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Oscillation and Nonoscillation for Caputo–Hadamard Impulsive Fractional Differential Equations

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Abstract: In this paper, the concept of the upper and lower solutions method combined with the fixed point theorem is used to investigate the existence of oscillatory and nonoscillatory solutions for a class of initial value problems for Caputo–Hadamard impulsive fractional differential equations.

Keywords: *impulsive fractional differential equations; Caputo–Hadamard fractional derivative; fixed point; upper solution; lower solution, oscillation, nonoscillation.*

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1 Introduction

Fractional differential equations and integrals are valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. In the monographs [1, 3, 4, 11, 12, 15], we can find the mathematical background and various applications of fractional calculus. Recently, many researchers studied different fractional problems involving the Riemaan-Liouville, Caputo and Hadamard derivatives; see, for example, the papers [2, 17]. Sufficient conditions for the oscillation of solutions of differential equations are given in [9, 14, 16].

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The method of upper and lower solutions has been successfully applied to study the existence of solutions for ordinary and fractional differential equations and inclusions. See the monograph [13], and the papers [8, 18, 19], and the references therein.

This paper deals with the existence of oscillatory and nonoscillatory solutions for the following class of initial value problems for Caputo–Hadamard impulsive fractional differential equations:

$$^{Hc}D^{\alpha}_{t_k}y(t) = f(t, y(t)),\ t \in (t_k, t_{k+1}), \tag{1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, 2, \ldots, \tag{2}$$

$$y(1) = y_*, \tag{3}$$

where $^{Hc}D^{\alpha}_{t_k}$ is the Caputo–Hadamard fractional derivative of order $0 < \alpha \leq 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_* \in \mathbb{R}$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $1 = t_0 < t_1 < \ldots < t_m < t_{m+1} < \cdots < \infty$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k,\ k = 1, 2, \ldots$

This paper initiates the study of oscillatory and nonoscillatory solutions for impulsive fractional differential equations involving the Caputo–Hadamard fractional derivative.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J, \mathbb{R})$ be the space of all continuous functions from J into $\mathbb{R}$.

$$\|y\|_\infty = \sup_{t\in J} |y(t)|.$$

Let $BC(J, \mathbb{R})$ be the Banach space of all continuous and bounded functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup_{t\in J} |y(t)|,$$

and let $L^1(J, \mathbb{R})$ be the Banach space of Lebesgue integrable functions $y : J \longrightarrow \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_1^T |y(t)|dt.$$

Denote by $AC(J, \mathbb{R})$ the space of absolutely continuous functions from J into $\mathbb{R}$.

Let us recall some definitions and properties of the Hadamard fractional integration and differentiation. Let $\delta = t\dfrac{d}{dt}$, and set

$$AC^n_\delta(J, \mathbb{R}) = \{y : J \longrightarrow \mathbb{R}, \delta^{n-1}y(t) \in AC(J, \mathbb{R})\}.$$

Definition 2.1 [12] The Hadamard fractional integral of order $r > 0$ for a function $h \in L^1([1, +\infty), \mathbb{R})$ is defined as

$$^{H}I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds,$$

provided the integral exists for a.e. $t > 1$.

Example 2.1 Let $q > 0$. Then

$$^{H}I_1^q \ln t = \frac{1}{\Gamma(2+q)}(\ln t)^{1+q};\ \textit{for a.e. } t \in [1, +\infty).$$

Definition 2.2 [12] The Hadamard fractional derivative of order $r > 0$ applied to the function $h \in AC_\delta^n([1, +\infty), \mathbb{R})$ is defined as

$$(^{H}D_1^q h)(t) = \delta^n(^{H}I_1^{n-r}h)(t),$$

where $n - 1 < r < n,\ n = [r] + 1$, and $[r]$ is the integer part of r.

Definition 2.3 [10] For a given function $h \in AC_\delta^n([a, b], \mathbb{R})$, such that $0 < a < b$, the Caputo–Hadamard fractional derivative of order $r > 0$ is defined as follows:

$$^{Hc}D^r y(t) =^{H} D^r \left[y(s) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{s}{a}\right)^k \right](t),$$

where $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$.

Lemma 2.1 *[10] Let $y \in AC_\delta^n([a, b], \mathbb{R})$ or $C_\delta^n([a, b], \mathbb{R})$ and $\alpha \in \mathbb{C}$. Then*

$$^{H}I^r(^{Hc}D^r y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k.$$

3 Main Results

We consider the space

$$PC(J, \mathbb{R}) = \{y : J \to \mathbb{R},\ y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, 1, \ldots,$$

$$\text{and there exist}\, y(t_k^+) \text{ and } y(t_k^-),\ k = 1, 2, \ldots, \text{ with } y(t_k^-) = y(t_k)\}.$$

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J}|y(t)|.$$

Let us start by defining what we mean by a solution of problem (1)–(3).

Definition 3.1 A function $y \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R}),\ k = 0, 1, \ldots$, is said to be a solution of (1)–(3) if y satisfies the equation $^{Hc}D_{t_k}^\alpha y(t) = f(t, y(t))$ on (t_k, t_{k+1}) and conditions $y(t_k^+) = I_k(y(t_k^-))$, k=1,2,…, $y(1) = y_*$.

Definition 3.2 A function $u \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R}), k = 0, 1, \ldots$, is said to be a lower solution of (1)–(3) if $^{Hc}D_{t_k}^\alpha u(t) \leq f(t, u(t))$ on (t_k, t_{k+1}) and $u(t_k^+) \leq I_k(u(t_k)), k = 1, \ldots$. Similarly, a function $v \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R}), k = 0, \ldots$, is said to be an upper solution of (1)–(3) if $^{Hc}D_{t_k}^\alpha v(t) \geq f(t, v(t))$ on (t_k, t_{k+1}) and $v(t_k^+) \geq I_k(v(t_k)), k = 1, 2, \ldots$.

For the study of this problem we first list the following hypotheses:

(H1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous;

(H2) For all $r > 0$ there exists a function $h_r \in C(J, \mathbb{R}^+)$ such that

$$|f(t, y)| \le h_r(t) \text{ for all } t \in J \text{ and all } |y| \le r;$$

(H3) There exist u and $v \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R}),\ k = 0, \ldots,$ which are the lower and upper solutions for the problem (1)–(3) such that $u \le v$;

(H4)

$$u(t_k^+) \le \min_{y\in[u(t_k^-),v(t_k^-)]} I_k(y) \le \max_{y\in[u(t_k^-),v(t_k^-)]} I_k(y) \le v(t_k^+),\ k = 1, 2, \ldots.$$

Theorem 3.1 *Assume that hypotheses (H1)-(H4) hold. Then the problem (1)–(3) has at least one solution y such that*

$$u(t) \le y(t) \le v(t) \quad \textit{for all } t \in J.$$

Proof. The proof will be given in several steps.

Step 1: Consider the problem

$$^{Hc}D^{\alpha}_{t_0} y(t) = f(t, y(t)),\ t \in J_1 := [t_0, t_1], \tag{4}$$

$$y(1) = y_*. \tag{5}$$

Transform the problem (4)–(5) into a fixed point problem. Consider the following modified problem:

$$^{Hc}D^{\alpha}_{t_0} y(t) = f_1(t, y(t)),\ t \in J_1, \tag{6}$$

$$y(1) = y_*, \tag{7}$$

where

$$f_1(t, y) = f(t, \tau(t, y))$$

$$\tau(t, y) = \max\{u(t), \min(y, v(t))\}$$

and

$$\overline{y}(t) = \tau(t, y).$$

A solution to (6)–(7) is a fixed point of the operator $N : C([t_0, t_1], \mathbb{R}) \longrightarrow C([t_0, t_1], \mathbb{R})$ defined by

$$y(t) = y_* + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f_1(s, y(s)) \frac{ds}{s}.$$

Remark 3.1 (i) Notice that f_1 is a continuous function, and from (H2) there exists $M^* > 0$ such that

$$|f_1(t, y)| \le M^* \text{ for each } (t, y) \in J_1 \times \mathbb{R}.$$

(ii) By the definition of τ it is clear that

$$u(t_k^+) \le I_k(\tau(t_k, y(t_k))) \le v(t_k^+),\ k = 1, 2, \ldots.$$

In order to apply the nonlinear alternative of Leray–Schauder type, we first show that N is continuous and completely continuous.

Claim 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([t_0, t_1], \mathbb{R})$. Then

$$|N(y_n)(t) - N(y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |f_1(s, \overline{y}_n(s)) - f_1(s, \overline{y}(s))| \frac{ds}{s}.$$

Since f_1 is a continuous function, we have

$$\|N(y_n) - N(y)\|_\infty \leq \frac{\left(\log \dfrac{t_1}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} \|f_1(\cdot, \overline{y}_n(\cdot)) - f_1(\cdot, \overline{y}(\cdot))\|_\infty.$$

Thus

$$\|N(y_n) - N(y)\|_\infty \to 0 \ \ as \ n \to \infty.$$

Claim 2: *N maps bounded sets into bounded sets in $C([t_0, t_1], \mathbb{R})$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C([t_0, t_1], \mathbb{R}) : \|y\|_\infty \leq q\}$ one has $\|Ny\|_\infty \leq \ell$. Let $y \in B_q$. Then for each $t \in J_1$ we have

$$y(t) = y_* + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f_1(s, y(s)) \frac{ds}{s}.$$

By (H1) and Remark 3.1 we have for each $t \in J_1$

$$\begin{aligned} |Ny(t)| &\leq |y_*| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |f_1(s, y(s))| \frac{ds}{s} \\ &\leq |y_*| + \frac{M\left(\log \dfrac{t_1}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} := \ell. \end{aligned}$$

Thus $\|N(y)\|_\infty \leq \ell$.

Claim 3: *N maps bounded set into equicontinuous sets of PC.*

Let $\tau_1, \tau_2 \in J_1$, $\tau_1 < \tau_2$ and B_q be a bounded set of PC as in Claim 2. Let $y \in B_q$, then

$$|N(u_2) - N(u_1)| \leq \frac{M\left(\log \dfrac{\tau_2}{\tau_1}\right)^{\alpha}}{\Gamma(\alpha+1)}.$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Claims 1 to 3 together with the Arzela–Ascoli theorem we can conclude that $N : C([t_0, t_1], \mathbb{R}) \longrightarrow C([t_0, t_1], \mathbb{R})$ is continuous and completely continuous.

Claim 4: *A priori bounds on solutions.*

Let y be a possible solution of $y = \lambda N(y)$ with $\lambda \in [0,1]$. Then we have

$$y(t) = \lambda \left[|y_*| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} |f_1(s, y(s))| \frac{ds}{s} \right].$$

This implies by Remark 3.1 that for each $t \in J_1$ we have

$$\begin{aligned} |Ny(t)| &\leq |y_*| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} |f_1(s, y(s))| \frac{ds}{s} \\ &\leq |y_*| + \frac{M \left(\log \frac{t_1}{t_0} \right)^{\alpha}}{\Gamma(\alpha+1)} := M_1. \end{aligned}$$

Set

$$U = \{ y \in C([t_0, t_1], \mathbb{R}) : \|y\|_\infty < M_1 + 1 \}.$$

From the choice of U there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type, we deduce that N has a fixed point y in U which is a solution of the problem (6)–(7).

Claim 5: *Every solution y of* $(6) - (7)$ *satisfies*

$$u(t) \leq y(t) \leq v(t) \textit{ for all } t \in J_1.$$

Let y be a solution of $(6) - (7)$. We prove that

$$u(t) \leq y(t) \text{ for all } t \in J_1.$$

Suppose not. Then there exist τ_1, τ_2 with $\tau_1 < \tau_2$ such that $u(\tau_1) = y(\tau_1)$ and

$$u(t) > y(t) \text{ for all } t \in (\tau_1, \tau_2).$$

In view of the definition of τ one has

$${}^{Hc}D^{\alpha} y(t) = f(t, u(t)) \text{ for all } t \in (\tau_1, \tau_2).$$

An integration on $(\tau_1, t]$ with $t \in (\tau_1, \tau_2)$ yields

$$y(t) - y(\tau_1) = \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, u(s)) \frac{ds}{s}.$$

Since u is a lower solution to $(4) - (5)$, we have

$$u(t) - u(\tau_1) \leq \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, u(s)) \frac{ds}{s};\ t \in (\tau_1, \tau_2).$$

It follows from $y(\tau_1) = u(\tau_1)$ that

$$u(t) \leq y(t);\ \text{for all } t \in (\tau_1, \tau_2),$$

which is a contradiction, since $u(t) > y(t)$ for all $t \in (\tau_1, \tau_2)$. Consequently,

$$u(t) \le y(t) \text{ for all } t \in J_1.$$

Analogously, we can prove that

$$y(t) \le v(t) \text{ for all } t \in J_1.$$

This shows that

$$u(t) \le y(t) \le v(t) \text{ for all } t \in J_1.$$

Consequently, the problem $(4) - (5)$ has a solution y satisfying $u \le y \le v$. Denote this solution by y_0.

Step 2: Consider the following problem:

$$^{Hc}D^{\alpha}_{t_1} y(t) = f(t, y(t)),\ t \in J_2 := [t_1, t_2], \tag{8}$$

$$y(t_1^+) = I_1(y_0(t_1^-)). \tag{9}$$

Consider the following modified problem:

$$^{Hc}D^{\alpha}_{t_1} y(t) = f_1(t, y(t)),\ t \in J_2, \tag{10}$$

$$y(t_1^+) = I_1(y_0(t_1^-)). \tag{11}$$

A solution to (10)–(11) is a fixed point of the operator $N_1 : C([t_1, t_2], \mathbb{R}) \longrightarrow C([t_1, t_2], \mathbb{R})$ defined by

$$N_1(y)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f_1(s, y(s)) \frac{ds}{s} + I_1(y_0(t_1^-)).$$

Since $y_0(t_1) \in [u(t_1^-), v(t_1^-)]$, (H4) implies that

$$u(t_1^+) \le I_1(y_0(t_1^-)) \le v(t_1^+),$$

that is

$$u(t_1^+) \le y(t_1^+) \le v(t_1^+).$$

Using the same reasoning as that used for problem (4)–(5), we can conclude the existence of at least one solution y to (10)–(11). We now show that this solution satisfies

$$u(t) \le y(t) \le v(t) \quad \text{for all} \quad t \in J_2.$$

Let y be the above solution to (10)–(11). We show that

$$u(t) \le y(t) \quad \text{for all} \quad t \in J_2.$$

Let y be a solution of $(6) - (7)$. We prove that

$$u(t) \le y(t) \text{ for all } t \in J_1.$$

Suppose not. Then there exist τ_3, τ_4 with $\tau_3 < \tau_4$ such that $u(\tau_3) = y(\tau_4)$ and

$$u(t) > y(t) \text{ for all } t \in (\tau_3, \tau_4).$$

In view of the definition of τ one has

$$^{Hc}D^{\alpha}y(t) = f(t, u(t)) \text{ for all } t \in (\tau_3, \tau_4).$$

An integration on $(\tau_3, t]$ with $t \in (\tau_3, \tau_4)$ yields

$$y(t) - y(\tau_3) = \frac{1}{\Gamma(\alpha)} \int_{\tau_3}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, u(s)) \frac{ds}{s}.$$

Since u is a lower solution to (4) – (5), we have

$$u(t) - u(\tau_3) \leq \frac{1}{\Gamma(\alpha)} \int_{\tau_3}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, u(s)) \frac{ds}{s}; \; t \in (\tau_3, \tau_4).$$

It follows from $y(\tau_3) = u(\tau_3)$ that

$$u(t) \leq y(t) \text{ for all } t \in (\tau_3, \tau_4),$$

which is a contradiction, since $u(t) > y(t)$ for all $t \in (\tau_3, \tau_4)$. Consequently,

$$u(t) \leq y(t) \text{ for all } t \in J_2.$$

Analogously, we can prove that

$$y(t) \leq v(t) \text{ for all } t \in J_2.$$

This shows that

$$u(t) \leq y(t) \leq v(t) \text{ for all } t \in J_2.$$

Denote this solution by y_1.

Step 3: We continue this process and take into account that $y_m := y|_{[t_{m-1}, t_m]}$ is a solution to the problem

$$^{Hc}D^{\alpha}_{t_{m-1}}y(t) = f(t, y(t)), \; t \in J_m := [t_{m-1}, t_m], \tag{12}$$

$$y(t_m^+) = I_m(y_{m-1}(t_{m-1}^-)). \tag{13}$$

Consider the following modified problem:

$$^{Hc}D^{r}_{t_{m-1}}y(t) = f_1(t, y(t)), \; t \in J_m, \tag{14}$$

$$y(t_m^+) = I_m(y_{m-1}(t_{m-1}^-)). \tag{15}$$

A solution to (14)–(15) is a fixed point of the operator $N_m : C([t_{m-1}, t_m], \mathbb{R}) \longrightarrow C([t_{m-1}, t_m], \mathbb{R})$ defined by

$$N_m(y)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s} + I_m(y(t_{m-1}^-)).$$

Using the same reasoning as that used for problems (4)-(5) and (8)–(9) we can conclude the existence of at least one solution y to (12)–(13). Denote this solution by y_{m-1}.

The solution y of the problem (1)–(3) is then defined by

$$y(t) = \begin{cases} y_0(t), & t \in [t_0, t_1], \\ y_2(t), & t \in (t_1, t_2], \\ . \\ . \\ . \\ y_{m-1}(t), & t \in (t_{m-1}, t_m], \\ . \\ . \\ . \end{cases}$$

The proof is complete.

3.1 Nonoscillation and oscillation of solutions

The following theorem gives sufficient conditions to ensure the nonoscillation of solutions of problem (1)–(3).

Theorem 3.2 *Let u and v be lower and upper solutions, respectively, of (1)–(3) with $u \leq v$ and assume that*

(H5) u is eventually positive nondecreasing, or v is eventually negative nonincreasing.

Then every solution y of (1)–(3) such that $y \in [u, v]$ is nonoscillatory.

Proof. Assume that u is eventually positive. Thus there exists $T_u > t_0$ such that

$$u(t) > 0 \;\text{ for all }\; t > T_u.$$

Hence $y(t) > 0$ for all $t > T_u$, and $t \neq t_k, k = 1, \ldots$ For some $k \in N$ and $t > t_u$, we have $y(t_k^+) = I_k(y(t_k))$. From (H4) we get $y(t_k^+) > u(t_k^+)$. Since for each $h > 0, u(t_k + h) \geq u(t_k) > 0$, one has $I_k(y(t_k)) > 0$ for all $t_k > T_u, k = 1, \ldots,$ which means that y is nonoscillatory. Analogously, if v is eventually negative, then there exists $T_v > t_0$ such that

$$y(t) < 0 \;\text{ for all }\; t > T_v,$$

which means that y is nonoscillatory. This completes the proof.

The following theorem discusses the oscillation of solutions of problem (1)–(3).

Theorem 3.3 *Let u and v be lower and upper solutions, respectively, of (1)–(3), and assume that the sequences $u(t_k)$ and $v(t_k), k = 1, 2, \ldots,$ are oscillatory. Then every solution y of (1)–(3) such that $y \in [u, v]$ is oscillatory.*

Proof. Suppose on the contrary that y is a nonoscillatory solution of (1)–(3). Then there exists $T_y > 0$ such that $y(t) > 0$ for all $t > T_y$, or $y(t) < 0$ for all $t > T_y$. In the case when $y(t) > 0$ for all $t > T_y$ we have $v(t_k) > 0$ for all $t_k > T_y, k = 1, 2, \ldots,$ which is a contradiction since $v(t_k)$ is an oscillatory upper solution. Analogously, in the case $y(t) < 0$ for all $t > T_y$ we have $u(t_k) < 0$ for all $t_k > T_y, k = 1, 2, \ldots,$ which is also a contradiction, since $u(t_k)$ is an oscillatory lower solution.

3.2 An example

We consider the following impulsive fractional differential equation:

$$ {}^{Hc}D^{\alpha}y(t) = f(t, y(t)), \quad \text{for each } t \in (t_k, t_{k+1}),\ 0 < \alpha < 1,\ \ k = 1, 2, \ldots, \tag{16} $$

$$ y(t_k^+) = I_k(y(t_k^-)),\ k = 1, 2, \ldots, \tag{17} $$

$$ y(1) = y_*, \tag{18} $$

where $f : J \times \mathbb{R} \to \mathbb{R}$. Assume that there exist $g_1(\cdot),\ g_2(\cdot) \in C(J, \mathbb{R})$ such that

$$ g_1(t) \le f(t, y) \le g_2(t) \ \text{ for all } \ t \in J, \ \text{ and } \ y \in \mathbb{R}, $$

and for each $t \in J$

$$ \int_1^t g_1(s)\frac{ds}{s} \le I_k\left(\int_1^t g_1(s)\frac{ds}{s}\right),\ k \in \mathbb{N}, $$

$$ \int_1^t g_2(s)\frac{ds}{s} \ge I_k\left(\int_1^t g_2(s)\ \frac{ds}{s}\right),\ \ k \in \mathbb{N}. $$

Consider the functions $u(t) := \displaystyle\int_1^t g_1(s)\frac{ds}{s}$ and $v(t) := \displaystyle\int_1^t g_2(s)\frac{ds}{s}$. Clearly, u and v are lower and upper solutions of the problem (16)-(18), respectively; that is,

$$ {}^{Hc}D^{\alpha}u(t) \le f(t, u(t)) \ \text{ for all } \ t \in J, $$

and

$$ {}^{Hc}D^{\alpha}v(t) \ge f(t, v(t)) \ \text{ for all } \ t \in J. $$

Since all the conditions of Theorem 3.1 are satisfied, the problem (16)-(18) has at least one solution y on J with $u \le y \le v$. If $g_1(t) > 0$, then u is positive and nondecreasing, thus y is nonoscillatory. If $g_2(t) < 0$, then v is negative and nonincreasing, thus y is nonoscillatory. If the sequences $u(t_k)$ and $v(t_k)$ are both oscillatory, then y is oscillatory.

4 Conclusion

In this paper, we have provided some sufficient conditions guaranteeing the existence of the oscillatory and nonoscillatory solutions of a class of impulsive differential equations involving the Caputo–Hadamard fractional derivative. We use the concept of the upper and lower solutions method combined with the nonlinear alternative of Leray–Schauder type.

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