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Diagonal Riccati Stability of a Class of Matrices and Applications

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Abstract: Necessary and sufficient conditions of the diagonal Riccati stability are derived for a class of pairs of matrices with special structures. The obtained conditions are used in the problems of analysis and synthesis of some types of time-delay systems. Results of numerical simulation are presented to illustrate the effectiveness of the proposed approaches.

Keywords: *diagonal stability; delay; Lyapunov–Krasovskii functional; complex system; asymptotic stability; consensus.*

Mathematics Subject Classification (2010): 34D20, 34K20.

1 Introduction

The problem of diagonal Riccati stability was introduced in [15] and is motivated by the construction of the diagonal Lyapunov–Krasovskii functionals for linear time-delay systems.

In [4], a criterion for a given pair of matrices to be diagonally Riccati stable has been derived. This result extended the well known condition of Barker, Berman and Plemmons for the diagonal Lyapunov stability [7]. With the aid of this criterion, necessary and sufficient conditions of the existence of diagonal Lyapunov–Krasovskii functionals were found for linear positive differential and difference systems with delay [3, 4].

However, it should be noted that the conditions of the above criterion are not constructive enough. Therefore, an actual problem is to determine the classes of matrices for which simple and constructively verified necessary and sufficient conditions of the diagonal Riccati stability can be obtained. Some of such classes were found in [2, 5].

In the present paper, a class of pairs of matrices is studied. These matrices can be used for the modeling of complex systems composed of second order subsystems with a special structure of connections between the subsystems and with a delay in the feedback law. A criterion of the diagonal Riccati stability is derived for the matrices under consideration. Moreover, it is shown that the obtained result can be applied to the problems of analysis and synthesis of some types of time-delay systems.

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2 Statement of the Problem

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the vector spaces of n-tuples of real numbers and $n \times n$ matrices, respectively, $\|\cdot\|$ be the Euclidean norm of a vector.

For a matrix $C \in \mathbb{R}^{n\times n}$, we use the notation $C^\top$ for the transpose of C. The matrix C is Hurwitz if all of its eigenvalues have negative real parts, C is Metzler if its off-diagonal entries are all nonnegative, C is nonnegative if all of its entries are nonnegative. Let $\mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ be the diagonal matrix with the elements $\lambda_1, \dots, \lambda_n$ along the main diagonal.

Let matrices $A, B \in \mathbb{R}^{n\times n}$ be given.

Definition 2.1 (see [15]) The pair of matrices (A, B) is diagonally Riccati stable if there exist diagonal positive definite matrices $P = \mathrm{diag}\{p_1, \dots, p_n\}$ and $Q = \mathrm{diag}\{q_1, \dots, q_n\}$ such that the matrix

$$R = A^\top P + PA + Q + PBQ^{-1}B^\top P \tag{1}$$

is negative definite.

In [4, 5] the following results were obtained.

Proposition 2.1 *Let the matrix $A \in \mathbb{R}^{n\times n}$ be Metzler and the matrix $B \in \mathbb{R}^{n\times n}$ be nonnegative. Then the pair (A, B) is diagonally Riccati stable if and only if the matrix $A + B$ is Hurwitz.*

Proposition 2.2 *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ be given and let $D = \mathrm{diag}\{d_1, \dots, d_n\}$, $E = \mathrm{diag}\{e_1, \dots, e_n\}$ with $d_i \in \{-1; +1\}$, $e_i \in \{-1; +1\}$ for $i = 1, \dots, n$. The pair (A, B) is diagonally Riccati stable if and only if (DAD, DBE) is diagonally Riccati stable.*

In this contribution, we will look for the conditions of the diagonal Riccati stability for a special class of pairs of matrices. Assume that n is an even number ($n = 2k$, k is a positive integer), and the matrices A and B have the following forms:

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_1 & a_{43} & a_{44} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & a_{65} & a_{66} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_{k-1} & a_{n\,n-1} & a_{n\,n} \end{pmatrix}, \tag{2}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & b \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3}$$

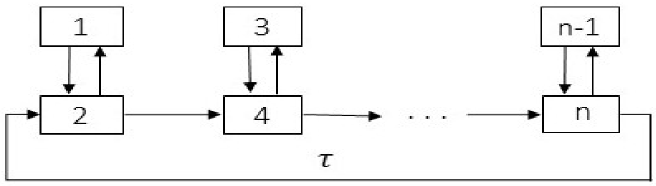

Figure 1: Structure of connections in a complex system.

Such matrices can be used for the modeling of complex systems composed of second order subsystems with a special structure of connections between the subsystems and with a delay in the feedback law (see Fig. 1).

Furthermore, we will apply the obtained conditions of the diagonal Riccati stability to the problems of analysis and synthesis for some classes of linear and nonlinear difference-differential systems.

3 A Criterion of the Diagonal Riccati Stability

Construct the auxiliary matrices

$$\widetilde{A} = \begin{pmatrix} a_{11} & \tilde{a}_{12} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \tilde{a}_{21} & a_{22} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_{33} & \tilde{a}_{34} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & |c_1| & \tilde{a}_{43} & a_{44} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & \tilde{a}_{56} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & |c_2| & \tilde{a}_{65} & a_{66} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1\,n-1} & \tilde{a}_{n-1\,n} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & |c_{k-1}| & \tilde{a}_{n\,n-1} & a_{n\,n} \end{pmatrix},$$

$$\widetilde{B} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & |b| \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Here $\tilde{a}_{2j-1\,2j} = \tilde{a}_{2j\,2j-1} = 0$ for $a_{2j-1\,2j}a_{2j\,2j-1} < 0$, and $\tilde{a}_{2j-1\,2j} = |a_{2j-1\,2j}|$, $\tilde{a}_{2j\,2j-1} = |a_{2j\,2j-1}|$ for $a_{2j-1\,2j}a_{2j\,2j-1} \geq 0$, $j = 1, \ldots, k$.

Denote $\widetilde{\Delta}_{2j-1\,2j} = a_{2j-1\,2j-1}a_{2j\,2j} - \tilde{a}_{2j-1\,2j}\tilde{a}_{2j\,2j-1}$, $j = 1, \ldots, k$.

Theorem 3.1 *Let the matrices A and B be of the form (2) and (3), respectively. Then the pair (A, B) is diagonally Riccati stable if and only if the inequalities*

$$a_{ii} < 0, \quad i = 1, \ldots, n, \qquad \widetilde{\Delta}_{2j-1\,2j} > 0, \quad j = 1, \ldots, k, \tag{4}$$

$$\widetilde{\Delta}_{12}\widetilde{\Delta}_{34} \ldots \widetilde{\Delta}_{n-1\,n} > |a_{11}a_{33} \ldots a_{n-1\,n-1}c_1c_2 \ldots c_{k-1}b| \tag{5}$$

hold.

Proof. Let $P = \operatorname{diag}\{p_1, \ldots, p_n\}$ and $Q = \operatorname{diag}\{q_1, \ldots, q_n\}$ be positive definite diagonal matrices. Without loss of generality, assume that $q_n = 1$.

If the matrices A and B are defined by the formulae (2) and (3), then the matrix (1) can be represented in the form $R = \widetilde{R} + \operatorname{diag}\{q_1, q_2, \ldots, q_{n-1}, 0\}$, where

$$\widetilde{R} = \begin{pmatrix} R_{12} & L_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ L_1 & R_{34} & L_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & L_2 & R_{56} & L_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & R_{n-3\,n-2} & L_{k-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & L_{k-1} & R_{n-1\,n} \end{pmatrix},$$

$$R_{12} = \begin{pmatrix} 2p_1a_{11} & p_1a_{12} + p_2a_{21} \\ p_1a_{12} + p_2a_{21} & 2p_2a_{22} + p_2^2b^2 \end{pmatrix},$$

$$R_{2j-1\,2j} = \begin{pmatrix} 2p_{2j-1}a_{2j-1\,2j-1} & p_{2j-1}a_{2j-1\,2j} + p_{2j}a_{2j\,2j-1} \\ p_{2j-1}a_{2j-1\,2j} + p_{2j}a_{2j\,2j-1} & 2p_{2j}a_{2j\,2j} \end{pmatrix}, \quad j = 2, \ldots, k-1,$$

$$R_{n-1\,n} = \begin{pmatrix} 2p_{n-1}a_{n-1\,n-1} & p_{n-1}a_{n-1\,n} + p_na_{n\,n-1} \\ p_{n-1}a_{n-1\,n} + p_na_{n\,n-1} & 2p_na_{nn} + 1 \end{pmatrix},$$

$$L_j = \begin{pmatrix} 0 & 0 \\ 0 & c_jp_{2j+2} \end{pmatrix}, \quad j = 1, \ldots, k-1.$$

Thus, the pair (A, B) is diagonally Riccati stable if and only if there exist positive numbers $p_1, \ldots, p_n$ for which the matrix $\widetilde{R}$ is negative definite.

Let Δ_i be the leading principal minor of the i-th order of the matrix $\widetilde{R}$, $i = 1, \ldots, n$. Necessary and sufficient conditions of the negative definiteness of $\widetilde{R}$ can be formulated as follows: $a_{ii} < 0$, $i = 1, \ldots, n$,

$$\det R_{2j-1\,2j} > 0, \quad \Delta_{2j} > 0, \quad j = 1, \ldots, k. \tag{6}$$

Choose a number $l \in \{1, \ldots, k\}$. Consider the inequalities from (6) depending on the corresponding parameter p_{2l-1}. We obtain $\det R_{2l-1\,2l} > 0$,

$$\Delta_{2j} > 0, \quad j = l, \ldots, k. \tag{7}$$

Developing Δ_{2j} by the $(2l-1)$-th and $2l$-th columns, rewrite (7) in the form

$$\alpha_{lj}\frac{\det R_{2l-1\,2l}}{p_{2l-1}} > \beta_{lj}, \quad j = l, \ldots, k,$$

where α_{lj} and β_{lj} are independent of p_{2l-1}, and $\alpha_{lj} > 0$.

Thus, to derive less conservative restrictions on the entries of the matrices A and B, one should take a value of p_{2l-1} for which $\det R_{2l-1\,2l}/p_{2l-1}$ is minimal. Hence,

$$p_{2l-1} = \begin{cases} p_{2l}a_{2l\,2l-1}/a_{2l-1\,2l} & \text{for} \quad a_{2l-1\,2l}a_{2l\,2l-1} > 0, \\ -p_{2l}a_{2l\,2l-1}/a_{2l-1\,2l} & \text{for} \quad a_{2l-1\,2l}a_{2l\,2l-1} < 0. \end{cases}$$

Moreover, taking into account Proposition 2.2, we can assume that $b \geq 0$, $c_s \geq 0$, $s = 1, \ldots, k-1$, and $a_{2j-1\,2j} \geq 0$, $a_{2j\,2j-1} \geq 0$ for $a_{2j-1\,2j}a_{2j\,2j-1} \geq 0$.

As a result, we obtain that conditions of the diagonal Riccati stability for the pair (A, B) coincide with those for the pair $(\widetilde{A}, \widetilde{B})$.

The matrix $\widetilde{A}$ is Metzler and the matrix $\widetilde{B}$ is nonnegative. Hence (see Proposition 2.1), $(\widetilde{A}, \widetilde{B})$ is diagonally Riccati stable if and only if the matrix $\widetilde{A} + \widetilde{B}$ is Hurwitz. Verifying the Sevastyanov–Kotelyanskii conditions [10] for the matrix $\widetilde{A} + \widetilde{B}$, we arrive at the inequalities (4), (5). □

4 Applications

In this section, we will show how the result described above can be applied to some problems of analysis and synthesis of time-delay systems.

4.1 Absolute stability of the Persidskii-type systems

Let the nonlinear time-delay system

$$\dot{x}(t) = Af(x(t)) + Bf(x(t-\tau)) \tag{8}$$

be given. Here $x(t) = (x_1(t), \dots, x_n(t))^\top$ is the state vector; $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^n$ are constant matrices; τ is a constant nonnegative delay. The nonlinearity $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, diagonal $f(x) = (f_1(x_1), \dots, f_n(x_n))^\top$ and satisfies the sector-like conditions $x_i f_i(x_i) > 0$ for $x_i \neq 0$, $i = 1, \dots, n$. Such a nonlinearity is said to be admissible.

The system (8) is a well-known Persidskii-type system [12, 14]. Such systems are widely used for the modeling of automatic control systems and neural networks.

From the properties of functions $f_1(x_1), \dots, f_n(x_n)$ it follows that the system (8) possesses the zero solution.

We assume that the initial functions for (8) belong to the space $C([-\tau, 0], \mathbb{R}^n)$ of continuous functions $\varphi(\theta) : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm $\|\varphi\|_\tau = \sup_{\theta \in [-\tau,0]} \|\varphi(\theta)\|$. In addition, let x_t stand for the restriction of a solution $x(t)$ of (8) to the segment $[t-\tau, t]$, i.e., $x_t : \theta \to x(t+\theta)$, $\theta \in [-\tau, 0]$.

Definition 4.1 The system (8) is absolutely stable if its zero solution is asymptotically stable for any admissible nonlinearity and any constant nonnegative delay τ.

Theorem 4.1 *Let $n = 2k$, k be a positive integer, and the matrices A and B in* (8) *be of the form* (2) *and* (3)*, respectively. If the inequalities* (4) *and* (5) *are fulfilled, then the system* (8) *is absolutely stable.*

Proof. Under conditions (4) and (5), the pair (A, B) is diagonally Riccati stable. Choose positive definite diagonal matrices $P = \mathrm{diag}\{p_1, \dots, p_n\}$ and $Q = \mathrm{diag}\{q_1, \dots, q_n\}$ for which the matrix (1) is negative definite.

Using diagonal elements of P and Q, construct a Lyapunov–Krasovskii functional for (8) in the form

$$V(x_t) = \sum_{i=1}^{n} \left(2p_i \int_0^{x_i(t)} f_i(u)du + q_i \int_{t-\tau}^{t} f_i^2(x_i(\theta))d\theta \right).$$

It is easy to verify that there exists a number $\gamma > 0$ such that

$$\dot{V}\big|_{(8)} \leq -\gamma \left(\|f(x(t))\| + \|f(x(t-\tau))\| \right).$$

Hence (see [11]), the system (8) is absolutely stable. □

4.2 Stability analysis of a mechanical system

Consider a complex system describing the interaction of k mechanical systems with two degrees of freedom. Let equations of motion be of the form

$$\begin{aligned} &\ddot{x}_1(t) + h\alpha_1\dot{x}_1(t) + \beta_{11}x_1(t) + \beta_{12}x_2(t) = 0, \\ &\ddot{x}_2(t) + h\alpha_2\dot{x}_2(t) + \beta_{21}x_1(t) + \beta_{22}x_2(t) = \omega_1 x_{2k}(t-\tau), \\ &\ddot{x}_{2j-1}(t) + h\alpha_{2j-1}\dot{x}_{2j-1}(t) + \beta_{2j-1\,2j-1}x_{2j-1}(t) + \beta_{2j-1\,2j}x_{2j}(t) = 0, \\ &\ddot{x}_{2j}(t) + h\alpha_{2j}\dot{x}_{2j}(t) + \beta_{2j\,2j-1}x_{2j-1}(t) + \beta_{2j\,2j}x_{2j}(t) = \omega_j x_{2j-2}(t), \ \ j = 2,\dots,k. \end{aligned} \tag{9}$$

Here $x_i(t) \in \mathbb{R}$, $\alpha_i, \beta_i, \omega_j$ are constant coefficients, $i = 1,\dots,2k$, $j = 1,\dots,k$, h is a positive parameter, τ is a constant nonnegative delay.

Denote $n = 2k$, $x(t) = (x_1(t),\dots,x_n(t))^\top$. Then the equations (9) can be rewritten as follows:

$$\ddot{x}(t) + hD\dot{x}(t) + C_1x(t) + C_2x(t-\tau) = 0. \tag{10}$$

Here $D = \operatorname{diag}\{\alpha_1,\dots,\alpha_n\}$, and C_1, C_2 are constant matrices with the structures similar to those of (2) and (3), respectively.

We assume that the initial functions for (10) belong to the space $C^1([-\tau,0],\mathbb{R}^n)$ of continuously differentiable functions $\varphi(\theta) : [-\tau,0] \to \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_\tau = \sup_{\theta\in[-\tau,0]} \|\varphi(\theta)\| + \sup_{\theta\in[-\tau,0]} \|\dot{\varphi}(\theta)\|.$$

To derive delay-independent stability conditions for (10), we will use the decomposition method [13, 19, 20] and the approach proposed in [1, 6].

Consider the auxiliary isolated subsystems

$$\dot{y}(t) = Ay(t) + By(t-\tau), \tag{11}$$

$$\dot{z}(t) = -Dz(t), \tag{12}$$

where $y(t), z(t) \in \mathbb{R}^n$, $A = -D^{-1}C_1$, $B = -D^{-1}C_2$.

Assumption 4.1 Let $\alpha_i > 0$, $i = 1,\dots,n$.

Remark 4.1 Under Assumption 4.1, the system (12) is asymptotically stable.

Assumption 4.2 The inequalities (4) and (5) are valid for the entries of the matrices A and B.

Remark 4.2 Under Assumption 4.1, the subsystem (11) possesses a diagonal Lyapunov–Krasovskii functional of the form

$$V(y_t) = y^\top(t)Py(t) + \int_{t-\tau}^{t} y^\top(\theta)Qy(\theta)d\theta,$$

where P and Q are constant positive definite diagonal matrices.

Applying Theorem 1 from [6], we arrive at the following result.

Theorem 4.2 *Let Assumptions 4.1 and 4.2 be fulfilled. Then there exists a number $h_0 > 0$ such that if $h \geq h_0$, then the system* (10) *is asymptotically stable for any nonnegative delay.*

4.3 Synthesis of the decentralized control for a multiagent system

The problems of cooperative control of multiagent systems have attracted considerable attention in the last decade due to their wide applicability [8, 9, 17, 18]. The key goal of cooperative control is to reach a desired global group behavior by using global/local information shared among neighboring agents in a distributed fashion. One of the important cooperative control problems is that of consensus [8, 16].

In the present subsection, we are going to design a decentralized control ensuring consensus for a group of n mobile agents on a line with a special structure of communication topology.

Let $x_i(t) \in \mathbb{R}$ be the position of the i-th agent at time $t \geq 0$, $i = 1, \ldots, n$. We will assume that the following conditions are fulfilled:

(i) $n = 2k$, where k is a positive integer;

(ii) the $(2j-1)$-th agent is a satellite of the $2j$-th agent, and it receives information about the distance $x_{2j-1}(t) - x_{2j}(t)$, $j = 1, \ldots, k$;

(iii) the $2j$-th agent receives information about the distances $x_{2j}(t) - x_{2j-1}(t)$ and $x_{2j}(t) - x_{2j-2}(t)$, $j = 2, \ldots, k$;

(iv) the 2-th agent receives information about the distances $x_2(t) - x_1(t)$ and $x_2(t) - x_n(t-\tau)$, where τ is a constant nonnegative delay;

(iv) the 2-th agent is a leader: it knows the distance between itself and a desired position ξ.

Thus, the communication topology of the system has the structure depicted in Fig. 1.

First, consider the case where the dynamics of agents are described by the first order integrators

$$\dot{x}_i(t) = u_i, \quad i = 1, \ldots, n. \tag{13}$$

Here $u_i \in \mathbb{R}$ denotes the control input (or protocol) of agent i. We will say that the multiagent system achieves a consensus if $x_i(t) \to \xi$ as $t \to +\infty$, $i = 1, \ldots, n$.

Under conditions (i)–(iv), the control law can be chosen as follows:

$$\begin{aligned} &u_{2j-1} = \alpha_{2j-1}(x_{2j} - x_{2j-1}), \quad j = 1, \ldots, k, \\ &u_{2s} = \alpha_{2s}(x_{2s-1}(t) - x_{2s}(t)) + \beta_s(x_{2s-2}(t) - x_{2s}(t)), \quad s = 2, \ldots, k, \\ &u_2 = \alpha_2(x_1(t) - x_2(t)) + \beta_1(x_n(t-\tau) - x_2(t)) + \gamma(\xi - x_2(t)), \end{aligned} \tag{14}$$

where α_i, β_j, γ are constant coefficients, $i = 1, \ldots, n$, $j = 1, \ldots, k$.

Let $x(t) = (x_1(t), \ldots, x_n(t))^\top$. Then the system (13) closed by the control (14) takes the form

$$\dot{x}(t) = Ax(t) + Bx(t-\tau). \tag{15}$$

Here A and B are constant matrices with the structures similar to those of (2) and (3), respectively. The system (15) admits the equilibrium position $x = \bar{x}$, where $\bar{x} = (\xi, \ldots, \xi)^\top$.

Applying Theorem 4.1, we arrive at the following result.

Theorem 4.3 *Let the inequalities*

$$\begin{aligned} &\gamma > 0, \quad \alpha_{2j-1} > 0, \quad j = 1, \ldots, k, \\ &\beta_1 + \gamma + \min\{\alpha_2; 0\} > 0, \quad \beta_j + \min\{\alpha_{2j}; 0\} > 0, \quad j = 2, \ldots, k, \\ &(\beta_1 + \gamma + \min\{\alpha_2; 0\}) \prod_{j=2}^{k} (\beta_j + \min\{\alpha_{2j}; 0\}) > |\beta_1| \beta_2 \ldots \beta_k \end{aligned} \tag{16}$$

be valid. Then the equilibrium position $x = \bar{x}$ *of* (15) *is asymptotically stable for any nonnegative delay* τ*.*

Next, assume that the dynamics of agents are described by the double integrators

$$\ddot{x}_i(t) + h\dot{x}_i(t) = u_i, \quad i = 1, \dots, n. \tag{17}$$

Here $u_i \in \mathbb{R}$ is the control input of agent i, and h is a constant positive damping coefficient. We will say that the multiagent system achieves a consensus if $x_i(t) \to \xi$, $\dot{x}_i(t) \to 0$ as $t \to +\infty$, $i = 1, \dots, n$.

Choose a control law for (17) in the form (14). Then the corresponding closed-loop system can be rewritten as follows:

$$\ddot{x}(t) + h\dot{x}(t) = Ax(t) + Bx(t - \tau), \tag{18}$$

where A and B are constant matrices with the structures similar to those of (2) and (3), respectively.

With the aid of Theorem 4.2, it can be shown that the following theorem is valid.

Theorem 4.4 *Let the inequalities* (16) *hold. Then there exists a number* $h_0 > 0$ *such that if* $h \geq h_0$*, then the equilibrium position* $x = \bar{x}$*,* $\dot{x} = 0$ *of* (18) *is asymptotically stable for any nonnegative delay* τ*.*

5 Results of Numerical Simulation

To illustrate the effectiveness of the proposed approaches, consider a group consisting of six agents. Let the control law be of the form (14).

For the simulation, we choose $\alpha_1 = 1$, $\alpha_2 = -0.25$, $\alpha_3 = 1$, $\alpha_4 = -0.25$, $\alpha_5 = 1$, $\alpha_6 = -0.1$, $\beta_1 = -0.5$, $\beta_2 = 0.5$; $\beta_3 = 1$, $\gamma = 2$, $\tau = 10$, $\xi = 0.5$. In addition, it is assumed that $x(t) \equiv (0.1, 0.4667, 0.7, 0.2, 0.5, 0.2)^\top$ for $t \in [-10, 0]$.

Figure 2 corresponds to the case where the agent dynamics are described by the first order integrators. We can see the convergence of agents to the desired equilibrium position.

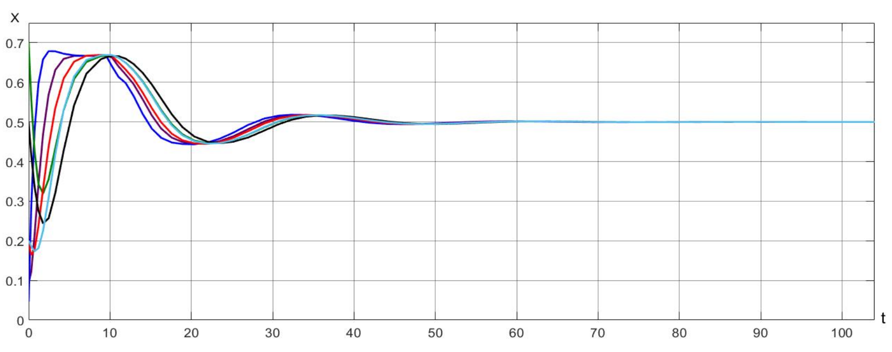

Figure 2: The agent time history (first order integrators).

Next, consider the double integrators (17). Figures 3 and 4 demonstrate that if $h = 0.2$, then the equilibrium position is unstable, whereas if $h = 2$, then the agents achieve the consensus.

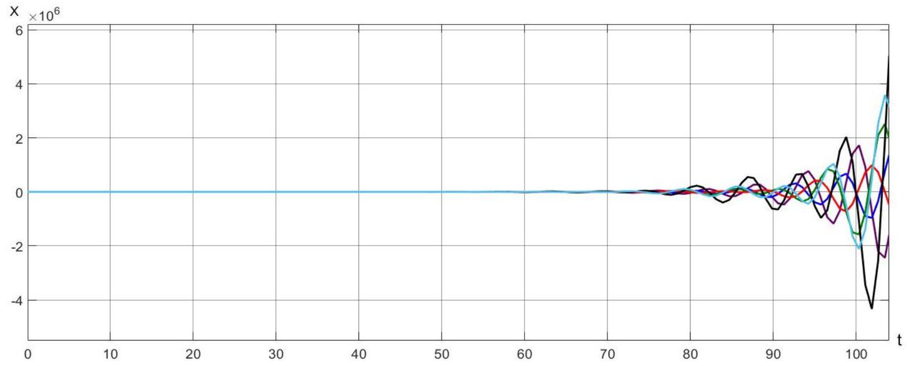

Figure 3: The agent time history (double integrators, $h = 0.2$).

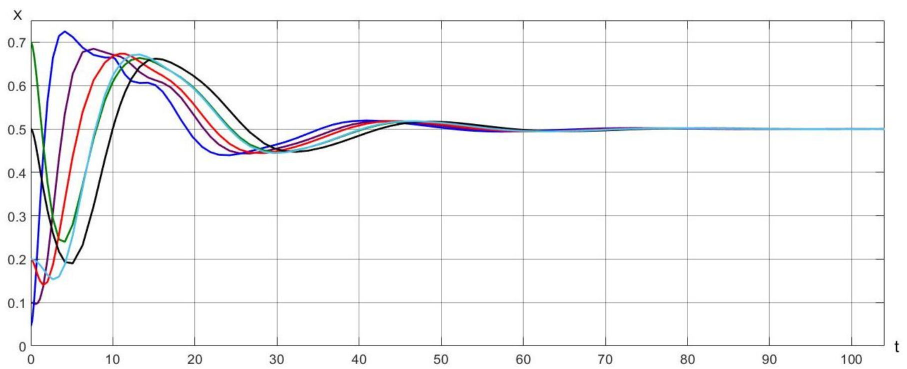

Figure 4: The agent time history (double integrators, $h = 2$).

6 Conclusion

In the present paper, simple necessary and sufficient conditions of the diagonal Riccati stability are derived for a class of pairs of matrices with special structures. These conditions are formulated in terms of algebraic inequalities for the entries of the matrices under consideration. We have shown that the obtained result can be used for the analysis of absolute stability of the Persidskii-type systems, the determination of delay-independent stability conditions for a mechanical system with a special structure of connections and the construction of decentralized controls providing the achievement of a consensus for some types of multiagent systems.

An application of the developed approaches to wider classes of matrices and time-delay systems is our future work.

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