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On Three Definitions of Chaos

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Abstract: We discuss in this paper three notions of chaos which are commonly used in the mathematical literature, namely those being introduced by Li & Yorke, Block & Coppel and Devaney, respectively. We in particular show that for continuous mappings of a compact interval into itself the notions of chaos due to Block & Coppel and Devaney are equivalent while each of these is sufficient but not necessary for chaos in the sense of Li & Yorke. We also give an example indicating that in the general context of continuous mappings between compact metric spaces the relation between these three notions of chaos is more involved.

Keywords: *Iteration; chaos; chaotic map.*

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1 Introduction

As a mathematical notion the term *chaos* has first been used in 1975 by Li & Yorke in their paper [13] "Period three implies chaos", but even before it has been observed that very simple functions may give rise to very complicated dynamics. One of the cornerstones in the development of *chaotic dynamics* is the 1964 paper [15] "Coexistence of cycles of a continuous mapping of the line into itself" (in Russian) by Šarkovskii. During the seventies and eighties the interest in *chaotic dynamics* has been exploding and various attempts have been made to give the notion of *chaos* a mathematically precise meaning. Outstanding works in this context are the 1980 book [6] "Iterated Maps on the Interval as Dynamical Systems" by Collet & Eckmann, the 1989 book [16] "Dynamics of One-dimensional Mappings" (in Russian) by Šarkovskii, Kolyada, Sivak & Fedorenko and the 1992 Lecture Notes [4] "Dynamics in One Dimension" by Block & Coppel. While up to the end of the eighties the subject of *chaotic dynamics* was restricted mainly to research oriented publications, the 1986 book [7] "An Introduction to Chaotic Dynamical Systems" by Devaney marked the point where *chaos* (as a mathematical notion) became popular and began to enter university textbooks such as [9] "A First Course in Discrete Dynamical Systems" by Holmgren (1994) or [8] "Discrete Chaos" by Elaydi (1999).

The different definitions of *chaos* being around at the turn of the century have been designed to meet different purposes and they are based on very different backgrounds and levels of mathematical sophistication. Therefore it is not obvious how these notions of *chaos* relate to each other and whether there is a chance that – in the long run – a universally accepted definition of *chaos* might evolve. With this paper we want to make a contribution to this question by picking three of the most popular definitions of *chaos* and investigating their mutual interconnections. After listing the technical prerequisites in Section 2, in Section 3 we give the precise definitions of the notions of *chaos* due to Li & Yorke [13], Block & Coppel [4] and Devaney [7], respectively. In Section 4 we then show that in the case of continuous maps on a compact interval the notions of *chaos* in the sense of Block & Coppel and Devaney are equivalent and that, on the other hand, each of these two notions is sufficient for *chaos* in the sense of Li & Yorke. In Section 5 we discuss the familiy of so-called truncated tent maps by means of which we in particular demonstate that *chaos* in the sense of Block & Coppel and Devaney is not necessary for *chaos* in the sense of Li & Yorke. Finally, in Section 6 we indicate by means of an example that the previously described simple relations for interval maps do not carry over to maps between general compact metric spaces. In fact, we exhibit a map which is chaotic both in the sense of Li & Yorke and Block & Coppel but not chaotic in the sense of Devaney.

Before going into detail we want to mention that the majority of results we describe and prove in this paper can be found – more or less explicitly stated – in the literature, in particular in the Lecture Notes [4] of Block & Coppel. Since those results, on the other hand, appear in the broad context of *discrete dynamics* and since they partly are formulated with notations which subtly differ from each other it is the purpose of this paper to narrow the view and sketch a clearer picture of the notion of *chaos* in a unified way.

2 Preliminaries

In this section we collect some notation and a few facts from topological and symbolic dynamics which are used in this paper.

Throughout this paper let (X, d) be a compact metric space and $f\colon X \to X$ a continuous mapping. A nonempty subset Y of X satisfying $f(Y) \subseteq Y$ is said to be (f-)*invariant* and it is called *strongly* (f-)*invariant* if $f(Y) = Y$. The set Y is called *minimal* if it is compact and does not contain any nonempty compact invariant proper subset. By f^n, $n \in \mathbb{N}$, we denote the mapping which is defined recursively by $f \circ f^{n-1}$ and $f^0 = \mathrm{id}$. The sequence $\gamma(x, f) := (f^n(x))_{n\geq 0}$ is the *trajectory* of $x \in X$, the set $\mathrm{O}(x, f) := \{f^n(x) \mid n \geq 0\}$ the (*forward-*)*orbit* of x and $\omega(x, f) := \bigcap_{m\geq 0} \overline{\mathrm{O}(f^m(x), f)}$ is the (ω-)*limit set* of x.

A point $x \in X$ and its orbit is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$, x is called *recurrent* if $x \in \omega(x, f)$. $\mathrm{P}(f)$ denotes the set of periodic points in X and $\mathrm{R}(f)$ the set of recurrent points. A point $x \in X$ is *finally periodic* if $f^n(x)$ is periodic for some $n \in \mathbb{N}$, it is *asymptotically periodic* if there exists a periodic point $z \in X$ such that $\lim_{n\to\infty} d(f^n(x), f^n(z)) = 0$, and it is *approximately periodic* if for any $\varepsilon > 0$ there exists a periodic point $z \in X$ such that $\limsup_{n\to\infty} d(f^n(x), f^n(z)) < \varepsilon$.

If $\omega(x, f) = X$ for some $x \in X$, the mapping f is called (*topologically*) *transitive*. The map f is said to have *sensitive dependence on initial conditions* if there exists a $\delta > 0$

such that for any $x \in X$ and any $\varepsilon > 0$ there exists a $y \in \{z \in X \mid d(x,z) < \varepsilon\}$ and an $n \in \mathbb{N}$ with $d(f^n(x), f^n(z)) > \delta$.

In the sequel we need a few results on the notions just introduced. They are described in the following remarks.

Remarks 2.1

(1) X always contains a minimal subset Y (see [4, V Lemma 3]) and $Y \subseteq X$ is minimal if and only if $\omega(x,f) = Y$ for all $x \in Y$ (see [4, V Lemma 1]).

(2) Any limit set $\omega(x,f)$ is nonempty, compact and strongly invariant (see [4, IV Lemma 2]), $\omega(x,f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n)$ and $f(\omega(x,f^n)) = \omega(f(x), f^n)$ for any $n \in \mathbb{N}$ (see [4, p.70/71]).

(3) $\mathrm{P}(f) = \mathrm{P}(f^n)$ and $\mathrm{R}(f) = \mathrm{R}(f^n)$ for all $n \in \mathbb{N}$ (see [4, I Lemma 10 and IV Lemma 25]).

(4) f is transitive if and only if for any two open sets U and V in X there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ (see [4, VI Prop. 39]).

Let $\Sigma := \{\alpha = (a_0, a_1, \dots) \mid a_i \in \{0,1\}\}$ be the space of sequences with entries 0 or 1. The map $d_\Sigma \colon \Sigma \times \Sigma \to \mathbb{R}$ defined by $(\alpha,\beta) \mapsto 0$ if $\alpha = \beta$ and $(\alpha,\beta) \mapsto 2^{-j}$ if $\alpha = (a_0, a_1, \dots) \neq (b_0, b_1, \dots) = \beta$ and $a_i = b_i$ for $i = 0, \dots, j-1$ and $a_j \neq b_j$ defines a metric on Σ under which Σ becomes a cantor set [4, p.34]. The *shift* operator $\sigma \colon \Sigma \to \Sigma$ is defined by $(a_0, a_1, \dots) \mapsto (a_1, a_2, \dots)$. This shift operator is continuous and surjective (see e.g. [7, Proposition 6.5]).

Let (Y, d_Y) be another compact metric space and let $g \colon Y \to Y$ be continuous. If there exists a continuous surjection $h \colon X \to Y$ such that $h \circ f = g \circ h$ on X then f is said to be (*topologically*) *semi-conjugate* to g via the (*topological*) *semi-conjugacy* h.

3 Three Definitions of Chaos

We now describe three commonly used definitions of chaos for continuous maps. To begin with we consider the general case of maps from a compact metric space (X,d) into itself and later we concentrate on the special case where X is a real interval.

Definition 3.1 [L/Y-chaos] A continuous map $f \colon X \to X$ on a compact metric space (X,d) is called *chaotic in the sense of Li and Yorke* – or just *L/Y-chaotic* – if there exists an uncountable subset S (called a *scrambled set*) of X with the following properties:

(i) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ for all $x, y \in S,\ x \neq y$,

(ii) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ for all $x, y \in S,\ x \neq y$,

(ii) $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$ for all $x \in S,\ p \in X,\ p$ periodic.

Remarks 3.1

(1) In Li & Yorke's original definition of chaos there is the additional condition that f has periodic points of any period in $\mathbb{N}$ [13, Theorem 1]. In the literature, however, most authors (see e.g. [12, Definition. 1.1]) refer to chaos in the sense of Li & Yorke without this condition.

(2) That condition (iii) in the definition of L/Y-chaos is redundant can be seen as follows. Two approximately periodic points x, y cannot satisfy both (i) and (ii) in the definition of a scrambled set (see [4, VI Lemma 28]). Consequently there exists at most one approximately periodic point in any set satisfying conditions (i) and (ii) of Definition 3.1. Removing this point the new set also satisfies (iii).

Definition 3.2 [B/C-chaos] A continuous map $f\colon X \to X$ on a compact metric space X is called *chaotic in the sense of Block and Coppel* – or just *B/C-chaotic* – if there exists an $m \in \mathbb{N}$ and a compact f^m-invariant subset Y of X such that $f^m|_Y$ is semi-conjugate to the shift on Σ, i.e. if there exists a continuous surjection $h\colon Y \to \Sigma$ satisfying

$$h \circ f^m = \sigma \circ h \quad \text{on } Y.$$

Remarks 3.2

(1) In [4, p.127/128] it has been described that the definition of B/C-chaos is equivalent to the property that there exist an $m \in \mathbb{N}$ and two compact disjoint sets X_0, X_1 in X such that given any $(a_0, a_1, \dots) \in \Sigma$ there is an $x \in X$ such that $f^{mn}(x) \in X_{a_n}$ for all $n \in \mathbb{N}_0$ (see also [10, Theorem 2.2.3]).

(2) If $m = 1$ in Defintion3.2 then this notion of chaos is also known as *chaos in the sense of coin tossing* [11, Definition 1]. It should be noted here that not every B/C-chaotic map is also chaotic in the sense of coin tossing. In order to see this consider the subset $T := \{(0, a_0, 0, a_1, 0, \dots) \mid (a_0, a_1, \dots) \in \Sigma\} \cup \{(a_0, 0, a_1, 0, a_2, \dots) \mid (a_0, a_1, \dots) \in \Sigma\}$ of Σ. This set is σ-invariant and the map $\sigma^2|_T$ is semi-conjugate to $\sigma\colon \Sigma \to \Sigma$. However, there is no σ-invariant subset W of T such that $\sigma|_W$ is semi-conjugate to $\sigma\colon \Sigma \to \Sigma$ (see [10, Example 2.2.5] for more details).

Definition 3.3 [D-chaos] A continuous map $f\colon X \to X$ on a compact metric space X is called *chaotic in the sense of Devaney* – or just *D-chaotic* – if there exists a compact invariant subset Y (called a *D-chaotic set*) of X with the following properties:

(i) $f|_Y$ is transitiv,
(ii) $\overline{\mathrm{P}(f|_Y)} = Y$,
(iii) $f|_Y$ has sensitive dependence on initial conditions.

Remarks 3.3

(1) In [7, Definition 8.5] Devaney originally defined f to be chaotic if $Y = X$ in Definition 3.3. In the literatur, however, D-chaos is usually meant in the more general sense with $Y \subseteq X$ (see e.g. [12, Definition 1.3]).

(2) Condition (iii) in Definition 3.3 turned out to be redundant in the nontrivial case where Y is infinite (see [2]).

The three notions of chaos just described have the nice property of being invariant under conjugation and iteration. The corresponding statements are as follows.

Proposition 3.1 *Let (X, d_X) and (Y, d_Y) be compact metric spaces and suppose that a continuous map $f\colon X \to X$ is conjugate to a continuous map $g\colon Y \to Y$. Then f is D-chaotic (or B/C-chaotic or L/Y-chaotic) if and only if g is D-chaotic (or B/C-chaotic or L/Y-chaotic, respectively).*

Proof See [11, Proposition 1].

Proposition 3.2 *Let (X,d) be compact and suppose $f\colon X \to X$ is continuous. Then for any $n \in \mathbb{N}$ the map f is D-chaotic (or B/C-chaotic or L/Y-chaotic) if and only if f^n is D-chaotic (or B/C-chaotic or L/Y-chaotic, respectively).*

Proof D-chaos: If f is D-chaotic with D-chaotic set $Y \subseteq X$ then there is an $x \in Y$ such that $\omega(x, f|_Y) = Y$. Since $\mathrm{R}(f|_Y) = \mathrm{R}(f^n|_Y)$ we have $x \in W := \omega(x, f^n|_Y)$, so $f^n|_W$ is transitive. Since $\mathrm{P}(f|_Y) = \mathrm{P}(f^n|_Y)$ and $Y = \bigcup_{i=0}^{n-1} f^i(W)$ we get $\overline{\mathrm{P}(f^n|_W)} = \overline{\mathrm{P}(f^n|_Y)} \cap W = Y \cap W = W$ and W is infinite because Y is infinite ($f^n|_Y$ has sensitive dependence on initial conditions). So also $f^n|_W$ sensitively depends on initial conditions. Conversely, let $f^n|_W$ satisfy the conditions in the definition of D-chaos for an f^n-invariant compact set W in X. Then it is easy to see that also $f|_Y$ with $Y := \bigcup_{i=0}^{n-1} f^i(W)$ satisfies these conditions (see also [12, Proposition 4.10]).

B/C-chaos: Let f be B/C-chaotic, $m \in \mathbb{N}$, $Y \subseteq X$ compact and f^m-invariant and let $f^m|_Y$ be semi-conjugate to σ via $h\colon Y \to \Sigma$. Then defining the continuous surjection $t\colon \Sigma \to \Sigma$ by $(a_0, a_1, a_2, \dots) \mapsto (a_0, a_n, a_{2n}, \dots)$ and $\bar{h} := t \circ h$ we get $\bar{h} \circ (f^n)^m = \sigma \circ \bar{h}$ on Y and f^n is B/C-chaotic. The converse immediately follows from the definition of B/C-chaos.

L/Y-chaos: It is easy to see that a set $S \subseteq X$ is a scrambled set with respect to f if and only if it is a scrambled set with respect to f^n (see also [10, Proposition 2.3.8]). This completes the proof of Proposition 3.2.

For the remainder of this section we concentrate on the special case where the compact metric space X is a nontrivial real interval I (i.e. nonempty and not a singleton). In this case B/C-chaos originally (see [4, p.33]) has been defined differently from Definition 3.2. In fact, the original definition of a B/C-chaotic map was based on the notion of *tubulence* which is defined as follows (see [4, p.25]). A map $f\colon I \to I$ is called *turbulent* if there exist compact subintervals J, K of I with at most one common point such that $J \cup K \subseteq f(J) \cap f(K)$. If J and K can be chosen disjoint then f is said to be *strictly turbulent*. The relation between turbulence and B/C-chaos is described in the following result (see [4, p.33/128]).

Proposition 3.3 *A continuous map $f\colon I \to I$ on a nontrivial compact interval I is B/C-chaotic if and only if one of the following equivalent conditions is satisfied:*

(i) *f^m is turbulent for some $m \in \mathbb{N}$,*
(ii) *f^m is strictly turbulent for some $m \in \mathbb{N}$,*
(iii) *f has a periodic point whose period is not a power of 2.*

Three more results are needed in order to reach the goals of this paper.

Proposition 3.4 *f is L/Y-chaotic if and only if not every point in I is approximately periodic.*

Proof See [4, p.145]).

Lemma 3.1 *f is B/C-chaotic if and only if there exists a $c \in I$ such that $\omega(c, f)$ contains a periodic orbit as a proper subset.*

Proof See [4, VI Proposition 6]).

Lemma 3.2 *Let J and K be two compact subintervals of I having the property $K \subseteq f(J)$. Then there exists a compact subinterval L of J such that $f(L) = K$ and that f maps the endpoints of L onto the endpoints of K.*

Proof Let $K = [a,b]$ for two points $a, b \in I$ and let c be the largest point in J with $f(c) = a$. If there exists an $x \in J,\ x > c$, with $f(x) = b$, let d be the smallest x with this property. Then with $L := [c,d]$ the claim follows. On the other hand, if there exists an $x \in J,\ x < c$, with $f(x) = b$ we define $\tilde{c}$ as the largest x with this property. Let $\tilde{d}$ be the smallest $x \in (\tilde{c}, c]\ (\subset J)$ satisfying $f(x) = a$. Then the interval $L := [\tilde{c}, \tilde{d}]$ has the claimed property and the proof of the lemma is complete.

4 The Mutual Relations for Interval Maps

In this section we discuss the mutual relations between the three notions of chaos described in the previous section for the special case of interval maps. In fact, throughout the present section we consider continuous maps $f\colon I \to I$ from a nontrivial compact interval $I = [a,b],\ a < b$, into itself. The main results of this section say that in this case B/C-chaos and D-chaos are equivalent while, on the other hand, B/C-chaos and D-chaos are sufficient for L/Y-chaos. For the lacking necessity in the last statement we refer to the counterexample g_{λ^*} presented in the next section.

Theorem 4.1 *A continuous map $f\colon I \to I$ on an interval I is D-chaotic if and only if it is B/C-chaotic.*

Proof ($\Rightarrow$) Let f be D-chaotic with compact D-chaotic set $Y \subseteq I$. Then Y is infinite since $f|_Y$ has sensitive dependence on initial conditions. Furthermore, since $f|_Y$ is transitive there is a $c \in Y$ with $\omega(c,f) = Y$, and because of the relation $\overline{\mathrm{P}(f|_Y)} = Y$ the map $f|_Y$ has a periodic orbit. As a finite set this periodic orbit is a proper subset of $Y = \omega(c,f)$, and this implies (by Lemma 3.1) that f is B/C-chaotic.

($\Leftarrow$) Now suppose f is B/C-chaotic. Then because of Proposition 3.3 the map f^m is strongly turbulent for some $m \in \mathbb{N}$, i.e. there exist two disjoint compact subintervals X_0 and X_1 of I with the property that for $g := f^m$ we have

$$X_0 \cup X_1 \subseteq g(X_0) \cap g(X_1). \tag{1}$$

The idea of proceeding from here is to first derive from (1) the existence of a compact g-invariant subset X of $X_0 \cup X_1$ with the property that the map $g|_X\colon X \to X$ is semi-conjugate to the shift σ via a continuous surjection $s\colon X \to \Sigma$ and then to show that there exists a compact g-invariant subset Z of X on which g is D-chaotic. We carry out this program in 5 steps.

Step 1. Construction of X and s: Starting with the above X_0 and X_1 and using mathematical induction, for each $\alpha = (a_1, a_2, \dots) \in \Sigma$ Lemma 3.2 yields a sequence of compact, pairwise disjoint intervals $X_{a_1 a_2 \dots a_k},\ (a_1, a_2, \dots, a_k) \in \{0,1\}^k,\ k \geq 1$ in $X_0 \cup X_1$ having the following properties:

$$\begin{gathered} X_{a_1 a_2 \dots a_k} \subseteq X_{a_1 a_2 \dots a_{k-1}}, \quad g(X_{a_1 a_2 \dots a_k}) = X_{a_2 a_3 \dots a_k} \quad \text{and} \\ g \text{ maps endpoints of } X_{a_1 a_2 \dots a_k} \text{ onto endpoints of } X_{a_2 a_3 \dots a_k}. \end{gathered} \tag{2}$$

Then for each $\alpha=(a_1,a_2,\dots)\in\Sigma$ the set

$$X_\alpha := \bigcap_{k=1}^{\infty} X_{a_1\dots a_k} \tag{3}$$

is either a singleton or a nontrivial compact interval. Furthermore we have

$$X_\alpha \cap X_\beta = \emptyset \quad \text{for all} \quad \alpha,\, \beta\in\Sigma, \quad \alpha\neq\beta \tag{4}$$

since the sets $X_{a_1a_2\dots a_k}$, $(a_1,a_2,\dots,a_k)\in\{0,1\}^k$, are pairwise disjoint and

$$g(X_\alpha) = X_{\sigma(\alpha)} \quad \text{for all} \quad \alpha\in\Sigma. \tag{5}$$

Next we define the set

$$\tilde X := \bigcup_{\alpha\in\Sigma} X_\alpha \tag{6}$$

which turns out to be strongly g-invariant and compact. Also the set

$$X := \{x\in I \mid x \ \text{ is an endpoint of } X_\alpha \text{ for some } \alpha\in\Sigma\}$$

is compact (even if $X_\alpha=\{x\}$ for some x we call x an endpoint of X_α). From (2) and (5) we conclude that for any $\alpha\in\Sigma$ the map g maps the endpoints of X_α onto the endpoints of $X_{\sigma(\alpha)}$ and that X is strongly g-invariant. On X we define the map

$$s\colon X\to\Sigma, \quad x\mapsto\alpha \quad \text{if} \quad x\in X_\alpha.$$

Obviously, this map is well defined, continuous and onto and each point of Σ is the s-image of at most two points of X. Finally, because of (5) and the definition of s we have

$$s\circ g|_X = \sigma\circ s \quad \text{on } X. \tag{7}$$

Step 2. Construction of Z: For any $\alpha\in\Sigma$ the set X_α defined in (3) is a nonempty compact interval. Since the X_α's are pairwise disjoint (see (4)) there exist at most countably many α's in Σ such that X_α is not a singleton. Therefore the set

$$R := \{x\in X \mid X_\alpha=\{x\} \text{ for some } \alpha\in\Sigma\}$$

is nonempty and consists of all but countably many points of X. Because of (5) the set R is g-invariant and the set

$$Z := \bar R$$

and the map $g|_Z\colon Z\to Z$ are well defined.

Step 3. Transitivity of $g|_Z$: Let U be an arbitrary open nonempty subset of Z. Then there exists a point $x\in U\cap R$ and some $\alpha=(a_1,a_2,\dots)\in\Sigma$ with $X_\alpha=\{x\}$. Because of definition (3) of X_α and the openness of U in Z there exists a $k\in\mathbb{N}$ with $Z\cap X_{a_1a_2\dots a_k}\subseteq U$. Therefore, in order to prove the transitivity of $g|_Z$ it suffices to prove the relation

$$g^k(Z\cap X_{a_1a_2\dots a_k}) = Z. \tag{8}$$

Since $Z = \bar{R}$ and since the set $Z \cap X_{a_1 a_2 \dots a_k}$ is compact, it even suffices to find a g^k-preimage of an arbitrary point $y \in R$ in the set $Z \cap X_{a_1 a_2 \dots a_k}$. Due to the definition of R, for any $y \in R$ there exists a $\beta = (b_1, b_2, \dots) \in \Sigma$ with $\{y\} = X_\beta$. With the aid of this β we define

$$\gamma := (a_1, a_2, \dots, a_k, b_1, b_2, \dots) \in \Sigma$$

and use (5) to get the relation

$$g^k(X_\gamma) = X_\beta = \{y\}. \tag{9}$$

If X_γ consists of a single point we get the inclusion $X_\gamma \subseteq Z$ and the claim (8) is proved, since X_γ is a subset of $X_{a_1 a_2 \dots a_k}$. If, on the other hand, X_γ is a nontrivial interval then at least one of its endpoints is contained in Z. This can be shown as follows: For any $n \in \mathbb{N}$ there exists (because of (3)) a number $m_n \in \mathbb{N}$ with

$$X_{a_1 a_2 \dots a_k b_1 b_2 \dots b_{m_n}} \subseteq \{x \in I \mid \operatorname{dist}(x, X_\gamma) < \tfrac{1}{n}\}, \tag{10}$$

and since the set

$$\{(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_{m_n}, *, *, \dots) \in \Sigma \mid * \in \{0, 1\}\}$$

is uncountable we can find a point γ_n in this set such that $X_{\gamma_n} = \{y_n\}$. By (4) the sets X_γ and X_{γ_n} are disjoint and by (10) the distance of the point y_n from at least one of the endpoints of X_γ is less than $\frac{1}{n}$ (since $y_n \in X_{a_1 a_2 \dots a_k b_1 b_2 \dots b_{m_n}}$). Because the relation $X_{\gamma_n} \subset R$ holds for all $n \in \mathbb{N}$, the sequence $(y_n)_{n \in \mathbb{N}}$ in R converges, w.l.o.g., to one of the endpoints of X_γ. On the other hand, because of $Z = \bar{R}$ this endpoint is contained in Z and it is mapped via g^k to y according to (9). In both cases we thus can find a g^k-preimage of the point y in $Z \cap X_{a_1 a_2 \dots a_k}$, and this proves claim (8).

Step 4. $\overline{\mathrm{P}(g|_Z)} = Z$: Let U again be an arbitrary open nonempty set in Z and x a point in $U \cap R$ with $\{x\} = X_\alpha$ for some $\alpha = (a_1, a_2, \dots) \in \Sigma$. As in the proof of Step 3, given any $n \in \mathbb{N}$ there is an $m_n \in \mathbb{N}$ with

$$X_{a_1 a_2 \dots a_{m_n}} \subseteq \{x \in I \mid \operatorname{dist}(x, X_\alpha) < \tfrac{1}{n}\}. \tag{11}$$

We now consider the periodic point

$$\gamma_n := (\overline{a_1, a_2, \dots, a_{m_n}}, \dots) \in \Sigma$$

and notice that because of $\sigma^{m_n}(\gamma_n) = \gamma_n$ and (5) we get $g^{m_n}(X_{\gamma_n}) = X_{\gamma_n}$. Furthermore, the two endpoints of X_{γ_n} are periodic with respect to g, since g^{m_n} maps the endpoints of X_{γ_n} onto the endpoints of $g^{m_n}(X_{\gamma_n})$ $(= X_{\gamma_n})$. In case X_{γ_n} is a nontrivial interval then at least one of its (periodic) endpoints is contained in Z. This can be seen as in the previous Step 3. So in any case, for any $n \in \mathbb{N}$ we get a g-periodic point $x_n \in X_{\gamma_n} \cap Z$ and the sequence $(x_n)_{n \in \mathbb{N}}$ converges to x because of $X_{\gamma_n} \subseteq X_{a_1 \dots a_{m_n}}$ and (11). This implies the relation $x \in \overline{\mathrm{P}(g|_Z)}$ and completes the proof of Step 4.

Step 5. Conclusion: The set Z is infinite (because R is infinite), and therefore the map g is D-chaotic on Z. By Proposition 3.2 then f is D-chaotic on $Y := \bigcup_{i=0}^{n-1} f^i(Z)$. This completes the proof of Theorem 4.1.

Remark 4.1 Theorem 4.1 can also be proved by using the notion of *positive entropy* (see [4, VIII] and [12]). The argument is as follows: The map f is D-chaotic if and only if it has positive entropy (see [12]) and positive entropy in turn is equivalent to B/C-chaos (see e.g. [4, VII Theorem 24]).

Theorem 4.2 *If a continuous map* $f: I \to I$ *on an interval* I *is B/C-chaotic then it is also L/Y-chaotic.*

Proof See [4, VI Proposition 27]).

As mentioned above the implication of Theorem 4.2 cannot be reversed. In fact, in the next section we present an example of a map which is L/Y-chaotic but not B/C-chaotic. Before doing this, however, we want to show that all maps which are L/Y- but not B/C-chaotic have an interesting property in common. In fact, we show that for any map of this kind there exists an infinite compact invariant set such that the restriction of the map to this set is transitive but does not have periodic points. This shows that the result "On intervals transitivity = chaos" [3] (earlier proved in [4, VI Lemma 41] and stating that $\overline{\mathrm{P}(f)} = I$ if f is transitive) cannot be generalized from intervals to disconnected compact subsets of $\mathbb{R}$.

Proposition 4.1 *If a continuous map* $f: I \to I$ *on a compact interval* I *is L/Y-chaotic but not B/C-chaotic then there exists a compact infinite invariant subset* Y *of* I *such that* $f|_Y: Y \to Y$ *is transitive but does not have periodic points.*

Proof Since f is L/Y-chaotic there exists (by Proposition 3.4) a point $x \in I$ which is not approximately periodic. So the limit set $\omega(x, f)$ of x is infinite by [4, IV Lemma 4]. According to [4, VI Proposition 7] there exists a unique minimal set Y in $\omega(x, f)$ such that

$$Y = \omega(y, f) \quad \text{for some} \quad y \in Y. \tag{12}$$

Again using [4, IV Lemma 4] the set Y (as a subset of $\omega(x, f)$) is infinite. This is because f is not B/C-chaotic and therefore the infinite set $\omega(x, f)$ contains no periodic points by Proposition 3.4. So finally the well defined map $f|_Y: Y \to Y$ has no periodic points but is transitive by (12).

5 The Family of Truncated Tent Maps

In order to analyse the so-called *truncated tent maps* we need a famous theorem due to Šarkovskii and two lemmas. In any case we consider a continuous map $f: I \to I$ of a compact interval into itself.

Theorem 5.1 [Šarkovskii] *Let* $\mathbb{N}$ *be totally ordered in the following way:*

$$3 \prec 5 \prec 7 \prec \ldots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec 7 \cdot 2 \prec \ldots \prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec \ldots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Then if f *has a periodic orbit of period* $n \in \mathbb{N}$ *and if* $m \in \mathbb{N}$ *with* $n \prec m$*, then* f *also has a periodic orbit of period* m*.*

Proof See [15], also [4, I Theorem 1].

Lemma 5.1 *Suppose* $f: I \to I$ *is not B/C-chaotic. Then for any* $x \in I$ *with infinite* $\omega(x, f)$ *and any* $s \in \mathbb{N}$ *the intervals*

$$J_i^s := [\min \omega(f^i(x), f^{2^s}),\ \max \omega(f^i(x), f^{2^s})], \quad i = 0, 1, \ldots, 2^s - 1$$

have the following properties:

(i) $J_i^s \cap J_k^s = \emptyset$ *for all* $i, k \in \{0, 1, \ldots, 2^s - 1\}$ *with* $i \neq k$,
(ii) J_i^s *contains a* 2^s*-periodic point for* $i = 0, 1, \ldots, 2^s - 1$.

Proof See [4, VI Lemma 14].

Lemma 5.2 *If f is L/Y-chaotic then f has infinitely many periodic points.*

Proof We distinguish the two cases of f being B/C-chaotic or not.

If f is B/C-chaotic then by Proposition 3.3 f has an n-periodic point with n not being a power of 2. By Šarkovskii's Theorem then f has periodic points with periods 2^n for all $n \in \mathbb{N}$. So f has infinitely many periodic points.

In case f is not B/C-chaotic then there exists a point $x \in I$ which is not approximately periodic by Proposition 3.4. Therefore the limit set $\omega(x, f)$ of x is infinite (see [4, IV Lemma 4]). Now take any $s \in \mathbb{N}$ and define

$$J_i^s := [\min \omega(f^i(x), f^{2^s}),\ \max \omega(f^i(x), f^{2^s})] \quad \text{for} \quad i = 0, 1, \dots, 2^s - 1.$$

Then the intervals J_i^s are pairwise disjoint and each of them contains a 2^s-periodic point by Lemma 5.1. Therefore f has at least 2^s distinct periodic points and hence infinitely many since $s \in \mathbb{N}$ was arbitrary. This completes the proof of Lemma 5.2.

Now we are prepared to investigate the announced family of maps one member of which shows that the statement of Theorem 4.2 cannot be reversed.

Example 5.1 The piecewise linear map $g\colon [0,1] \to [0,1]$ with

$$g(0) = 0, \quad g\left(\tfrac{1}{2}\right) = 1, \quad g(1) = 0$$

is known as the (*standard*) *tent map*. Its graph is a "tent" with peak of height 1 at the point $\frac{1}{2}$. In order to modify this map to get a family of maps suitable for our purposes we cut the peak at any height $\lambda \in [0,1]$ and consider the family of *truncated tent maps* defined by

$$g_\lambda\colon [0,1] \to [0,1], \quad x \mapsto \min\{\lambda, g(x)\}, \quad \lambda \in [0,1].$$

It is apparent that for any $0 \le \lambda < \gamma \le 1$ the maps g_λ and g_γ coincide on the set

$$J_\lambda := \left[0, \tfrac{\lambda}{2}\right] \cup \left[1 - \tfrac{\lambda}{2}, 1\right], \quad \lambda \in [0,1]$$

and that (periodic) orbits of g_γ in J_λ are also (periodic) orbits of g_λ and vice versa. Furthermore, since g_λ is constant on the open interval

$$K_\lambda := \left(\tfrac{\lambda}{2}, 1 - \tfrac{\lambda}{2}\right), \quad \lambda \in [0,1],$$

the map g_λ has at most one periodic point in $\bar{K}_\lambda$.

For the original tent map g $(= g_1)$ the set $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ is obviously a 3-periodic orbit and therefore, by Šarkovskii's Theorem, it has 2^n-periodic points for all $n \in \mathbb{N}$. Furthermore, it is easy to see that

$$|\{x \in [0,1] \mid x \text{ is } m\text{-periodic with respect to } g\}| \le 2^m \text{ for all } m \in \mathbb{N}. \tag{13}$$

Therefore the number

$$\lambda_n := \min\{\lambda \in [0,1] \mid g \text{ has a } 2^n\text{-periodic orbit in } [0,\lambda]\}$$

is well defined and λ_n is a 2^n-periodic point of g for any $n \in \mathbb{N}$. Because of the relation $g(K_{\lambda_n}) = (\lambda_n, 1]$ we have $\mathrm{O}(\lambda_n, g) \subseteq J_{\lambda_n}$, and therefore λ_n is also periodic with respect to g_{λ_n} having the same periodic orbit as for g. By Šarkovskii's Theorem we have the identity

$$\{2^i \mid i = 0, 1, \dots, n\} = \{k \in \mathbb{N} \mid x \text{ is } k\text{-periodic w.r. to } g_{\lambda_n} \text{ for some } x \in [0,1]\} \quad (14)$$

because otherwise there were an m-periodic orbit M of g_{λ_n} for some $m \in \mathbb{N}$ with $m \prec 2^n$. Since g_{λ_n} has at most one periodic point in $\bar{K}_{\lambda_n}$ (the point λ_n) the inclusion $M \subseteq J_{\lambda_n}$ holds and with $\rho := \max M < \lambda_n$ the map g_ρ and hence also g_{λ_n} has a 2^n-periodic orbit in $[0, \rho] \cap J_\rho$. This contradicts the minimality of λ_n.

The sequence $(\lambda_n)_{n\in\mathbb{N}}$ is strongly increasing because otherwise there would exist numbers $n, m \in \mathbb{N}$, $m > n$ with $\lambda_m \le \lambda_n$ such that the map g_{λ_m} has a 2^n-periodic orbit in $[0, \lambda_m)$ and this would again contradict the minimality of λ_n. On the other hand, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is bounded above by 1 and therefore it has a limit

$$\lambda^* := \lim_{n\to\infty} \lambda_n$$

which is smaller then $\frac{6}{7}$ since the map $g_{\frac{6}{7}}$ has periodic points of any period $n \in \mathbb{N}$ (by Šarkovskii's Theorem). In addition, λ^* is greater than $\frac{4}{5}$, since $\lambda_2 = \frac{4}{5}$. Indeed, in [14, Remark 4] it has been mentioned that $\lambda^* = 0.8249080\dots$

We now determine for each member of the family of truncated tent maps which kind of chaos prevails.

For $0 \le \lambda < \lambda^*$ the map g_λ is not L/Y-chaotic:

For any $\lambda \in [0, \lambda^*)$ there exists an $n \in \mathbb{N}$ with $\lambda < \lambda_n$. Therefore, any periodic orbit of g_λ in J_λ is also a periodic orbit of g_{λ_n}. On the other hand, the map g_{λ_n} has finitely many periodic points because of (13) and (14), and therefore also g_λ has only finitely many periodic points, since at most one periodic orbit of g_λ has nonempty intersection with K_λ. So g_λ is not L/Y-chaotic by Lemma 5.2.

The map g_{λ^*} is L/Y-chaotic but not B/C-chaotic:

In [4, VI Example 29] it has been shown that not all points in $[0,1]$ are approximately periodic with respect to g_{λ^*}, and therefore g_{λ^*} is L/Y-chaotic by Proposition 3.4. On the other hand, assuming to the contrary that g_{λ^*} is B/C-chaotic, by Proposition 3.3 there exists an odd number $q > 1$ such that g_{λ^*} has a $q\,2^k$-periodic orbit P for some $k \ge 0$. In case $p := \max P < \lambda^*$ there is an $n \in \mathbb{N}$ with $\lambda_n > p$ such that P is a periodic orbit of g_{λ_n}. This contradicts (14). If, on the other hand, $p = \lambda^*$, by Šarkovskii's Theorem the map g_{λ^*} has a $(q+2)\,2^k$-periodic orbit Q. Because of $\max Q < \lambda^*$ this again leads to a contradiction.

For $\lambda^* < \lambda \le 1$ the map g_λ is B/C-chaotic:

The original tent map g $(= g_1)$ is D-chaotic on $[0,1]$ (see e.g. [7, III Example 9]) and therefore g has a periodic point $\rho \in [\lambda^*, \lambda]$.

We first prove now that the map g_ρ is B/C-chaotic. To this end we notice that ρ is a periodic point of g_ρ. This is due to the fact that either $\mathrm{O}(\rho, g) \cap K_\rho = \emptyset$, and thus $\mathrm{O}(\rho, g_\rho) = \mathrm{O}(\rho, g)$, or $g^j(\rho) \in K_\rho$ for some minimal $j \in \mathbb{N}$, and therefore $g_\rho^{j+1}(\rho) = \rho$. The case where ρ is m-periodic with respect to g_ρ for some $m \in \mathbb{N}$, $m \notin \{2^n \mid n \in \mathbb{N}_0\}$ is easily settled because in this case g_ρ is B/C-chaotic by Proposition 3.3. So from now on we may assume that ρ is a 2^n-periodic point of g_ρ for some $n \in \mathbb{N}_0$. We now define the intervals

$$K := [g_\rho^{2^n}(\lambda_{n+1}), \lambda_{n+1}] \quad \text{and} \quad J := [\lambda_{n+1}, \rho]$$

and prove the existence of an $N \in \mathbb{N}$ with the property

$$K \cup J \subseteq g_\rho^N(K) \cap g_\rho^N(J) \tag{15}$$

which in turn implies, by Proposition 3.3, that g_ρ is B/C-chaotic. To this end we first notice that the intervals K and J are well defined since the relations $\rho > \lambda^*$ and $\mathrm{O}(\lambda_{n+1}, g_\rho) = \mathrm{O}(\lambda_{n+1}, g)$ are valid. Since ρ is 2^n-periodic we get

$$K \cup J \subseteq g_\rho^{2^n}(J). \tag{16}$$

In order to prove that for some $r \in \mathbb{N}$ we have

$$K \cup J \subseteq g_\rho^r(K) \tag{17}$$

we state the following property which is valid for all $\tau \in [0,1]$ and $j \in \mathbb{N}$:

$$\begin{gathered} |g_\tau^j(x) - g_\tau^j(y)| = 2^j|x-y| \quad \text{for all} \quad x,\, y \in [0,1] \quad \text{such that} \\ g_\tau^i(x),\ g_\tau^i(y) \geq 1 - \tfrac{\tau}{2} \quad \text{or} \quad g_\tau^i(x),\ g_\tau^i(y) \leq \tfrac{\tau}{2} \quad \text{for} \quad i = 0, 1, \ldots, j. \end{gathered} \tag{18}$$

Using (18) and the fact that both λ_{n+1} and $g_\rho^{2^n}(\lambda_{n+1})$ are 2^{n+1}-periodic with respect to g_ρ, we see that there is some $j \in \{0, 1, \ldots, 2^n - 1\}$ with the following property:

$$\begin{aligned} & g_\rho^j(\lambda_{n+1}) < \tfrac{\rho}{2} && \text{and} \quad g_\rho^{2^n+j}(\lambda_{n+1}) > 1 - \tfrac{\rho}{2} \\ \text{or} \quad & g_\rho^j(\lambda_{n+1}) > 1 - \tfrac{\rho}{2} && \text{and} \quad g_\rho^{2^n+j}(\lambda_{n+1}) < \tfrac{\rho}{2}. \end{aligned}$$

So we get $\rho \in g_\rho^{j+1}(K)$ and therefore $[\lambda_{n+1}, \rho] \subseteq g_\rho^{j+1}(K)$. Thus with $r := 2^n + j + 1$ condition (17) is satisfied. Using (16), (17) and the definition $N := r + 2^n$ the claim (15) follows and g_ρ^N is turbulent. By Proposition 3.3 then g_ρ is B/C-chaotic.

Finally, since g_ρ is now known to be B/C-chaotic, there is a p-periodic orbit P of g_ρ in J_ρ for some $p \in \mathbb{N}$, $p \notin \{2^n \mid n \in \mathbb{N}_0\}$ (compare with the above proof that g_{λ^*} is not B/C-chaotic). Because of $\lambda > \rho$ the orbit P is also p-periodic with respect to g_λ and therefore g_λ is B/C-chaotic by Proposition 3.3.

Combining the previous considerations with the results of Section 4 we get the following summary for the family of truncated tent maps:

- For each $\lambda \in [0, \lambda^*)$ the map g_λ is not chaotic in any of the three senses considerd in this paper.
- The particular map g_{λ^*} is chaotic in the sense of Li & Yorke but neither in the sense of Block & Coppel nor Devaney.
- For each $\lambda \in (\lambda^*, 1]$ the map g_λ is chaotic in any of the three senses considered in this paper.

We conclude this section with a few additional remarks on the family of truncated tent maps.

Remarks 5.1

(1) The map g_{λ^*} first appeared in [14, Remark 4] as an example of a map of *typ* 2^∞ (i.e. the set of periods of its periodic points is $\{2^i \mid i \in \mathbb{N}_0\}$) having a scrambled set. Block and Coppel [4, VI Example 29] have proved that g_{λ^*} is L/Y-chaotic by showing that $\frac{\lambda^*}{2}, 1-\frac{\lambda^*}{2} \in \overline{\mathrm{P}(g_{\lambda^*})}$, but $\left[\frac{\lambda^*}{2}, 1-\frac{\lambda^*}{2}\right] \cap \mathrm{P}(g_{\lambda^*}) = \emptyset$. To this end additional results [4, VI Lemma 17 and Theorem 24] have been used.

(2) The fact that g_λ is B/C-chaotic for all $\lambda \in (\lambda^*, 1]$ can also be proved by using *kneading theory* for *unimodal* maps (see [6]). In this context one considers the restriction $g_\lambda|_{[0,\lambda]}$ of g_λ on $[0,\lambda]$ and defines the intervals $L := \left[0, \frac{\lambda}{2}\right)$, $C := \left[\frac{\lambda}{2}, 1-\frac{\lambda}{2}\right]$ and $R := \left(1-\frac{\lambda}{2}, \lambda\right]$, where $\{L, C, R\}^{\mathbb{N}}$ is the symbol space of the *itineraries* of g_λ. Then the corresponding results from [6] for unimodal maps are also valid for $g_\lambda|_{[0,\lambda]}$.

Using *kneading theory* one can also prove that g_{λ^*} is L/Y-chaotic. To this end one shows that λ^* is not finally periodic with respect to g_{λ^*} and hence not approximately periodic (see (18)). In addition one can see that the set of periodic points of g_{λ^*} is $\bigcup\limits_{n\in\mathbb{N}_0} \mathrm{O}(\lambda_n, g) \cup \{\frac{2}{3}\}$.

(3) Example 5.1 suggests that the set of B/C-chaotic maps on I is open in the set $\mathrm{C}_0(I,I)$ of continuous self maps of I (in the topology defined by the supremum norm). That this is indeed true can be seen in [4, Corrolary 20].

Example 5.1 might also suggest that the set of L/Y-chaotic maps on I is closed in $\mathrm{C}_0(I,I)$. This, however, is not true. In [5] the family $T_\lambda\colon [0,1] \to [0,1]$, $0 \le \lambda \le 2$, of tent maps is defined by $T_\lambda(x) := \lambda x$ for $x \in \left[0, \frac{1}{2}\right]$ and $T_\lambda(x) := \lambda(1-x)$ for $x \in \left[\frac{1}{2}, 1\right]$. It is proved then that for any $\lambda > 1$ the map T_λ is B/C- and therefore L/Y-chaotic. On the other hand, for $\lambda = 1$ all points of $[0,1]$ are obviously mapped on fixpoints in $\left[0, \frac{1}{2}\right]$ and this means that T_1 is not L/Y-chaotic.

6 An Example in a General Compact Metric Space

We finally show by means of an example that in the context of general compact metric spaces we canot expect the close relations between the three definitions of chaos as they appear in the particular case of interval maps. In fact, our example shows that even L/Y-chaos together with B/C-chaos does not imply D-chaos.

Example 6.1 The *adding machine* $\tau\colon \Sigma \to \Sigma$ is defined as follows: To any point $\alpha = (a_0, a_1, a_2, \dots) \in \Sigma$ it "adds" the particular point $(1,0,0,0,\dots)$ according to the following rule: If $\alpha = (1,1,1,\dots)$ define $\tau(\alpha) := (0,0,0,\dots)$, otherwise let all entries of α unchanged except the first a_n which vanishes; change this a_n to 1. It is well known that τ is a homeomorphism without periodic points (see e.g. [4, p.133/134]). So if we define the continuous map

$$f\colon \Sigma \times \Sigma \to \Sigma \times \Sigma, \quad (\alpha, \beta) \mapsto (\sigma(\alpha), \tau(\beta))$$

we see that f is semi-conjugate to σ via the projection $h\colon \Sigma \times \Sigma \to \Sigma$ from $\Sigma \times \Sigma$ to its first component. Therefore f is B/C-chaotic. Furthermore, if $S \subset \Sigma$ is a scrambled set for σ then for any $\alpha \in \Sigma$ the set $S_\alpha := \{(s, \alpha) \mid s \in S\}$ is obviously a scrambled set for f. Hence f is L/Y-chaotic. On the other hand, since τ has no periodic points f has none either and therefore f is not D-chaotic.

Remarks 6.1

(1) The previous example shows that condition (ii) of the Definition 3.3 of D-chaos (or more general, the existence of periodic points) may by too restrictive. Indeed, for a B/C-chaotic map $f\colon X \to X$ (X any compact metric space) there exists a compact invariant set $Y \subseteq X$ such that $f|_Y$ is B/C-chaotic, transitive and has sensitive dependence on initial conditions (use [1, Theorem. 3]).

(2) For maps defined on non-compact metric spaces in [10] we have given examples of maps which are D- but neither L/Y- nor B/C-chaotic (see [10, Theorem 3.3.3]) or D- and L/Y-chaotic but not B/C-chaotic (see [10, Theorem 3.3.5]).

We want to conclude this paper with raising the question about the relations between the three definitions of chaos considered above in the general case of mappings on arbitrary compact metric spaces. It is widely believed that B/C-chaos implies L/Y-chaos but we do not know if this is really true. And what is known about the other relations?

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We would like to thank Louis Block for pointing out that the truncated tent map g_λ should not be chaotic for $\lambda < \lambda^*$ but chaotic in every sense for $\lambda > \lambda^*$. The idea of using kneading theory for the proof of L/Y-chaos for g_{λ^*} and B/C-chaos for $g_\lambda,\ \lambda > \lambda^*$ is due to Michał Misiurewicz whome we also thank.

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