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Resonance in the Motion of a Geo-Centric Satellite Due to the Poynting-Robertson Drag and Oblateness of the Earth

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Abstract: In this paper, we have investigated resonances in a geo-centric satellite under the gravitational effect of the Sun, the Earth, oblateness of the Earth and the Poynting-Robertson (P-R) drag. It is found that resonances occur due to the commensurability between satellite's mean motion and average angular velocity of the Earth around the Sun, and also between the satellite's mean motion and average angular velocity of the regression angle. Amplitudes and time periods of the oscillation at the resonance points have been determined. Effects of oblateness and P-R drag on the amplitudes and time periods of oscillation at different resonance points have been analyzed graphically. We have also compared the values of the amplitude and time period of oscillations due to the oblateness parameter and P-R drag. We have observed that amplitude as well as the time period decreases as ϕ (an orbital angle of the Earth around the Sun) increases between -90^0 to 90^0, and the effect of the P-R drag parameter is minor on the amplitudes and time periods. Also, the amplitude and time-period decrease as ψ increases between -90^0 to 90^0.

Keywords: *three-body problem; ecliptic plane; orbital plane; resonance; Poynting-Robertson drag; oblateness.*

Mathematics Subject Classification (2010): 37N05.

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1 Introduction

One of the most important phenomena in the solar system is the occurrence of resonance which plays a significant role in the study of dynamical system. Resonance occurs when any two or more frequencies are commensurable in their ratio. The resonance in the orbital motion of the celestial bodies occurs not only due to the gravitational forces but also the non gravitational forces, for e.g., radiation pressures, oblateness, P-R drag, equatorial ellipticity of the Earth etc.

[3] discussed the motion of a geosynchronous satellite by taking the combined gravitational forces of the Sun (with radiation pressure), the Moon and the Earth. They showed for the geosynchronous satellite that angular velocity of the orbital plane lies between 0.042^0 to 0.58^0 degree per year.

[4] discussed numerically the effects of P-R drag on the equilibrium points of the photo-gravitational CR3BP including the P-R effect by taking the radiation of two massive bodies. They have used the modified bisection method to compute the position of the equilibrium points.

[7] studied the minimum fuel maneuvers to change the position of a spacecraft in orbit around the Earth. Bi-impulsive maneuver control is applied in the initial position of the satellite to send it to a transfer orbit that will cross the desired final position of the spacecraft where both the initial and the final position of satellite belong to the same Keplerian orbit.

[8] investigated the numerical search of bounded relative motion between two or more satellites. They studied the possibility of using global optimization technique to locate the initial conditions resulting into minimum drift per orbit as the perturbations such as the Earth oblateness and air drag effects are taken into account, an analytic solution appears to be more complicated.

Other pioneers in this field are [12], [6], [9], [11], [13], [5], [10], [14], [15].

The P-R effect in the three-body problem on numerical experiments in dynamical consequences has been discussed by many authors by taking only two of the three: (1) the P-R drag; (2) the three-body problem; (3) resonance. Taking all the three factors in this paper collectively, we have attempted to bridge the said gap. The motive of this paper is to investigate the resonance in the motion of geo-centric satellite due to the Poynting-Robertson drag and oblateness of the Earth in the framework of the three-body problem. Meticulous study of equations of motion in Section 2 of this paper reveals that if the regression angle is constant, there are five critical points $R_i's$, $i = 1-5$, at which resonance occurs in the motion of the orbiting satellite, between the mean motion of the satellite and the average angular velocity of the Earth around the Sun and if the regression angel is not constant, resonance occurs at six points ${R_j}''s$, $j = 1-6$, with two frequencies due to the oblateness of the Earth and at many points with three frequencies. Evaluation of the corresponding amplitude and time period at resonance points have been evaluated in Section 3. Discussion and conclusion are given in Section 4. In this section we have compared the amplitudes and time periods at same resonant point and for different values of q's, and also discussed the variation in the amplitudes and time periods for variation in q and ϕ at the resonant point $1:1$ and $1:2$ with the P-R drag and without the P-R drag. Further we have drawn graphs showing amplitudes and time periods due to oblateness of the Earth (J_2) at different resonant points.

2 Statement of the Problem and Equations of Motion

Let S represent the Sun, E be the Earth and $\bar{S}$ be the satellite with the masses M_S, M_E and M_P, respectively. The satellite moves around the Earth in orbital plane. Let the satellite be revolving about the Earth with the angular velocity $\vec{\omega}$ and the system be also revolving with the same angular velocity $\vec{\omega}$. Let $\vec{r}_E$, $\vec{r}_s$ and $\vec{r}$ represent the vectors from the Sun and the Earth, the Sun and the satellite and the Earth and the satellite, respectively; γ be the vernal equinox, α be the angle between the ecliptic plane and orbital plane, θ be the angle between the direction of ascending node and the direction of the satellite, ϕ be the angle between the direction of ascending node and the direction of the Sun, ψ be the regression angle, ϵ be the angle between the equatorial plane and ecliptic plane (obliquity) and c be the velocity of light. For convenience, let x, y, z be the co-ordinate system of the satellite with the origin at the center of the Earth with the unit vectors $\hat{I}$, $\hat{J}$ and $\hat{K}$ along the co-ordinates axes, respectively. Let x_0, y_0 and z_0 be another set of the co-ordinate system in the same plane, with the origin at the center of the Earth, with the unit vectors $\hat{I}_0$, $\hat{J}_0$ and $\hat{K}_0$ along the co-ordinate axes. Let X_G, Y_G and Z_G be the geo-centric reference system with the unit vectors $\hat{I_G}$, $\hat{J_G}$ and $\hat{K_G}$, respectively, along the co-ordinate axes, while the $X_G Y_G$ plane be the Earth's equatorial plane, which makes an angle $23^0 27'$ with the ecliptic plane (Figure 1).

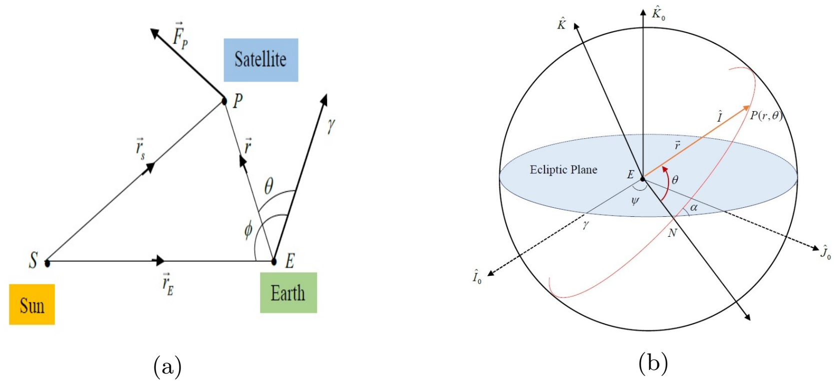

Figure 1: Configuration of the three-body problem; (a) in vector form; (b) with co-ordinate axis.

2.1 Equations of motion in polar form

Let $\vec{F}_P$ be the Poynting-Robertson drag per unit mass acting on the satellite due to the radiating body (the Sun) as shown in Figure 1, given by [4]

$$M_P \vec{F}_P = \vec{f}_1 + \vec{f}_2 + \vec{f}_3,$$

where

$\vec{f}_1 = F \dfrac{\vec{r}_s}{r_s}$ (the radiation pressure),

$\vec{f}_2 = -F \dfrac{(\vec{v} \cdot \vec{r}_s)}{c} \dfrac{\vec{r}_s}{r_s}$ (the Doppler shift owing to the motion),

$\vec{f}_3 = -F\dfrac{\vec{v}}{c}$ (the force due to the absorption and re-emission of part of the incident radiation),

$\vec{v} =$ the velocity of $\bar{S}$,

$c =$ the velocity of light,

$F =$ the measure of the radiation pressure.

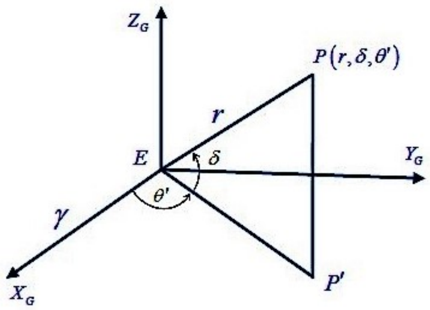

Figure 2: The coordinate system of the satellite in (X_G, Y_G, Z_G) system.

The relative motion of the satellite with respect to the Earth is obtained by

$$\ddot{\vec{r}} = \ddot{\vec{r}}_s - \ddot{\vec{r}}_E = \frac{\vec{F}_{SP} + \vec{F}_{EP} + \vec{F}_P\vec{M}_P}{M_P} - \frac{\vec{F}_{SE}}{M_E},$$

where

$$\vec{F}_{SP} = -G\frac{M_s M_p}{{r_s}^3}\vec{r}_s, \qquad \vec{F}_{SE} = -G\frac{M_s M_E}{{r_E}^3}\vec{r}_E.$$

Force of the Earth on the satellite: We take the potential of the Earth [2] at the point outside it in the form

$$U = \frac{GM_E M_P}{r}\left\{1 - \frac{J_2(R_\oplus)^2}{2r^2}\left(3\frac{(Z_G)^2}{r^2} - 1\right)\right\} + \dots,$$

$$\vec{F}_{EP} = \frac{\partial U}{\partial r}\vec{r} + \frac{\partial U}{\partial X_G}I_G + \frac{\partial U}{\partial Y_G}J_G + \frac{\partial U}{\partial Z_G}K_G$$

$$= -\frac{GM_E}{r^3}\left(\frac{3J_2(R_\oplus)^2}{2r^2}(5\frac{(Z_G)^2}{r^2} - 1) - 1\right)\vec{r} - \frac{3J_2(R_\oplus)^2}{r^2}Z_G\hat{K}_G,$$

$G =$ the gravitational constant,

$\theta' = \angle\gamma EP' =$ the angle between the projection of the line,

EP in the plane of the equator (EP') and the vernal (Figure 2) equinox,

$J_2 =$ the coefficient due to the oblateness of the Earth,

$R_\oplus =$ the mean radius of the Earth.

Thus,

$$\ddot{\vec{r}} = -qF_g\frac{\vec{r}_s}{r_s} - \frac{GM_E}{r^3}\left(\frac{3J_2(R_\oplus)^2}{2r^2}(5\frac{(Z_G)^2}{r^2} - 1) - 1\right)\vec{r}$$

$$-\frac{3J_2(R_\oplus)^2}{r^2}Z_G\hat{K}_G+\frac{GM_s}{(r_E)^3}\vec{r}_E-pF_g\left(\frac{(\vec{v}\cdot\vec{r}_s)\vec{r}_s}{cr_s}+\frac{\vec{v}}{c}\right),$$

where $q=1-F_p/F_g$ exhibits the relation between the gravitational force and the radiation pressure resulting from the Sun. Evidently, $0<q<1$ and $p=1-q$.

The motion of the Earth relative to the Sun is given by

$$\dot{\phi}^2=\frac{GM_s}{r_E^3},$$

also,

$$\vec{r}=r\hat{I},\ \vec{r}_E=r_E\hat{r}_E,\ \hat{r}_E=\cos\phi\hat{I}_\circ+\sin\phi\hat{J}_\circ,\ \vec{r}_E=r_E\cos\phi\hat{I}_\circ+r_E\sin\phi\hat{J}_\circ.$$

Using these values in the equation of motion of the satellite with respect to the Earth in vector form yields

$$\ddot{\vec{r}}=-qGM_s\frac{\vec{r}_s}{r_s^3}-\frac{GM_E}{r^3}\left\{\left(-1+\frac{3J_2(R_\oplus)^2}{2r^2}(5\frac{(Z_G)^2}{r^2}-1)\right)\vec{r}\right\}$$
$$-\frac{3J_2(R_\oplus)^2}{r^2}Z_G\hat{K}_G+\dot{\phi}^2r_E(\cos\phi\hat{I}_\circ+\sin\phi\hat{J}_\circ)-pF_g\left\{\frac{(\vec{v}\cdot\vec{r}_s)\,\vec{r}_s}{cr_s}+\frac{\vec{v}}{c}\right\}. \quad (1)$$

In the rotating frame of reference with angular velocity $\vec{\omega}$ of the satellite about the center of the Earth, we have

$$\ddot{\vec{r}}=\frac{\partial^2 r}{\partial t^2}\hat{I}+2\frac{\partial r}{\partial t}\left(\vec{\omega}\times\hat{I}\right)+r\left(\frac{\partial\vec{\omega}}{\partial t}\times\hat{I}\right)+r\left\{\left(\vec{\omega}\cdot\hat{I}\right)\vec{\omega}-(\vec{\omega}\cdot\vec{\omega})\,\hat{I}\right\}, \quad (2)$$

where $\vec{\omega}=\dot{\theta}\hat{K}+\dot{\psi}\hat{K}_0$. Taking dot products of equations (1) and (2) with $\hat{I}$ and $\hat{J}$ and equating the respective coefficients, we get the equations of motion of the satellite in the synodic coordinate system ([3])

$$\frac{d^2r}{dt^2}-r\dot{\theta}^2+\frac{GM_E}{r^2}=-qGM_s\frac{(\vec{r}_s\cdot\hat{I})}{r_s^3}+\dot{\phi}^2r_E\{\cos\theta\cos(\phi-\psi)+\cos\alpha\sin\theta\sin(\phi-\psi)\}$$
$$-\frac{3GM_EJ_2R_\oplus^2[1-3(\hat{I}.K_G)^2]}{2r^4}-p\frac{GM_s}{(r_s)^2}\left\{\frac{(\vec{v}\cdot\vec{r}_s)(\vec{r}_s\cdot\hat{I})}{cr_s}+\frac{(\vec{v}\cdot\hat{I})}{c}\right\}, \quad (3)$$

$$\frac{d(r^2\dot{\theta})}{dt}=-qGM_sr\frac{(\vec{r}_s\cdot\hat{J})}{r_s^3}-\dot{\phi}^2rr_E\left\{\sin\theta\cos(\phi-\psi)-\cos\alpha\cos\theta\sin(\phi-\psi)\right\}$$
$$-\frac{3GM_EJ_2R_\oplus^2(\hat{I}.\hat{K}_G)(\hat{J}.K_G)}{2r^3}-p\frac{GM_s}{r_s^2}\left\{\frac{(\vec{v}\cdot\vec{r}_s)(\vec{r}_s\cdot\hat{J})}{cr_s}+\frac{(\vec{v}\cdot\hat{J})}{c}\right\}. \quad (4)$$

	I_0	J_0	K_0
I	a_x	b_x	c_x
J	a_y	b_y	c_y
K	a_z	b_z	c_z

	I_0	J_0	K_0
I_G	1	0	0
J_G	0	$\cos\varepsilon$	$\sin\varepsilon$
K_G	0	$-\sin\varepsilon$	$\cos\varepsilon$

Table 1: Relation between coordinate system.

$$\begin{aligned}
a_x &= \cos\theta\cos\psi - \cos\alpha\sin\theta\sin\psi,\\
a_y &= -\sin\theta\cos\psi - \cos\alpha\cos\theta\sin\psi,\\
a_z &= \sin\alpha\sin\psi,\\
b_x &= \cos\theta\sin\psi - \cos\alpha\sin\theta\cos\psi,\\
b_y &= -\sin\theta\sin\psi + \cos\alpha\cos\theta\cos\psi,\\
b_z &= \cos\psi\sin\alpha,\\
c_x &= \sin\alpha\sin\theta, c_y = \sin\alpha\cos\theta,\\
c_z &= \cos\alpha.
\end{aligned}$$

Equations (3) and (4) are the required equations of motion of the satellite in polar form. These equations are not integrable, therefore we follow the perturbation technique and replace r, $\dot\theta$ and $\dot\psi$ by their steady state values r_0, $\dot\theta_0$, $\dot\psi_0$ and we may take $\theta = \dot\theta_0 t$, $\psi = \dot\psi_0 t$ and $\phi = \dot\phi t$, respectively. Putting the steady state values in the R.H.S of equations (3) and (4), we get

$$\begin{aligned}
\frac{d^2r}{dt^2} - r\dot\theta^2 + \frac{GM_E}{r^2} &= -qGM_s\frac{(\vec r_s\cdot\hat I)}{r_s^3} - \frac{3GM_EJ_2R_\oplus^2\{1-3(\hat I.\hat K_G)^2\}}{2r_0^4}\\
&+ \dot\phi^2 r_E\{\cos\dot\theta_0 t\cos(\dot\phi-\dot\psi_0)t + \cos\alpha\sin\dot\theta_0 t\sin(\dot\phi-\dot\psi_0)t)\}\\
&- p\frac{GM_s}{r_s^2}\left\{\frac{(\vec v\cdot\vec r_s)(\vec r_s\cdot\hat I)}{cr_s} + \frac{(\vec v\cdot\hat I)}{c}\right\}, \qquad (5)\\
\frac{d(r^2\dot\theta)}{dt} &= -qGM_sr_0\frac{(\vec r_s\cdot\hat J)}{r_s^3} - \frac{3GM_EJ_2R_\oplus^2(\hat I.\hat K_G)(\hat J.\hat K_G)}{2r_0^3}\\
&- \dot\phi^2 r_0 r_E\{\sin\dot\theta_0 t\cos(\dot\phi-\dot\psi_0)t + \cos\alpha_0\cos\dot\theta_0 t\sin(\dot\phi-\dot\psi_0)t\}\\
&- pr_0\frac{GM_s}{r_s^2}\left\{\frac{(\vec v\cdot\vec r_s)(\vec r_s\cdot\hat J)}{cr_s} + \frac{(\vec v\cdot\hat J)}{c}\right\}. \qquad (6)
\end{aligned}$$

Now

$$\begin{aligned}
\vec v &= \{-r_E(\dot\phi-\dot\psi_0)\cos\dot\theta_0 t\sin(\dot\phi-\dot\psi_0)t + r_E(\dot\phi-\dot\psi_0)\cos\alpha\sin\dot\theta_0 t\cos(\dot\phi-\dot\psi_0)t\}\hat I\\
&\quad \{r_0\dot\theta_0 + r_E(\dot\phi-\dot\psi_0)\sin\dot\theta_0 t\sin(\dot\phi-\dot\psi_0)t + r_E(\dot\phi-\dot\psi_0)\cos\dot\theta_0 t\cos(\dot\phi-\dot\psi_0)t\cos\alpha_0\}\hat J\\
&\quad - r_E(\dot\phi-\dot\psi_0)\sin\alpha_0\cos(\dot\phi-\dot\psi_0)t\hat K,\\
\hat K_G &= -\sin\epsilon\hat J_0 + \cos\epsilon\hat K_0\\
&= -\sin\epsilon(\hat Ib_x + \hat Jb_y + \hat Kb_z) + \cos\epsilon(C_x\hat I + C_y\hat J + C_z\hat K)\\
&= (-b_x\sin\epsilon + c_x\cos\epsilon)\hat I + (-b_y\sin\epsilon + c_y\cos\epsilon)\hat J + (-b_z\sin\epsilon + c_z\cos\epsilon)\hat K.
\end{aligned}$$

With the help of the above values, the transformations in Table 1 and taking $r^2\dot\theta =$ constant $= h$, $r = \frac{1}{u}$, we get

$$\frac{d^2u}{dt^2} + n^2u = K_1 + K_2\cos nt + K_3\sin nt + K_4\cos 2nt + k_5\sin 2nt$$

$$
\begin{aligned}
&+ K_6 \cos 3nt + K_7 \sin 3nt + K_8 \cos(\dot\phi + \dot\psi_0)t + K_9 \sin(\dot\phi - \dot\psi_0)t \\
&+ K_{10} \cos(n + \dot\phi - \dot\psi_0)t + K_{11} \sin(n + \dot\phi - \dot\psi_0)t + K_{12} \cos(n + \dot\phi + \dot\psi_0)t \\
&+ K_{13} \sin(n - \dot\phi + \dot\psi_0)t + K_{14} \cos(2n + \dot\phi - \dot\psi_0)t + K_{15} \sin(2n + \dot\phi - \dot\psi_0)t \\
&+ K_{16} \cos(2n - \dot\phi + \dot\psi_0)t + K_{17} \sin(2n - \dot\phi + \dot\psi_0)t + K_{18} \sin 2(\dot\phi - \dot\phi_0))t \\
&+ K_{19} \sin(2n+2\dot\phi - 2\dot\psi_0)t + K_{20} \sin(2n - 2\dot\phi + 2\dot\psi_0)t + K_{21} \sin(n + 2\dot\phi - 2\dot\psi_0)t \\
&+ K_{22} \sin(n - 2\dot\phi + 2\dot\psi_0)t + K_{23} \sin(3n+2\dot\phi - 2\dot\psi_0)t + K_{24} \sin(3n - 2\dot\phi + 2\dot\psi_0)t \\
&+ K_{25} \cos \dot\phi_0 t + K_{26} \cos 2\dot\phi_0 t + K_{27} \cos(2n + \dot\phi_0)t + K_{28} \cos(2n - \dot\psi_0)t \\
&+ K_{29} \cos(2n + 2\dot\psi_0)t + K_{30} \cos(2n - 2\dot\phi_0))t + K_{31} \cos(n + \dot\phi)t \\
&+ K_{32} \cos(n - \dot\phi_0)t + K_{33} \cos(n + 2\dot\psi_0)t + K_{34} \cos(n - 2\dot\psi_0)t \\
&+ K_{35} \cos(3n + 2\dot\phi_0)t + K_{36} \cos(3n - 2\dot\psi_0)t + K_{37} \cos(3n + \dot\psi_0)t \\
&+ K_{38} \cos(3n - \dot\phi_0)t. \qquad (7)
\end{aligned}
$$

The solution is given by

$$
\begin{aligned}
u = {} & A\cos(nt - \epsilon_1) + \frac{K_1}{n^2} - \frac{K_2 t \sin nt}{2n} + \frac{K_3 t \cos nt}{2n} + \frac{K_4 \cos 2nt}{n^2 - (2n)^2} + \frac{K_5 \sin 2nt}{n^2 - (2n)^2} \\
&+ \frac{K_6 \cos 3nt}{n^2 - (3n)^2} + \frac{K_7 \sin 3nt}{n^2 - (3n)^2} + \frac{K_8 (\dot\phi - \dot\psi_0)t}{n^2(\dot\phi - \dot\psi_0)^2 t} + \frac{K_9 \sin(\dot\phi - \dot\psi_0)t}{n^2(\dot\phi - \dot\psi_0)^2} + \frac{K_{10} \cos(n + \dot\phi - \dot\psi_0)t}{n^2 - (n + \dot\phi - \dot\psi_0)^2} \\
&+ \frac{K_{11} \sin(n + \dot\phi - \dot\psi_0)t}{n^2 - (n + \dot\phi - \dot\psi_0)t} + \frac{K_{12} \cos(n - \dot\phi + \dot\psi_0)t}{n^2 - (n - \dot\phi + \dot\psi_0)^2} + \frac{K_{13} \sin(n - \dot\phi + \dot\psi_0)t}{n^2 - (n - \dot\phi + \dot\psi_0)^2} \\
&+ \frac{K_{14} \cos(2n + \dot\phi - \dot\psi_0)t}{n^2 - (2n + \dot\phi - \dot\psi_0)^2} + \frac{K_{15} \sin(2n + \dot\phi - \dot\psi_0)t}{n^2 - (2n + \dot\phi - \dot\psi_0)^2} + \frac{K_{16} \cos(2n - \dot\phi + \dot\psi_0)t}{n^2 - (2n - \dot\phi + \dot\psi_0)^2} \\
&+ \frac{K_{17} \sin(2n - \dot\phi + \dot\psi_0)t}{n^2 - (2n - \dot\phi + \dot\psi_0)^2} + \frac{K_{18} \sin 2(\dot\phi - \dot\psi_0)t}{n^2 - 4(\dot\phi - \dot\psi_0)^2} + \frac{K_{19} \sin(2n + 2\dot\phi - 2\dot\psi_0)t}{n^2 - (2n + 2\dot\phi - 2\dot\psi_0)^2} \\
&+ \frac{K_{20} \sin(2n - 2\dot\phi + 2\dot\psi_0)t}{n^2 - (2n - 2\dot\phi + \dot\psi_0)^2} + K_{21} \frac{\sin(n + 2\dot\phi - 2\dot\psi_0)t}{n^2 - (n + 2\dot\phi - 2\dot\psi_0)^2} + K_{22} \frac{\sin(n + 2\dot\phi - 2\dot\psi_0)t}{n^2 - (n - 2\dot\phi + 2\dot\psi_0)^2} \\
&+ K_{23} \frac{\sin(3n + 2\dot\phi - 2\dot\psi_0)t}{n^2 - (3n + 2\dot\phi - 2\dot\psi_0)^2} + K_{24} \frac{\sin(3n - 2\dot\phi - 2\dot\psi_0)t}{n^2 - (3n - 2\dot\phi + 2\dot\psi_0)^2} + K_{25} \frac{\cos \dot\psi_0 t}{n^2 - \dot\psi_0^2} \\
&+ K_{26} \frac{\cos 2\dot\psi_0 t}{n^2 - (2\dot\psi_0)^2} + K_{27} \frac{\cos(2n + \dot\psi_0)t}{n^2 - (2n+\dot\psi_0)^2} + K_{28} \frac{\cos(2n + \dot\psi_0)t}{n^2 - (2n - \dot\psi_0)^2} + K_{29} \frac{\cos(2n + \dot\psi_0)t}{n^2 - (2n + \dot\psi_0)^2} \\
&+ K_{30} \frac{\cos(2n - \dot\psi_0)t}{n^2 - (2n - 2\dot\psi_0)^2} + K_{31} \frac{\cos(n+2\dot\psi_0)t}{n^2 - (n + 2\dot\psi_0)^2} + \frac{K_{32} \cos(n - \dot\psi_0)t}{n^2 - (n - \dot\psi_0)^2} + \frac{K_{33} \cos(n + \dot\psi_0)t}{n^2 - (n + \dot\psi_0)^2} \\
&+ \frac{K_{34} \cos(n - 2\dot\psi_0)t}{n^2 - (n - 2\dot\psi_0)^2} + \frac{K_{35} \cos(3n+2\dot\psi_0)t}{n^2 - (3n+2\dot\psi_0)^2} + \frac{K_{36} \cos(3n - 2\dot\psi_0)t}{n^2 - (3n - 2\dot\psi_0)^2} + \frac{K_{37} \cos(3n + \dot\psi_0)t}{n^2 - (3n + \dot\psi_0)^2} \\
&+ \frac{K_{38} \cos(3n - \dot\psi_0)t}{n^2 - (3n - \dot\psi_0)^2}. \qquad (8)
\end{aligned}
$$

The values of constant $K_i's$ are given in Appendix ‘A’ (which can be obtained from the authors).

2.2 Resonance

It is clear that the motion becomes indeterminate if any one of the denominator vanishes in equation (8), and hence the resonance occurs at these points, called the critical points. It is found that resonance occurs at many points with three frequencies and at six points $R'_1(n = \dot\psi)$, $R'_2(3n = \dot\psi)$, $R'_3(2n = \dot\psi)$, $R'_4(3n = 2\dot\psi)$, $R'_5(n = 2\dot\psi)$ and $R'_6(4n = \dot\psi)$ with two frequencies due to oblateness. Also, it is found that resonance occurs at five points $(n = \dot\phi)$, $(n = 2\dot\phi)$, $(3n = \dot\phi)$, $(2n = \dot\phi)$, $(3n = 2\dot\phi)$ in the frequencies n and $\dot\phi$. The 1 : 1 resonance repeated four times, 2 : 1 resonance occurs thrice while other four resonances occur once only. If we take the solar radiation pressure as a perturbing force, then there are only three points at which resonance occurs. If we consider the velocity dependent terms of the P-R drag, then five points of resonance occur, where three points of resonance are same, and 1 : 2 and 3 : 2 resonances occurs only due to the velocity dependent terms of the P-R drag.

3 Time Period and Amplitude at the Resonance Point

3.1 Time period and amplitude at $n = 2\dot\phi$

We follow the method in [1] to determine the time period and amplitude at $n = 2\dot\phi$. It is suggested to obtain the solution of (7) when that of

$$\frac{d^2u}{dt^2} + n^2u = 0 \tag{9}$$

is periodic and is known. The solution of (9) is

$$u = k\cos s,$$

where

$$s = nt + \epsilon, \quad n = \frac{k_1}{k} = \text{the function of } k; \tag{10}$$

k, k_1 and ϵ are arbitrary constants. As we are probing the resonance in the motion of the satellite at the point $n = 2\dot\phi$, in our case, the resulting equation. (7) can be written as

$$\frac{d^2u}{dt^2} + n^2u = HA'\cos n't = H\psi',$$

where

$$H = \frac{pF_g(r_E)^2\dot\phi}{4ca(r_s)(1-e^2)} = \text{constant}, n' = 2\dot\phi, \quad A' = -\sin^2\alpha,$$
$$\psi' = \frac{\partial\psi}{\partial u} = A'\cos 2n't, \psi = uA'\cos n't, \quad \psi = \frac{A'k}{2}\{\cos(2n't + s) + \cos(2n't - s)\}. \tag{11}$$

Then

$$\frac{dk}{dt} = \frac{H}{W}\frac{\partial u}{\partial s}\psi' = \frac{H}{W}\frac{\partial\psi}{\partial s}, \tag{12}$$

$$\frac{ds}{dt} = n - \frac{H}{W}\frac{\partial u}{\partial k}\psi' = n - \frac{H}{W}\frac{\partial \psi}{\partial k}, \tag{13}$$

where

$$W = \frac{\partial}{\partial k}(n\frac{\partial u}{\partial s})\frac{\partial u}{\partial s} - n\frac{\partial^2 u}{\partial s^2}\frac{\partial u}{\partial k} = \textbf{a function of } k \textbf{ only.}$$

Since n and W are the function of k only, we can put (12) and (13) into canonical form with new variables defined by

$$dk_1 = W dk, \tag{14}$$

$$dB = -n dk_1 = -nW dk, \tag{15}$$

(14) and (15) can be put in the form

$$\frac{dk_1}{dt} = \frac{\partial}{\partial s}(B + H\psi), \quad \frac{ds}{dt} = -\frac{\partial}{\partial s}(B + H\psi).$$

Differentiating (13) with respect to t and substituting the expression for $\frac{ds}{dt}$ and $\frac{dk}{dt}$, we have

$$\frac{d^2 s}{dt^2} = \frac{H}{W}\left(\frac{\partial n}{\partial k}\frac{\partial \psi}{\partial s} - n\frac{\partial^2 \psi}{\partial s \partial k} - \frac{\partial^2 \psi}{\partial k \partial t}\right) + \frac{H^2}{K^2}\left(\frac{\partial^2 \psi}{\partial s \partial k}\frac{\partial \psi}{\partial k} - W\frac{\partial}{\partial k}\left(\frac{1}{W}\frac{\partial \psi}{\partial k}\right)\frac{\partial \psi}{\partial s}\right). \tag{16}$$

Since the last expression of (16) has the factor H^2, it may, in general, be neglected in a first approximation. In (11) we find s and t are present in ψ' as a sum of the periodic terms with argument

$$s' = s - n't,$$

the affected term in our case is

$$\psi = \frac{kA' \cos s'}{2}. \tag{17}$$

Equation (16) for s' is then

$$\frac{d^2 s'}{dt^2} + (n - 2n')^2 \frac{H}{W}\frac{\partial}{\partial k}\left(\frac{1}{n - n'}\frac{\partial \psi}{\partial s'}\right) = 0 \tag{18}$$

or

$$\frac{d^2 s'}{dt^2} - (n - 2n')^2 \frac{H}{2W}\frac{\partial}{\partial k}\left(\frac{kA'}{n - n'}\right)\sin s' = 0. \tag{19}$$

At first approximation, we put constants $k = k_0, \quad n = n_0, \quad W = W_0$. Then (19) can be written as

$$\frac{d^2 s'}{dt^2} - (n - 2n')^2 \frac{H}{2W}\frac{\partial}{\partial k}\left(\frac{kA'}{n - n\prime}\right)\sin s' = 0. \tag{20}$$

If the oscillations are small intervals, then (20) may be put in the form

$$\frac{d^2 s'}{dt^2} - (n - 2n')^2 \frac{H}{2W}\frac{\partial}{\partial k}\left(\frac{kA'}{n - n\prime}\right)s' = 0$$

or

$$\frac{d^2s'}{dt^2} + p_1^2 s' = 0, \tag{21}$$

where

$$p_1 = \sqrt{\frac{pF_g(r_{_E})^2\dot{\phi}\sin^2\alpha}{8ca(r_s)(1-e^2)}}\sqrt{\frac{\sqrt{k}_1}{W_0k_0}}, \tag{22}$$

$$W_0 = (W)_0 = \frac{\partial}{\partial k}(n\frac{\partial y}{\partial s})\frac{\partial y}{\partial s} - n\frac{\partial^2 y}{\partial s^2}\frac{\partial y}{\partial k}_0 = (\sqrt{k}_1\cos^2(2n't+\epsilon)_0),$$
$$= \sqrt{k}_1\cos^2(2\dot{\phi}t+\epsilon_0). \tag{23}$$

The solution of (21) is given by

$$s' = A\sin(p_1t+\lambda_0),$$

where

$$A = \frac{\sqrt{k}_2}{p_1}, \qquad k_2, \lambda_0 = \text{the constants of integration}, \qquad s' = s - 2n't.$$

The equation for s gives

$$s = 2n't + A\sin(p_1t+\lambda_0). \tag{24}$$

Using (12), (19) and (22) the equation for k gives

$$k = k_0 + HA'\left(\frac{q}{W}\right)_0\frac{A}{p_1}\cos(p_1t+\lambda_0), \tag{25}$$

where k_0 is determined from $n_0 = n'$. Since n_0 is a known function of k_0, the amplitude 'A' and the time period T are given by

$$A = \frac{\sqrt{k}_2}{p_1}, \quad T = \frac{2\pi}{p_1},$$

where k_2 is an arbitrary constant,

$$p_1 = \frac{\sqrt{pF_g(r_{_E})^2\dot{\phi}\sin^2\alpha}}{\sqrt{8car_s(1-e^2)k_0}\cos(2\dot{\phi}+\epsilon_0)}.$$

Using equation (13), k_0 may be written as

$$k_0 = \frac{\sqrt{k}_1}{n_0}.$$

We may choose the constants of integration $k_1 = 1$, $k_2 = 1$, $\epsilon_0 = 0$ [12].

The amplitude and time period are given by

$$A = \frac{2\sqrt{2car_s(1-e^2)}}{\sqrt{pF_gn_0\dot{\phi}r_{_E}\sin\alpha}}\cos 2\phi, \quad T = \frac{4\pi\sqrt{2ca(r_s)(1-e^2)}}{\sqrt{pF_gn_0\dot{\phi}r_{_E}\sin\alpha}}\cos 2\phi.$$

In the same manner we have calculated the amplitudes and time periods at other points also. Thereafter two cases arise.

Case 1: Regression angle is constant.

- If we take the solar radiation pressure as a perturbing force, then there are only three points at which resonance occurs. The corresponding amplitudes and time periods are given in Table 2 below.

- In addition to the above, if we consider the velocity dependent terms of the P-R drag, then at five points $R_1(n = \dot{\phi})$, $R_2(3n = \dot{\phi})$, $R_3(2n = \dot{\phi})$, $R_4(3n = 2\dot{\phi})$, $R_5(n = 2\dot{\phi})$ resonance occurs, where three points of resonance are same as in subcase 1, and 1 : 2 and 3 : 2 resonances occur only due to the velocity dependent terms of the P-R drag. But the amplitudes and time periods at all resonance points are not same as in the case of the solar radiation pressure. The corresponding amplitude and time period are given in Table 3.

Case 2: Regression angle is not constant.

It is found that resonance occurs at many points with three frequencies and at six points $R'_1(n = \dot{\psi})$, $R'_2(3n = \dot{\psi})$, $R'_3(2n = \dot{\psi})$, $R'_4(3n = 2\dot{\psi})$, $R'_5(n = 2\dot{\psi})$ and $R'_6(4n = \dot{\psi})$ with two frequencies. The corresponding amplitudes and time-periods are given in Table 4.

	Resonance	Amplitude	Time Period
1	$n = \dot{\phi}$	A_1, A_2	T_1, T_2
2	$2n = \dot{\phi}$	A_5	T_5
3	$3n = \dot{\phi}$	A_9	T_9

Table 2: Amplitudes A_i's and time periods T_i's at resonance points with only radiation pressure as a perturbing force when the regression angle is constant.

	Resonance	Amplitude	Time Period
1	$n = \dot{\phi}$	A_3, A_4	T_3, T_4
2	$2n = \dot{\phi}$	A_6	T_6
3	$n = 2\dot{\phi}$	A_7, A_8	T_7, T_8
4	$3n = 2\dot{\phi}$	A_{10}	T_{10}

Table 3: Amplitudes A_i's and time periods T_i's at resonance points for the velocity dependent terms of the P-R drag when the regression angle is constant.

Resonance	Amplitude	Time Period
$n = \dot{\psi}$	$A_{11}, A_{12}, A_{13}, A_{14}$	$T_{11}, T_{12}, T_{13}, T_{14}$
$2n = \dot{\psi}$	A_{15}, A_{16}	T_{15}, T_{16}
$n = 2\dot{\psi}$	A_{17}, A_{18}	T_{17}, T_{18}
$3n = \dot{\psi}$	A_{19}	T_{19}
$3n = 2\dot{\psi}$	A_{20}	T_{20}
$4n = \dot{\psi}$	A_{21}	T_{21}

Table 4: Amplitudes A_i's and time periods T_i's at resonance points for two frequencies when the regression angle is not constant.

where A_i's and T_i's are are given in the Appendix A and Appendix B, respectively (which can be obtained from the authors).

4 Discussion and Conclusion

We have investigated the resonance in the motion of a satellite in the Earth-Sun system due to oblateness of the Earth and the P-R drag. Firstly, the equations of motion of the geo-centric satellite in vector as well as in polar form has been evaluated by taking the velocity of the satellite as v. Secondly, the velocity of the satellite in the P-R drag have been deduced by using an operator and then substituted in the equations of motion. We get resonances at many points with three frequencies, and at eleven points with two frequencies between n and $\dot{\phi}$ and n and $\dot{\psi}$.

Two resonance points 3 : 2 and 1 : 2 occur only due to the velocity dependent terms of the P-R drag. We have shown the effect of the P-R drag and oblateness on the amplitude and time period by using the following data of the satellite:

$$a = 6921000m; \quad e = .0065; n = 0.0628766\frac{deg}{sec}; \quad \dot{\phi} = 0.0000114077\frac{deg}{sec};$$
$$r_s = 149599 \times 10^6 m; \quad r_{_E} = 149.6 \times 10^9 m; \quad c = 3 \times 10^8 \frac{m}{sec}.$$

We make the above quantities dimensionless by taking

$$M_E + M_s = 1\text{unit}, \quad G = 1\text{unit},$$
$$r_s = \text{the distance between the Earth and the Sun} = 1\text{unit}.$$

From Figure 3, we observe that the amplitude and time period increase when q increases and it is maximum at $\phi = 0$. p is the factor of the velocity dependent terms of the P-R drag, when q increases, p decreases, and hence, when the P-R decreases, then the amplitude as well as the time period increase.

Figure 4 explains the variation in A_1 and time-period T_1, respectively, for $-90^0 < \phi < 90^0$ and $0 < q < 1$, at resonance 1 : 1 with the P-R drag. The below graphs show that the amplitude and time period decrease as ϕ increases.

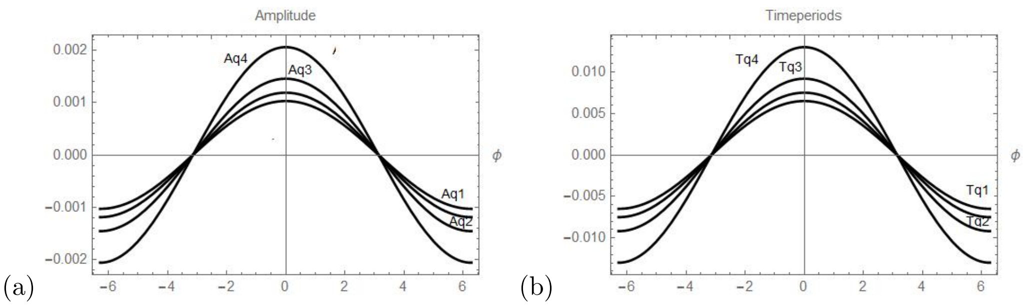

Figure 3: (a) Comparison of amplitudes at same resonant points and for different $q's$: $Aq1 = 0.20$ (Red); $Aq2 = 0.40$ (Green); $Aq3 = 0.60$ (Gray) and $Aq4 = 0.80$ (Blue). (b) Comparison of time periods at same resonant points and for different $q's$: $Tq1 = 0.20$ (Red); $Tq2 = 0.40$ (Green); $Tq3 = 0.60$ (Gray) and $Tq4 = 0.80$ (Blue), at resonance 1:1.

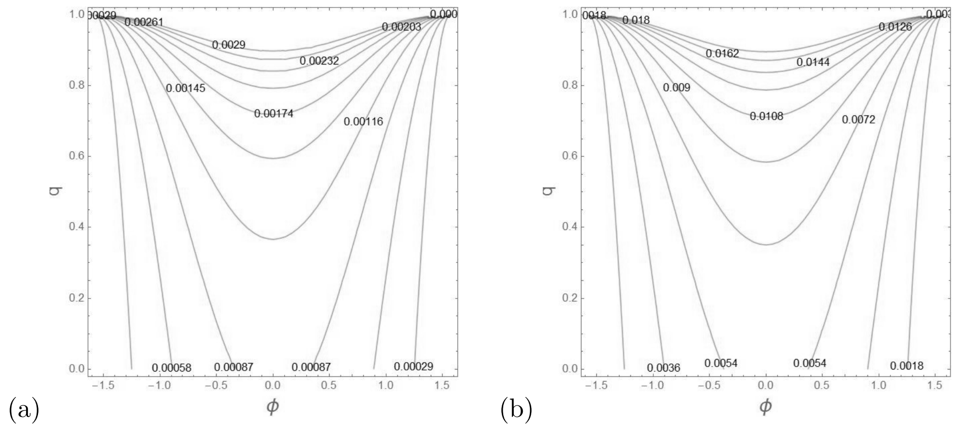

Figure 4: (a) Variation in amplitude, (b) variation in time period 'T' for $0^0 < \phi < 90^0$ and q $(0 < q < 1)$ at resonance 1:1 with the P-R drag.

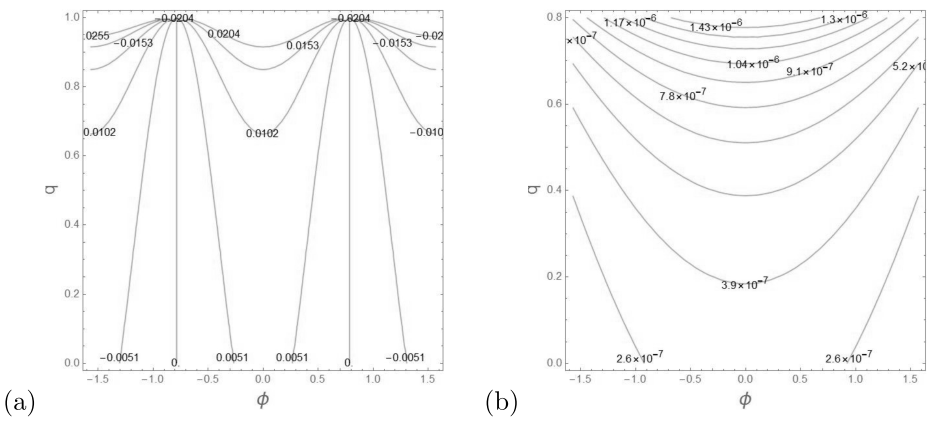

Figure 5: (a) Variation in amplitude 'A' w.r.t. ϕ , at resonance 1:2. (b) Time period 'T' at resonance 1:1, for $-1^0 < \phi < 1^0$ and q $(0 < q < 1)$ without the P-R drag

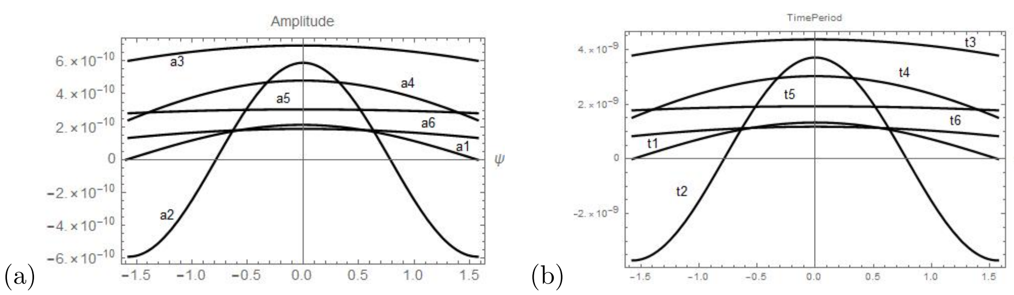

Figure 6: (a) Comparison of amplitude. (b) Comparison of time periods due to the coefficient of oblateness of the Earth (J_2) at different resonant points.

Figure 4 explains the variation in A_1 and time period T_1, respectively, for $-90^0 <$

$\phi < 90^0$ and $0 < q < 1$ at resonance $1:1$ with the P-R drag. The above graphs show that the amplitude and time-period decrease as ϕ increases.

Figure 5 also explains the amplitude and time period with respect to ϕ. In this case it can be observed that the amplitude becomes very high of greater range of ϕ but it is not in the case of the velocity dependent terms of the P-R drag. Similarly, Figure 5 explains the variation in amplitude for $-90^0 < \phi < 90^0$ and $0 < q < 1$ at resonance $1:2$. The graphs show that the amplitude is periodic with respect to ϕ and it increases (decreases) as q increases (decreases).

Figure 6 also explains the amplitudes and time periods due to oblateness of the Earth (J_2). In these graphs we have shown the comparison of the amplitude and time period at different critical points, where resonance occurs, and it is clear from the figures that the value of the amplitudes and time periods is different at different critical points. The present study is becoming of more interest in the commensurable orbits, for example, the interacting and navigation satellite system.

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