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Uniform Asymptotic Stability in Probability of Nontrivial Solution of Nonlinear Stochastic Systems

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Abstract: The aim of this paper is to study the uniform asymptotic stability in probability when a nonlinear stochastic differential equation does not have a trivial solution. For nontrivial solutions of a nonlinear stochastic differential equation, the problem of uniform asymptotic stability in probability is reformulated for a ball of radius $R > 0$. Based on this new formulation, a theorem for the uniform asymptotic stability in probability for this ball is proposed by using a Lyapunov approach.

Keywords: *stochastic systems; Itô formula; Brownian motions; stability in probability; asymptotic stability in probability.*

Mathematics Subject Classification (2010): 93E03, 93E15.

1 Introduction

In this paper, we discuss a new concept of uniform asymptotic stability in probability for stochastic systems which are described by stochastic differential equations (SDEs) driven by multiplicative noises. These systems differ from ordinary differential equations (ODEs) modeling deterministic processes. Unlike an ODE, a SDE contains two terms: the drift for the evolution of time and the diffusion for the action of the Brownian motion. These systems correspond to Itô processes, and the noises that affect them are Brownian motions, also called the Wiener processes. This kind of equations is extensively studied in [8, 16, 17] and references therein.

Numerous phenomena are described by this class of models when a deterministic description is not satisfactory: in finance (financial mathematics and stock prices), biology

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(geographical and population evolution), geology (earthquakes), engineering (synthesis of control taking into account the failures that may appear randomly), computing (modeling networks), electricity (modeling of electrical circuits taking account of noise of electrical circuits), physical and mechanical processes (particle movements in a gas or ionized medium, quantum physics), etc.

The notion of stability of the solutions of SDE was introduced by Kats and Krasovskii [9]. Then, in the works of Kushner [11, 12], Has'minski [8], Kozin [10], Wonham [19], Zakai [22, 23], Gikhman and Skorokhod [5] and Friedman [4], several types of stability have been defined for the SDE and a Lyapunov-type approach to study these stabilities has been developed and elaborated.

In the literature, the asymptotic stability in probability has been extensively studied in many works. Without exhaustivity, we can cite the textbooks [8, 16], the survey papers [10, 13] and the papers [6, 7, 14, 18, 20, 21, 24], with references therein. It is the asymptotic stability in probability of the equilibrium point $x = 0$ which is treated in the works mentioned above and the case where the stochastic differential equation has a nontrivial solution $x \neq 0$ is not considered.

In [2], the authors consider that the equilibrium point of the stochastic differential equation is not the origin and that the differential stochastic equation is perturbed by an external disturbance. There is a study of the stability of nontrivial solution in [1, 3, 15].

To deal with the existence of nontrivial solutions $x \neq 0$ of a stochastic differential equation, we propose to study the uniform asymptotic stability in probability for a ball with a given radius $R > 0$ and the classical concepts of uniform stability in probability and uniform asymptotic stability in probability are reformulated for this ball. Sufficient conditions are derived by using a Lyapunov approach in order to guarantee that a solution initialized outside of the ball converges asymptotically in probability to the border of the ball.

The paper is organized as follows. The concepts of uniform stability in probability and uniform asymptotic stability in probability for a ball of radius $R > 0$ are given in Section 2. A new theorem guaranteeing the uniform asymptotic stability in probability for this ball is proposed in Section 3. This theorem is illustrated by an example in Section 4.

Notations. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

$$\|A\| = \sqrt{\sum_{i,j} A_{i,j}^2} = \sqrt{\operatorname{tr}(A^T A)}$$

is the Euclidean norm of the matrix A, while $\|x\| = \sqrt{x^T x}$ is the Euclidean norm of the vector x. $a \vee b$ is the maximum of reals a and b. $a \wedge b$ is the minimum of reals a and b. The SDE means a stochastic differential equation. The probability measure associated with the random variable x is denoted $\mathbf{P}\{x\}$. $\mathbf{E}\{x\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathbf{P}\{x\}$. Let $\mathbf{K}$ denote the family of all continuous and nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$.

2 Concepts of Stability in Probability

Consider the following nonlinear stochastic differential equation (SDE):

$$\mathrm{d}\, x = f(x)\, \mathrm{d}\, t + g(x)\, \mathrm{d}\, w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^d$ is a multi-dimensional independent Wiener processes (or Brownian motions). The initial condition is given by $x_0 = x(t_0)$. We assume that $f(0) \neq 0$ or $g(0) \neq 0$, i.e. the stochastic differential equation (1) does not have the trivial solution $x = 0$.

The functions $f(x)$ and $g(x)$ verify the following standard assumptions for Itô calculus [8, 16, 17] :

$$\int_0^T \|f(x(s))\| \,\mathrm{d}\, s < \infty \qquad \text{a.s.} \qquad \forall\, T > 0, \tag{2a}$$

$$\int_0^T \|g(x(s))\|^2 \,\mathrm{d}\, s < \infty \qquad \text{a.s.} \qquad \forall\, T > 0. \tag{2b}$$

To guarantee the existence and the uniqueness of the solution $x(t)$ of the SDE (1), the functions $f(x)$ and $g(x)$ satisfy the following relations $\forall\, x \in \mathbb{R}^n$ and $\forall\, \overline{x} \in \mathbb{R}^n$ ([8, 16, 17]):

$$\|f(x)\|^2 + \|g(x)\|^2 \leqslant k_1(1 + \|x\|^2), \tag{3a}$$

$$\|f(x) - f(\overline{x})\| \vee \|g(x) - g(\overline{x})\| \leqslant k_2 \|x - \overline{x}\|, \tag{3b}$$

where k_1 and k_2 are given strictly positive reals.

In this paper, we study the uniform stability in probability of the solution of a SDE when the origin is not an equilibrium point. The considered stability is characterized by the convergence of the solution in probability to a border of a ball B_R defined as follows.

Definition 2.1 Let $R > 0$ be a real. The ball B_R is defined by

$$B_R = \{x \in \mathbb{R}^n : \|x\| \leqslant R\}. \tag{4}$$

Definition 2.2 The ball B_R is said to be uniformly stable in probability for the SDE (1) if, for any $0 < \varepsilon < 1$ and for any $r > 0$, there exists $\delta(\varepsilon, r, t_0) > 0$ such that

$$\mathbf{P}\{x(t) \in \mathbb{R}^n : R < \|x(t)\| \leqslant R + r \text{ for all } t \geqslant t_0\} \geqslant 1 - \varepsilon,$$
$$\forall\, x_0 \in \mathbb{R}^n \text{ and } R < \|x_0\| \leqslant R + \delta(t_0, \varepsilon, r). \tag{5}$$

Definition 2.3 The ball B_R is uniformly asymptotically stable in probability for the SDE (1) if the ball is uniformy stable in probability and, for any $0 < \varepsilon < 1$, there exists $\delta(\varepsilon, t_0) > 0$ such that

$$\mathbf{P}\{x(t) \in \mathbb{R}^n : \limsup_{t \to +\infty} \|x(t)\| = R\} \geqslant 1 - \varepsilon, \quad \forall\, x_0 \in \mathbb{R}^n, \ \ R < \|x_0\| \leqslant R + \delta(\varepsilon, t_0). \tag{6}$$

3 Uniform Asymptotic Stability in Probability of Nontrivial Solution of SDE

Let $V(x,t)$ be a function valued on $\mathbb{R}$ which is continuously twice differentiable in $x \in \mathbb{R}^n$ and once differentiable in $t \in \mathbb{R}_+$. Applying the Itô formula to $V(x,t)$ with SDE (1) yields [8, 16, 17]

$$\mathrm{d}\, V(x,t) = \mathfrak{L}V(x,t)\,\mathrm{d}\, t + \mathfrak{B}V(x,t)\,\mathrm{d}\, w \tag{7}$$

with

$$\mathfrak{L}V(x,t) = V_t(x,t) + V_x(x,t) f(x) + \frac{1}{2}\operatorname{tr}(g^T(x) V_{xx}(x,t) g(x)), \tag{8a}$$

$$\mathfrak{B}V(x,t) = V_x(x,t) g(x), \tag{8b}$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t},$$
$$V_x(x,t) = \begin{bmatrix} \frac{\partial V(x,t)}{\partial x_1} & \dots & \frac{\partial V(x,t)}{\partial x_n} \end{bmatrix},$$
$$V_{xx}(x,t) = \begin{bmatrix} \frac{\partial^2 V(x,t)}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 V(x,t)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V(x,t)}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 V(x,t)}{\partial x_n \partial x_n} \end{bmatrix}.$$

In the sequel, it is assumed that $V_t(x,t) = 0$, so $V(x,t)$ is replaced by $V(x)$.

The following theorem solves the problem of the uniform asymptotic stability in probability of SDE (1).

Theorem 3.1 *Let $V(x)$ be a positive definite Lyapunov function. If there exist three functions μ_1, μ_2 and μ_3 in $\mathbf{K}$ and a scalar $\gamma > 0$ and $R > 0$, such that*

$$\mu_1(\|x\|) \leqslant V(x) \leqslant \mu_2(\|x\|), \tag{9}$$
$$\mathfrak{L}V(x) \leqslant -\mu_3(\|x\|) + \gamma, \tag{10}$$
$$-\mu_3(\|x\|) + \gamma < 0, \quad \forall\ \|x\| > R, \tag{11}$$
$$\mu_1(R) = \mu_2(R), \tag{12}$$

then the ball B_R is uniformly asymptotically stable in probability for the SDE (1).

Proof. First step: The ball B_R is uniformly stable in probability.

Fix $t_0 \in \mathbb{R}^+$. Let $0 < \epsilon < 1$ and $r > 0$, we define $x(t)$ as the solution of the system (1), we suppose that $\forall t \geqslant t_0$, we have $\|x\| > R$.

The following stopping time

$$\tau_r = \inf\{t \geqslant t_0 \text{ such that } \|x(t)\| > R + r\} \tag{13}$$

is defined to eliminate all the solutions that go beyond $R + r$.

From (7), (8a), (10) and (11), we have

$$\mathrm{d}\, V(x) = \mathfrak{L}V(x)\,\mathrm{d}\, t + V_x g(x)\,\mathrm{d}\, w(t) \tag{14}$$

and

$$\mathfrak{L}V(x) \leqslant -\mu_3(\|x\|) + \gamma < 0, \quad \forall\ \|x\| > R. \tag{15}$$

Applying the expectation to the previous inequality leads to

$$\mathbf{E}\left\{\int_{t_0}^{t\wedge\tau_r} \mathrm{d}\, V(x(s))\right\} = \mathbf{E}\left\{\int_{t_0}^{t\wedge\tau_r} (\mathfrak{L}V(x(s)\,\mathrm{d}\, s + V_x g(x(s))\,\mathrm{d}\, w(s))\right\}. \tag{16}$$

In view of (15) and the following relation

$$\mathbf{E}\left\{\int_{t_0}^{t\wedge\tau_r} V_x g(x(s))\,\mathrm{d}\, w(s)\right\} = 0, \tag{17}$$

the equation (16) becomes

$$\mathbf{E}\left\{\int_{t_0}^{t\wedge\tau_r} \mathrm{d}\, V(x(s))\right\} = \mathbf{E}\left\{\int_{t_0}^{t\wedge\tau_r} \mathfrak{L}V(x(s))\,\mathrm{d}\, s\right\} \leqslant 0. \tag{18}$$

Integrating (18) gives

$$\mathbf{E}\{V(x(t\wedge\tau_r)) - V(x_0)\} \leqslant 0. \tag{19}$$

The previous inequality is equivalent to

$$\mathbf{E}\{V(x(t\wedge\tau_r)) - \mu_1(R)\} \leqslant \mathbf{E}\{V(x_0) - \mu_1(R)\}, \tag{20}$$

and, if $\tau_r \leqslant t$, we obtain

$$\mathbf{P}\{\tau_r \leqslant t\}\ \mathbf{E}\{V(x(\tau_r)) - \mu_1(R)\} \leqslant \mathbf{E}\{V(x_0) - \mu_1(R)\}. \tag{21}$$

Condition (9) yields $\mu_1(R+r) \leqslant V(x(\tau_r))$, then we have

$$\mathbf{P}\{\tau_r \leqslant t\}(\mu_1(R+r) - \mu_1(R)) \leqslant (V(x_0) - \mu_1(R)). \tag{22}$$

So, if $t \to +\infty$, the following inequality holds true:

$$\mathbf{P}\{\tau_r \leqslant +\infty\} \leqslant \frac{V(x_0) - \mu_1(R)}{\mu_1(R+r) - \mu_1(R)}. \tag{23}$$

Since we have $\mu_1(R) = \mu_2(R)$ (because of the continuity at the point R of μ_2), there exists $0 < \delta_0(t_0, \epsilon, r) < r$ such that $R < \|x_0\| < R + \delta_0(t_0, \epsilon, r)$, then we have

$$\mu_2(\|x_0\|) - \mu_2(R) \leqslant \epsilon(\mu_1(R+r) - \mu_1(R)). \tag{24}$$

Using the inequality above we have

$$V(x_0) - \mu_1(R) \leqslant \epsilon(\mu_2(\|x_0\|) - \mu_2(R)) \leqslant \epsilon(\mu_1(R+r) - \mu_1(R)).$$

From (**??**) we can deduce that

$$\frac{V(x_0) - \mu_1(R)}{\mu_1(R+r) - \mu_1(R)} \leqslant \epsilon. \tag{25}$$

Fix $x_0 \in \mathbb{R}^n$ such that $R < \|x_0\| < R + \delta_0(\varepsilon, r, t_0)$. From the existence and uniqueness of the solution x to the SDE (1) (see (2a) and (2b)), we assume that there exists $x(t)$ such that $\|x(t)\| > R$, $\forall t \geqslant t_0$.

Using x_0 chosen above and the inequality (25), we have

$$\mathbf{P}\{\tau_r \leqslant +\infty\} \leqslant \varepsilon. \tag{26}$$

Finally, we have $\forall R < \|x_0\| < (R + \delta_0(\epsilon, t_0, r)) \leqslant R + r$. Then, we obtain

$$\mathbf{P}\{x(t) \in \mathbb{R}^n : R < \|x(t)\| \leqslant R + r \text{ for all } t \geqslant t_0\} \geqslant 1 - \varepsilon \tag{27}$$

and the condition (5) in Definition 2.2 is proved.

Second step: The ball B_R is uniformly asymptotically stable in probability.

Since the ball B_R is uniformly stable in probability (see step 1), $\forall \varepsilon$ and r with $0 < \varepsilon < 1$ and $r > 0$, $\exists\, \delta_0(\varepsilon, r, t_0)$ such that $R < \|x_0\| < R + \delta_0(\varepsilon, r, t_0)$. From the existence and uniqueness of the solution x to the SDE (1) (see (2a) and (2b)), we assume that there exists $x(t)$ such that $\|x(t)\| > R$, $\forall t \geqslant t_0$. So, the following relation

$$\mathbf{P}\Big\{x(t) \in \mathbb{R}^n : R < \|x(t)\| \leqslant R + \frac{r}{2}, t \geqslant t_0\Big\} \geqslant 1 - \frac{\varepsilon}{4} \tag{28}$$

holds.

In view of the theorem of continuity on $\mu_2(R)$ and since $R < \|x_0\|$, for all β with $R < \beta < \|x_0\|$, there exists α such that $R < \alpha < \beta$ and

$$\frac{\mu_2(\alpha) - \mu_2(R)}{\mu_1(\beta) - \mu_1(R)} \leqslant \frac{\varepsilon}{4}. \tag{29}$$

We define the stopping times as follows:

$$\tau_\alpha = \inf\{t \geqslant t_0 \text{ such that } \|x(t)\| \leqslant \alpha\}, \tag{30}$$

$$\tau_r = \inf\Big\{t \geqslant t_0 \text{ such that } \|x(t)\| > R + \frac{r}{2}\Big\}. \tag{31}$$

Applying the Itô formula to $V(x)$, we have

$$\mathbf{E}\{V(x(\tau_\alpha \wedge \tau_r \wedge t))\} = \mathbf{E}\{V(x(t_0))\} + \mathbf{E}\Big\{\int_{t_0}^{\tau_\alpha \wedge \tau_r \wedge t} \mathfrak{L}V(x(s))\,\mathrm{d}\,s\Big\}. \tag{32}$$

If $t < \tau_\alpha \wedge \tau_r$, we have $\|x(t)\| > \alpha$ and, using (10), we obtain

$$\mathfrak{L}V(x(t)) \leqslant -\mu_3(\|x\|) + \gamma \leqslant -\mu_3(\alpha) + \gamma. \tag{33}$$

Integrating (32) and using (33) yield

$$\mathbf{E}\{V(x(\tau_\alpha \wedge \tau_r \wedge t))\} \leqslant \mathbf{E}\{V(x(t_0))\} + (\gamma - \mu_3(\alpha))(t - t_0)\mathbf{P}\{t < \tau_\alpha \wedge \tau_r\}. \tag{34}$$

Since the left term in inequality (34) is positive, we obtain

$$(t - t_0)(\mu_3(\alpha) - \gamma)\mathbf{P}\{t < \tau_\alpha \wedge \tau_r\} \leqslant \mathbf{E}\{V(x_0)\}. \tag{35}$$

Since the term $(t - t_0)(\mu_3(\alpha) - \gamma)\mathbf{P}\{t < \tau_\alpha \wedge \tau_r\}$ is bounded and using condition (11), we have

$$\mathbf{P}\{\tau_\alpha \wedge \tau_r = +\infty\} = 0 \tag{36}$$

when $t \to +\infty$. So we deduce

$$\mathbf{P}\{\tau_\alpha \wedge \tau_r < +\infty\} = 1. \tag{37}$$

Also, from the beginning of the second step of the proof, we get

$$\mathbf{P}\{\tau_r < +\infty\} \leqslant \frac{\varepsilon}{4}. \tag{38}$$

Then, using (37) and (38), the following relation

$$1 = \mathbf{P}\{\tau_\alpha \wedge \tau_r < +\infty\} \leqslant \mathbf{P}\{\tau_\alpha < +\infty\} + \mathbf{P}\{\tau_r < +\infty\} \leqslant \mathbf{P}\{\tau_\alpha < +\infty\} + \frac{\varepsilon}{4} \tag{39}$$

is obtained. Finally, we deduce

$$\mathbf{P}\{\tau_\alpha < +\infty\} \geqslant 1 - \frac{\varepsilon}{4}. \tag{40}$$

Now, we choose $\theta > 0$ sufficiently large such that

$$\mathbf{P}\{\tau_\alpha < \theta\} \geqslant 1 - \frac{\varepsilon}{2}, \tag{41}$$

and relations (38) and (41) lead to

$$\begin{aligned}\mathbf{P}\{\tau_\alpha < \tau_r \wedge \theta\} &\geqslant \mathbf{P}\{(\tau_\alpha < \theta) \cap (\tau_r = +\infty)\}\\ &\geqslant \mathbf{P}\{(\tau_\alpha < \theta)\}\mathbf{P}\{\tau_r = +\infty\}\\ &= \mathbf{P}\{(\tau_\alpha < \theta)\}(1 - \mathbf{P}\{\tau_r < +\infty\})\\ &\geqslant \mathbf{P}(\tau_\alpha < \theta) - \mathbf{P}\{\tau_r < +\infty\}\\ &\geqslant 1 - \frac{3\varepsilon}{4}.\end{aligned} \tag{42}$$

We define the time σ and the stopping time τ_β as follows:

$$\sigma = \begin{cases} \tau_\alpha, & \text{if} \quad \tau_\alpha < \tau_r \wedge \theta,\\ +\infty, & \end{cases} \tag{43}$$

$$\tau_\beta = \inf\{t > \sigma, \|x(t)\| > \beta\}. \tag{44}$$

Taking $t \geqslant \theta$, applying the Itô formula to the function $V(x)$ and using conditions (10) and (11), the following relation

$$\mathbf{E}\left\{\int_{\sigma\wedge t}^{\tau_\beta\wedge t} \mathrm{d}\, V(x(s))\right\} = \mathbf{E}\left\{\int_{\sigma\wedge t}^{\tau_\beta\wedge t} \mathfrak{L}V(x(s))\,\mathrm{d}\, s\right\} \leqslant 0 \tag{45}$$

is obtained.

Integrating (45) gives

$$\mathbf{E}\{V(x(\tau_\beta \wedge t))\} \leqslant \mathbf{E}\{V(x(\sigma \wedge t))\}. \tag{46}$$

The previous inequality is equivalent to

$$\mathbf{E}\{V(x(\tau_\beta \wedge t)) - \mu_2(R)\} \leqslant \mathbf{E}\{V(x(\sigma \wedge t)) - \mu_2(R)\}. \tag{47}$$

If $\tau_\alpha < \tau_r \wedge \theta$, in view of (43), the inequality (47) becomes

$$\mathbf{E}\{V(x(\tau_\beta \wedge t)) - \mu_2(R)\} \leqslant E\{V(x(\tau_\alpha \wedge t)) - \mu_2(R)\}. \tag{48}$$

If $\tau_\beta < t$, then $\tau_\alpha < t$, and the previous inequality becomes

$$\mathbf{P}\{\tau_\beta < t\}\mathbf{E}\{V(x(\tau_\beta)) - \mu_2(R)\} \leqslant \mathbf{E}\{V(x(\tau_\alpha)) - \mu_2(R)\}. \tag{49}$$

So, using condition (9), we obtain

$$\mathbf{P}\{\tau_\beta < t\}(\mu_1(\beta) - \mu_2(R)) \leqslant \mu_2(\alpha) - \mu_2(R). \tag{50}$$

Then, using (29), the following inequality

$$\mathbf{P}\{\tau_\beta < t\} \leqslant \frac{\mu_2(\alpha) - \mu_2(R)}{\mu_1(\beta) - \mu_2(R)} \leqslant \frac{\varepsilon}{4} \tag{51}$$

is satisfied and, if $t \to +\infty$, we have

$$\mathbf{P}\{\tau_\beta < +\infty\} \leqslant \frac{\varepsilon}{4}. \tag{52}$$

Then, relations (42) and (52) lead to

$$\begin{aligned}\mathbf{P}\{(\sigma < +\infty) \cap (\tau_\beta = +\infty)\} &\geqslant \mathbf{P}\{\sigma < +\infty\}\mathbf{P}\{\tau_\beta = +\infty\} \\ &= \mathbf{P}\{\sigma < +\infty\}(1 - \mathbf{P}\{\tau_\beta < +\infty\}) \\ &\geqslant \mathbf{P}\{\sigma < +\infty\} - \mathbf{P}\{\tau_\beta < +\infty\} \\ &\geqslant \mathbf{P}\{\tau_\alpha < \tau_r \wedge \theta\} - \mathbf{P}\{\tau_\beta < +\infty\} \\ &\geqslant 1 - \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = 1 - \varepsilon, \end{aligned} \tag{53}$$

and, with (43) and (44), we obtain

$$\mathbf{P}\{x(t) \in \mathbb{R}^n : R \leqslant \lim_{t\to+\infty} \sup \|x(t)\| \leqslant \beta\} \geqslant 1 - \varepsilon, \qquad \forall\, \beta > R. \tag{54}$$

Since β is arbitrary, if $\beta \to R$, we obtain

$$\mathbf{P}\{x(t) \in \mathbb{R}^n : \limsup_{t\to+\infty} \|x(t)\| = R\} \geqslant 1 - \varepsilon. \tag{55}$$

The proof is ended.

4 Example

To illustrate Theorem 3.1, we consider the SDE (1) given by

$$\mathrm{d}\, x = (-x + 1)\,\mathrm{d}\, t + \beta\,\mathrm{d}\, w. \tag{56}$$

This SDE does not have the trivial solution $x = 0$ since $f(0) = 1$.

Let $V(x)$ be the following Lyapunov function

$$V(x) = \frac{1}{2}x^2 \tag{57}$$

and the functions $\mu_1(x)$, $\mu_2(x)$ and $\mu_3(x)$ be given by

$$\mu_1(x) = \mu_2(x) = \mu_3(x) = \frac{1}{2}x^2. \tag{58}$$

Applying the Itô formula to $V(x)$ leads to

$$\mathfrak{L}V(x) = -x^2 + x + \frac{1}{2}\beta^2 \leqslant \frac{-1}{2}x^2 + \frac{1}{2}(1 + \beta^2).$$

Then we have

$$\mathfrak{L}V(x) < 0,\ \forall\ |x| > \sqrt{1+\beta^2}. \tag{59}$$

The radius of the ball B_R is equal to $R = \sqrt{1+\beta^2}$.

Finally, the ball $S_{\sqrt{1+\beta^2}} = \{x \in \mathbb{R}, |x| = \sqrt{1+\beta^2}\}$ is uniformly asymptotically stable in probability as can be seen in Figure 1 with $\beta = 0.01$.

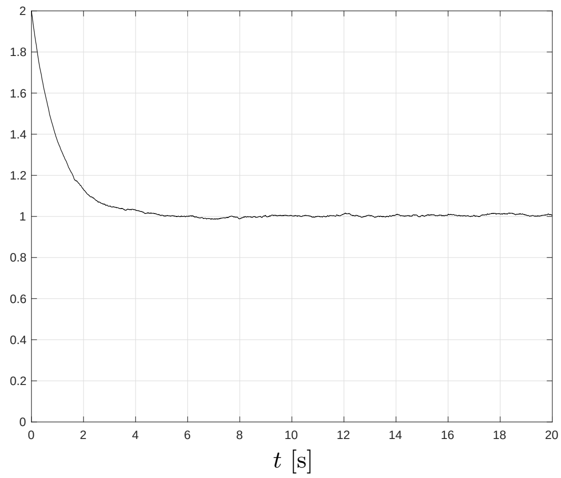

Figure 1: State x.

5 Conclusion

In this paper, the concept of uniform asymptotic stability in probability of nontrivial solutions of a SDE is studied. A new theorem is proposed to check this uniform asymptotic stability in probability. This theorem is based on sufficient conditions to be verified for a given Lyapunov function. An example is given to illustrate our approach.

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