice of rotor rotation. Thus, unstable rotation of a body around the middle axis can be made stable under the corresponding choice of a gyrostatic moment. Moreover, any permanent rotation of a rigid body can be made stable under the corresponding choice of a gyrostatic moment.

Rotations Around an Arbitrary Axis. Sufficient stability conditions of a gyrostat around an arbitrary axis of permanent rotations cone were obtained by Anchev [33] and Druzhinin [34]. In paper [34] necessary stability conditions were also obtained. Using these conditions in paper [35] the stability and instability domains are shown on the permanent rotations cone.

It is convenient to describe the permanent rotations considered using the Kuz'min parametrization [9]

$$\nu_1 = \frac{\omega\lambda_1 + \Gamma e_1}{\omega^2(\rho - A_1)} \quad (123), \tag{22}$$

where ρ is an auxiliary parameter and angular velocity satisfies the equation

$$\omega^4 - \sum_{(123)} \frac{(\omega\lambda_1 + \Gamma e_1)^2}{\omega^2(\rho - A_1)^2} = 0.$$

Sufficient stability conditions have the following form [34]

$$\begin{gathered} D = A_1 A_2 A_3 \omega^6 (L + \omega^2 JM) > 0, \qquad D_1 = \omega^2 L > 0, \\ L = \sum_{(123)} (\rho - A_1)[2\omega(A_2 - A_3)\nu_2\nu_3 + \lambda_3\nu_2 - \lambda_2\nu_3]^2, \\ M = \sum_{(123)} (\rho - A_2)(\rho - A_3)\nu_1^2, \qquad J = \sum_{(123)} A_1\nu_1^2. \end{gathered} \tag{23}$$

Necessary stability conditions are such [34]

$$D > 0, \qquad N > 2\sqrt{D}, \tag{24}$$

$$N = \omega^2 \sum_{(123)} [\omega^2 A_1(\rho - A_1)(A_2\nu_2^2 + A_3\nu_3^2) + A_1(\omega(A_1 - A_2 - A_3)\nu_1 + \lambda_1)^2].$$

Conditions (23) and (24) were analyzed in paper [35]. From the conclusions obtained there we note that the permanent rotations around the axes near to the middle principal axis are unstable. On the stability of the permanent rotations around the axes close to the major axis it is impossible to come to a conclusion on the basis of conditions (23) and (24) because the necessary conditions are fulfilled, but the sufficient conditions (23) are not. For their study Theorem 3.2 is applied.

Analytical difficulties connected with the investigation of the general case have led to the necessity of the use of computer methods for its analysis. The appearance of the computing normalization algorithms for Hamilton systems promotes this analysis. A numerical algorithm of the investigation of stability of the permanent rotations of a gyrostat was created by Chudnenko [29]. Using the parametrization (22) he introduces the partition $\rho_1, \dots, \rho_n$ of the set of permanent rotations. For every ρ_k conditions (23) and (24) are checked. If conditions (24) are fulfilled and conditions (23) are not, conditions of Theorem 3.2 are checked. In addition the values of parameter ρ are noted for which Theorem 3.2 does not solve the stability question. Thus the obtained algorithm permits us to solve practically completely the stability problem of permanent rotations of a gyrostat in the case of general mass distribution.

6 Permanent Rotations of a Rigid Body with Vortex Filling

The permanent rotations of a rigid body with vortex filling are realized only around an axis coincident with the vertical under the condition that vortex components in the moving coordinate system are constant [12]. Because of the complexity of the problem on the distribution of the permanent rotation axes with respect to a body geometrically visual turned out the approach which was based on the construction the set of axes corresponding to the permanent rotation velocity given. These sets are depicted on the plane Ouv defined by the mapping

$$\Gamma^{-1}u = e_1\nu_1^1 - e_3\nu_3^{-1}, \quad \Gamma^{-1}v = e_2\nu_2^{-1} - e_3\nu_3^{-1}, \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

On the basis of the fulfilled analysis interesting effects connected with the vortex presence are obtained. In particular it has been established that a body can rotate permanently around the principal axis when its mass center doesn't belong to the principal axis, and conversely permanent rotations around nonprincipal axes are possible when the mass center belongs to the principal axis.

One of the most interesting effects from the application point of view is the permanent rotation of a shell around the principal axis carring the mass center with angular velocity ω and permanent rotation of a liquid as a rigid body around the same axis with angular velocity Ω. The case of a symmetric body is studied the most completely. On the basis of analysis of necessary stability conditions it is fixed that appearance of codirected vorticity ($\omega\Omega > 0$) extends with respect to ω the domain of fulfilment of necessary stability conditions and contrarily directed vorticity ($\omega\Omega < 0$) constricts. Note separately the case when the motion of a body-liquid system is unstable under any value of ω. This takes place for a top with a cavity that is a rotated ellipsoid stretched along the axis of a body symmetry under the fulfilment of some additional conditions. For example, for the case when the mass of a shell is negligibly small with respect to the mass of a liquid in the cavity and the distance from the fixed point to the center of the cavity is equal to the major semiaxis of an ellipsoid c_3 with $c_1 < c_3 < 1.26c_1$, such a "slightly" stretched fluid rotated ellipsoid is unstable no matter with what angular velocity it is rotated. This effect was experimentally found by Lord Kelvin [36].

The sufficient stability conditions and formal stability of permanent rotations around principal and nonprincipal axes are investigated [12]. It has been established that the presence of vortex motion of a liquid in a cavity leads to the appearance of such conditions of shell motion that are absent for a rigid body and a gyrostat. If it is necessary these rotations can be made stable.

7 Permanent Rotations of Multibody Systems

The essential influence on the development of multibody systems dynamics was rendered by the stability problem of permanent rotations of two heavy Lagrange gyros connected by ideal spherical hinges one of which has a fixed point. Ishlinskii called attention to this problem in 1972 at the 13th IUTAM Congress although the characteristic equation for it was obtained in 1898 and was published in Lur'e's monograph [37]. The solution of this problem is given in paper [38] where equations (8) are used for the description of the motion. The permanent rotation of the system of two Lagrange gyros represents such motion in which every from the gyros is rotated with permanent angular velocity about its dynamic symmetry axis collinear to the gravity force direction. Under the stabiliity of this motion is understood the stability of the corresponding solution of equations (8) with respect to some of the variables – namely, to the angular velocities of the bodies S_1, S_2 and to the parameters defining the position in a space of symmetry axes of the bodies S_1 and S_2. Sufficient stability conditions are obtained by the Chetaev method. When both gyros are unbalanced $(c_i > 0)$ they have the form

$$\omega_1\omega_2 > 0, \quad A_2^2\omega_2^2 - 4\mu_2 B_2 > 0, \quad A_1\omega_1^2 - 4\mu_1 B_1 > 0, \tag{24}$$

where $\mu_1 = m_1c_1 + m_2s_1 > 0$, $\mu_2 = m_2c_2 > 0$; where A_1 and B_1 are the axial and equatorial inertia moments of i-th body, respectively. These conditions mean that the bodies are rotated in the same direction. For the second body the Majevskii stability criterion of permanent rotations for one unbalanced Lagrange gyro is fulfilled, the first gyro S_1 with point mass m_2 at the point O_2 has also to satisfy the Majevskii criterion.

The necessary stability conditions are analyzed in detail and are compared with the sufficient ones. Here the interesting stabilization effect is found when one of the remaining unbalanced Lagrange gyros becomes stable under the definite rotation velocity of the second one. This effect calls to mind the stabilization effect of an unbalanced remaining gyro on the oscillating base. But in this case the oscillations of the fixed point arise not at the expense of exterior forces but are, in a definite sense, the self-vibrating ones.

The permanent rotations are also considered for an n-bodies system. It has proved ([13]) to be possible (for $\lambda = 0$) only under the condition when the vectors c_k, s_k, ν are collinear. Equations (8) admit the solution $\omega_k = \omega_k\nu$ which corresponds to the permanent rotations of every body S_k around the axis carring its mass center and coinciding with vertical. In this connection the angular velocities ω_k for each body can be different. Under the additional assumption that the bodies considered are the Lagrange gyros the stability of these rotations was investigated [39] with respect to the angular velocities and the parameters defined the position of the rotation axes in the space. The notion of the "enlarged" body S_k' is introduced which is obtained from the body S_k by adding to it at the point O_k the point mass equals to $\sum\limits_{i=k+1}^{m} m_i$. Such a body is characterized by the parameters A_k, B_k', a_k:

$$B_k' = B_k + s_k^2 \sum_{i=k+1}^{n} m_i, \quad a_k = m_kc_k + s_k \sum_{i=k+1}^{n} m_i, \quad k = 1, \dots, n-1,$$

where A_k, B_k are the axial and equatorial inertia moments of the body S_k with respect to its suspension point O_k.

For the case when all "the enlarged" bodies are unbalanced $(a_k > 0)$ the sufficient stability conditions are obtained

$$A_k^2\omega_k^2 - 4B_k^{'}a_k g > 0, \quad \omega_k\omega_i > 0; \quad k,\, i = 1, \dots, n$$

which generalize the Majevskii criterion and conditions (24). Necessary stability conditions of permanent rotations and regular precessions have been considered. More complicated motions named "similar" ones [12] when symmetry axes of Lagrange gyros belong to one plane during the entirety of the motion are found and investigated.

New effects were discovered during the analysis of the influence of the stiffness in the elastic joints on the stability of the permanent rotations of a multibody system. In particular, under specific values of the stiffness instability interval appears in the problem which classical analog is Euler case. However, when the stiffness is rather great this system behaves as a single rigid body and under the fulfilment of the Majevskii criterion for the body obtained from the system considered by the change of the joints on rigid fixings permanent rotations are stable.

In conclusion, we note a new direction in the investigation of the stability of permanent rotations connected with studying the influence of small nonsymmetry on the stability of the motion of the system of the bodies connected by elastic joints. The analysis of the motion of a single unsymmetric body has already showed that its stable permanent rotations about the symmetry axis after introducing the system debalance pass into unstable ones in the neighborhoods of some frequencies named resonance ones [39, 40]. The research into multibody systems discovered similar situations. A general approach for finding the resonance frequencies was offered and a constructive algorithm for finding two groups of such frequencies was created which gave the possibility to obtain the analytical expressions for them in some cases. With its help the motion of multibody systems was studied with different ways of connection and the force action and the critical operating conditions of the elastic objects motion were established.

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