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Extensions of Schauder's and Darbo's Fixed Point Theorems

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Abstract: In this paper, some new extensions of Schauder's and Darbo's fixed point theorems are given. As applications of the main results, the existence of global solutions for first-order nonlinear integro-differential equations of mixed type in a real Banach space is investigated.

Keywords: *nonlinear integro-differential equation; Darbo fixed point theorem; Schauder fixed point theorem; Kuratowksi measure of noncompactness.*

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1 Introduction

It is well known that the following two fixed points are very important.

Theorem 1.1 (Schauder's fixed point theorem) *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in Ω.*

The Schauder fixed point theorem plays an important role in nonlinear analysis. In 1955, Darbo [9] proved a fixed point property for set-contraction on a closed, bounded and convex subset of Banach spaces in terms of the measure of noncompactness, which was first defined by Kuratowski [17].

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Theorem 1.2 (Darbo's fixed point theorem) *Let $\Omega \neq \emptyset$ be a bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\alpha(TX) \leq k\alpha(X) \tag{1}$$

for any nonempty subset $X \subset \Omega$, where α is a measure of noncompactness defined in E. Then T has a fixed point in Ω.

Darbo's fixed point theorem is a significant extension of the Schauder fixed point theorem, and it also plays a key role in nonlinear analysis, especially in proving the existence of solutions for many classes of nonlinear equations. Since then, some generalizations of Darbo's fixed point theorem have appeared. For example, we refer the reader to [1–3, 6, 13, 23] and the references therein.

Recently, the authors of [21] established the following new fixed point theorem, which is a generalization of Darbo's fixed point theorem.

Theorem 1.3 (See [21, Lemma 2.4]) *Let F be a closed and convex subset of a real Banach space E, $A : F \to F$ be a continuous operator, and $A(F)$ be bounded. For any bounded subset $B \subset F$, put*

$$\widetilde{A}^1(B) = A(B) \quad \text{and} \quad \widetilde{A}^{n+1}(B) = A(\overline{\mathrm{co}}(\widetilde{A}^n(B))), \quad n \in \mathbb{N}.$$

If there exist a constant $0 \leq k < 1$ and $n_0 \in \mathbb{N}$ such that for any bounded subset $B \subset F$,

$$\alpha(\widetilde{A}^{n_0}(B)) \leq k\alpha(B), \tag{2}$$

then A has a fixed point in F.

As an application of their result, the authors in [21] investigated the existence of global solutions of the Volterra type integral equation

$$u(t) = h(t) + \int_0^t G(t, s)f(s, u(s), (Tu)(s), (Su)(s))\mathrm{d}s, t \in J, \tag{3}$$

where $J = [0, a]$, $a > 0$, $f \in \mathrm{C}(J \times E \times E \times E, E)$,

$$(Tu)(t) = \int_0^t k(t, s)u(s)\mathrm{d}s, \quad (Su)(t) = \int_0^a h(t, s)u(s)\mathrm{d}s, \quad t \in J,$$

$k \in \mathrm{C}(D, \mathbb{R})$, $h \in \mathrm{C}(D_0, \mathbb{R})$,

$$D = \{(t, s) \in \mathbb{R}^2 :\ 0 \leq s \leq t \leq a\}, \quad D_0 = \{(t, s) \in \mathbb{R}^2 :\ 0 \leq t, s \leq a\},$$

and $\mathbb{R}$ denotes the set of real numbers. The main results of [21] extend and improve related results in [11, 12, 20–22]. For other results concerning integro-differential equations, we refer to [4, 5, 7, 8, 10, 14–16].

Motivated by the above works, in this paper, we first establish a new fixed point theorem (Theorem 2.1), which is an extension of Schauder's fixed point theorem. Then, by using this extended Schauder fixed point theorem, we get a new extension of Darbo's fixed point theorem (Theorem 2.2). As an application of the new extended Darbo fixed point theorem, we obtain the existence of global solutions of (3). The existence result (Theorem 3.1) includes and extends and improves related results in [11, 12, 20–22].

This paper is organized as follows. In Section 2, we present our main results, the extensions of Schauder's and Darbo's fixed point theorems. In Section 3, in order to demonstrate the applicability of our main results, we obtain the existence of global solutions of (3).

2 Main Results

Throughout this paper, let $\mathrm{C}(J,E)$ denote the Banach space of all continuous mappings $u : J \to E$ with norm $\|u\|_c = \max_{t\in J} \|u(t)\|$, while $\mathrm{C}^1(J,E)$ denotes the Banach space of all $u \in \mathrm{C}(J,E)$ such that u' is continuous on J with norm $\|u\|_{c^1} = \max\{\|u\|_c, \|u'\|_c\}$. Let α denote the Kuratowski measure of noncompactness in E and $\mathrm{C}(J,E)$. Please, see [18] for more details on the Kuratowski measure of noncompactness. For any $B \in \mathrm{C}(J,E)$, $t \in J$, let

$$
\begin{aligned}
B(t) &= \{u(t) : \ u \in B\} \subset E,\\
(TB)(t) &= \left\{\int_0^t k(t,s)u(s)\mathrm{d}s : \ u \in B\right\},\\
(SB)(t) &= \left\{\int_0^a h(t,s)u(s)\mathrm{d}s : \ u \in B\right\}.
\end{aligned}
$$

For any $R > 0$, let

$$T_R = \{x \in E : \ \|x\| \le R\} \quad \text{and} \quad B_R = \{u \in \mathrm{C}(J,E) : \ \|u\|_c \le R\}.$$

Lemma 2.1 (See [12]) *If $B \subset \mathrm{C}(J,E)$ is bounded and equicontinuous, then $\overline{\mathrm{co}}(B) \subset \mathrm{C}(J,E)$ is also bounded and equicontinuous.*

Lemma 2.2 (See [12]) *If $B \subset \mathrm{C}(J,E)$ is bounded and equicontinuous, then $\alpha(B(t))$ is continuous on J and*

$$\alpha\left(\int_J B(s)\mathrm{d}s\right) \le \int_J \alpha(B(s))\mathrm{d}s.$$

Lemma 2.3 (See [19]) *If f is bounded and uniformly continuous on $J \times T_R \times T_R \times T_R$ for all $R > 0$ and $B \subset \mathrm{C}(J,E)$ is bounded and equicontinuous, then $\{f(t,u(t),(Tu)(t),(Su)(t)) : \ u \in B\}$ is bounded and equicontinuous in $\mathrm{C}(J,E)$.*

First, we give the extension of Schauder's fixed point theorem.

Theorem 2.1 *Let D be a closed and convex subset of a real Banach space E. Suppose that the operator $A : D \to D$ is continuous. If there exists $n_0 \in \mathbb{N}$ such that $\widetilde{A}^{n_0-1}(D)$ is bounded and $\alpha(\widetilde{A}^{n_0}(D)) = 0$, where*

$$\widetilde{A}^0(D) = D \quad \text{and} \quad \widetilde{A}^n(D) = \overline{\mathrm{co}}(A(\widetilde{A}^{n-1}(D))), \quad n \in \mathbb{N},$$

then A has a fixed point in D.

Proof. Since $A(D) \subset D$ and D is a closed convex subset, we have

$$\widetilde{A}^1(D) = \overline{\mathrm{co}}(A(D)) \subset \overline{\mathrm{co}}(D) = D = \widetilde{A}^0(D).$$

Hence,

$$\widetilde{A}^2(D) = \overline{\mathrm{co}}(A(\widetilde{A}^1(D))) \subset \overline{\mathrm{co}}(A(D)) = \widetilde{A}^1(D).$$

By the method of mathematical induction, we can deduce that

$$\widetilde{A}^n(D) \subset \widetilde{A}^{n-1}(D), \quad n \in \mathbb{N}.$$

Thus,

$$A(\widetilde{A}^{n_0-1}(D)) \subset \overline{\text{co}}(A(\widetilde{A}^{n_0-1}(D))) = \widetilde{A}^{n_0}(D) \subset \widetilde{A}^{n_0-1}(D).$$

Consequently, $A : \widetilde{A}^{n_0-1}(D) \to \widetilde{A}^{n_0-1}(D)$ is continuous. Moreover, for any bounded subset $S \subset \widetilde{A}^{n_0-1}(D)$, we get

$$A(S) \subset A(\widetilde{A}^{n_0-1}(D)) \subset \widetilde{A}^{n_0}(D),$$

and hence,

$$\alpha(A(S)) \leq \alpha(\widetilde{A}^{n_0}(D)) = 0.$$

Noting that $\widetilde{A}^{n_0-1}(D)$ is a closed bounded convex subset of E, we know from Schauder's fixed point theorem that A has a fixed point in $\widetilde{A}^{n_0-1}(D) \subset D$.

Remark 2.1 The well-known Schauder fixed point theorem is the special case $n_0 = 1$ of Theorem 2.1.

By using Theorem 2.1, we now present a new extension of Darbo's fixed point theorem.

Theorem 2.2 *Let D be a closed and convex subset of a real Banach space E. Suppose that the operator $A : D \to D$ is continuous. For any bounded subset $B \subset E$, put*

$$\widetilde{A}^0(B) = B \quad \text{and} \quad \widetilde{A}^n(B) = A(\overline{\text{co}}(\widetilde{A}^{n-1}(B))), \quad n \in \mathbb{N}. \tag{4}$$

If there exists $n_0 \in \mathbb{N}$ such that $\widetilde{A}^{n_0-1}(D)$ is bounded and for any decreasing sequence of sets $\{B_n\} \subset D$, $n \in \mathbb{N}$,

$$\alpha\left(\widetilde{A}^{n_0}\left(\bigcap_{n=1}^{\infty} B_n\right)\right) = 0, \tag{5}$$

then A has a fixed point in D.

Proof. Let

$$B_0 = D \quad \text{and} \quad B_n = \overline{\text{co}}(\widetilde{A}^{n_0}(B_{n-1})), \quad n \in \mathbb{N}. \tag{6}$$

Then (6) and $A : D \to D$ imply that

$$B_1 = \overline{\text{co}}(\widetilde{A}^{n_0}(B_0)) \subset D = B_0.$$

Hence, $\widetilde{A}^{n_0}(B_1) \subset \widetilde{A}^{n_0}(B_0)$. Therefore,

$$B_2 = \overline{\text{co}}(\widetilde{A}^{n_0}(B_1)) \subset \overline{\text{co}}(\widetilde{A}^{n_0}(B_0)) = B_1.$$

By the method of mathematical induction, we can prove

$$B_n \subset B_{n-1}, \quad n \in \mathbb{N}. \tag{7}$$

If we set

$$\widehat{B} = \bigcap_{n=0}^{\infty} B_n, \tag{8}$$

where $\{B_n\}$ is defined as in (6), then $\widehat{B}$ is a nonempty and convex subset in D. Hence, (5), (7), and (8) imply

$$\alpha(\widetilde{A}^{n_0}(\widehat{B})) = 0. \tag{9}$$

Since $\widetilde{A}^{n_0-1}(D)$ is bounded, we get that

$$\widetilde{A}^{n_0-1}(\widehat{B}) \text{ is bounded.} \tag{10}$$

Next, we shall prove

$$A(\widehat{B}) \subset \widehat{B}. \tag{11}$$

In fact, from $B_1 \subset \overline{\text{co}}(\widetilde{A}^{n_0-1}(B_0))$, we have

$$A(B_1) \subset A(\overline{\text{co}}(\widetilde{A}^{n_0-1}(B_0))) = \widetilde{A}^{n_0}(B_0) \subset \overline{\text{co}}(\widetilde{A}^{n_0}(B_0)) = B_1.$$

By the same method, we can prove $A(B_n) \subset B_n$, $n \in \mathbb{N}$. Hence, we get

$$A(\widehat{B}) = \bigcap_{n=0}^{\infty} A(B_n) \subset \bigcap_{n=0}^{\infty} B_n = \widehat{B}.$$

Then (11) holds. From (9), (10), (11), and Theorem 2.1, we deduce that A has a fixed point in $\widehat{B} \subset D$.

Remark 2.2 When

$$B_n \equiv D, \quad n \in \mathbb{N}$$

in Theorem 2.2, then Theorem 2.1 is obtained.

Remark 2.3 When

$$n_0 = 1, \quad B_n \equiv D, \quad n \in \mathbb{N}$$

in Theorem 2.2, then Theorem 1.2, i.e., Darbo's fixed point theorem is obtained. So Theorem 2.2 includes and extends Darbo's fixed point theorem.

Remark 2.4 Comparing with [21, Lemma 2.4], i.e., Theorem 1.3, the conclusion of Theorem 2.2 is the same. But the conditions are different. First, the assumption that $\widetilde{A}^{n_0-1}(D)$ is bounded is weaker than that $A(D)$ is bounded in [21, Lemma 2.4]. After that, we only need to consider the decreasing sequences without the boundedness $\{B_n\} \subset D$ in (5), while [21, Lemma 2.4] needs to consider all bounded sets $B \subset F$ in (2). Finally, (5) and (2) cannot be deduced from each other. Above all, Theorem 2.2 is a good supplement to the extension of Darbo's fixed point theorem.

3 Applications

Now, as an application of Theorem 2.2, we give an existence theorem for global solutions of (3).

Theorem 3.1 *Let E be a real Banach space. Assume*

(H_1) *For any $R > 0$, f is bounded and uniformly continuous on $J \times T_R \times T_R \times T_R$, and*

$$\limsup_{R\to\infty} \frac{M(R)}{R} < \frac{1}{aa_0b}, \tag{12}$$

where

$$a_0 = \max\{1, ak_0, ah_0\}, \quad k_0 = \max\{|k(t,s)| : (t,s) \in D\},$$
$$h_0 = \max\{|h(t,s)| : (t,s) \in D_0\}, \quad b = \max\{G(t,s) : (t,s) \in D\},$$
$$M(R) = \sup\{\|f(t,x,y,z)\| : (t,x,y,z) \in J \times T_R \times T_R \times T_R, t \in J\}.$$

(H_2) *There exist nonnegative Lebesgue integrable functions* $L_i \in \mathrm{L}(J, R^+)$ *such that for any decreasing sequences of bounded sets* $\{D_{in}\} \subset E$, $\alpha(D_{in}) \to 0$, $n \to \infty$, $i = 1, 2, 3$ *and* $t \in J$,

$$\alpha(f(t, D_{1n}, D_{2n}, D_{3n})) \le \sum_{i=1}^{3} L_i(t)\alpha(D_{in}). \tag{13}$$

Then (3) *has at least one global solution in* $\mathrm{C}^1(J, E)$.

Proof. First, we define an operator $A : \mathrm{C}(J, E) \to \mathrm{C}(J, E)$ by

$$(Au)(t) = h(t) + \int_0^t G(t, s)f(s, u(s), (Tu)(s), (Su)(s))\mathrm{d}s, \quad u \in \mathrm{C}(J, E). \tag{14}$$

Note that since $u \in \mathrm{C}^1(J, E)$ is a solution of (3) if and only if $u \in \mathrm{C}(J, E)$ is a solution of the integral equation

$$u(t) = (Au)(t), \quad t \in J,$$

we only need to prove that A has a fixed point. Since f is uniformly continuous on $J \times T_R \times T_R \times T_R$, we can easily see that $A : \mathrm{C}(J, E) \to \mathrm{C}(J, E)$ is continuous and bounded. On account of (12), there exist $0 < r < (aa_0b)^{-1}$ and $R_0 > 0$ such that for any $R \ge a_0R_0$,

$$\frac{M(R)}{R} < r. \tag{15}$$

Let

$$R^* = \max\left\{R_0, \|h\|_c (1 - aa_0b)^{-1}\right\}. \tag{16}$$

Then, by using (14) and (15), it is not difficult to verify that $A(B_{R^*}) \subset \mathrm{C}(J, E)$ is equicontinuous and bounded, and $A : B_{R^*} \to B_{R^*}$ is bounded and continuous. Set $D = \overline{\mathrm{co}}(A(B_{R^*}))$. Then, from Lemma 2.1, we get that $D \subset B_{R^*}$ is bounded and equicontinuous and

$$A : D \to D \text{ is continuous and bounded.} \tag{17}$$

For any decreasing sequence of bounded sets $\{B_m\} \subset D$, $m \in \mathbb{N}$, by (H_1) and (14), we have that $A(B_m)$ is bounded and equicontinuous. Hence, from Lemma 2.1, Lemma 2.3, and (4), we get for any $n \in \mathbb{N}$ that $\widetilde{A}^n(B_m)$ is bounded and equicontinuous on J, and so

$$\alpha(\widetilde{A}^n(B_m)) = \max_{t \in J} \alpha((\widetilde{A}^n(B_m))(t)), \quad m \in \mathbb{N}.$$

Next, we show that for any $n_0 \in \mathbb{N}$, we have

$$\widetilde{A}^{n_0-1}(D) \text{ is bounded,} \tag{18}$$

and for any decreasing sequence of sets $\{B_m\} \subset D$, $\alpha(B_m) \to 0$, $m \in \mathbb{N}$,

$$\alpha\left(\widetilde{A}^{n_0}\left(\bigcap_{m=1}^{\infty} B_m\right)\right) = 0. \tag{19}$$

Indeed, (18) follows from the fact that $D \subset B_{R^*}$ is bounded. Furthermore, from (13), (14), and (15), we get

$$\begin{aligned}
&\alpha((\widetilde{A}^1(B_m))(t)) \\
&= \alpha\left(\int_0^t G(t,s)f(s,(\overline{\mathrm{co}}B_m)(s),(T(\overline{\mathrm{co}}B_m))(s),(S(\overline{\mathrm{co}}B_m))(s))\mathrm{d}s\right) \\
&\le b\int_0^t [L_1(s)\alpha((\overline{\mathrm{co}}B_m)(s)) + L_2(s)\alpha((T(\overline{\mathrm{co}}B_m))(s)) + L_3(s)\alpha((S(\overline{\mathrm{co}}B_m))(s))]\mathrm{d}s \\
&\le b\int_0^t [L_1(s)\alpha((\overline{\mathrm{co}}B_m)(s)) + L_2(s)k_0\alpha((\overline{\mathrm{co}}B_m)(s)) + L_3(s)h_0\alpha((\overline{\mathrm{co}}B_m)(s))]\mathrm{d}s \\
&= b\int_0^t [L_1(s)\alpha((B_m(s)) + L_2(s)k_0\alpha(B_m(s)) + L_3(s)h_0\alpha(B_m(s))]\mathrm{d}s \\
&\to 0, \quad m \to \infty.
\end{aligned}$$

Assume

$$\alpha((\widetilde{A}^k(B_m))(t)) \to 0, \quad m \to \infty, \quad k \in \mathbb{N} \setminus \{1\}.$$

Then

$$\begin{aligned}
&\alpha((\widetilde{A}^{k+1}(B_m))(t)) \\
&= \alpha\left(\int_0^t G(t,s)f(s,(\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s),T(\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s),S(\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s))\mathrm{d}s\right) \\
&\le b\int_0^t \Big[L_1(s)\alpha((\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s)) + L_2(s)\alpha(T(\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s)) \\
&\quad + L_3(s)\alpha(S(\overline{\mathrm{co}}\widetilde{A}^k(B_m)(s)))\Big]\mathrm{d}s \\
&\le b\int_0^t \Big[L_1(s)\alpha((\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s)) + L_2(s)k_0\alpha((\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s)) \\
&\quad + L_3(s)h_0\alpha((\overline{\mathrm{co}}\widetilde{A}^k(B_m))(s))\Big]\mathrm{d}s \\
&= b\int_0^t \Big[L_1(s)\alpha((\widetilde{A}^k(B_m))(s)) + L_2(s)k_0\alpha((\widetilde{A}^k(B_m))(s)) \\
&\quad + L_3(s)h_0\alpha((\widetilde{A}^k(B_m))(s))\Big]\mathrm{d}s \\
&\to 0, \quad m \to \infty.
\end{aligned}$$

Consequently,

$$\alpha(\widetilde{A}^n(B_m)) \to 0, \quad m \to \infty, \quad n \in \mathbb{N},$$

and so $\alpha(\widetilde{A}^{n_0}(B_m)) \to 0$, $m \to \infty$. Thus, we have

$$\alpha\left(\widetilde{A}^{n_0}\left(\bigcap_{m=1}^{\infty} B_m\right)\right) \le \alpha(\widetilde{A}^{n_0}(B_m)) \to 0, \quad m \to \infty.$$

Hence, (19) holds. It follows from Theorem 2.2, (17), (18), and (19) that A has a fixed point in D. Thus, (3) has at least one global solution in $\mathrm{C}^1(J,E)$.

Remark 3.1 The main result of [21], i.e., [21, Theorem 3.1] is as follows: *Let E be a real Banach space. Assume*

(H_3) *For any $R > 0$, f is bounded and uniformly continuous on $J \times T_R \times T_R \times T_R$, and*

$$\limsup_{R\to\infty} \frac{M(R)}{R} < \frac{1}{aa_0b},$$

where

$$a_0 = \max\{1, ak_0, ah_0\}, \quad b = \max\{G(t,s) : \ (t,s) \in D\},$$
$$M(R) = \sup\{\|f(t,x,y,z)\| : \ (t,x,y,z) \in J \times T_R \times T_R \times T_R, t \in J\}.$$

(H_4) *There exist nonnegative Lebesgue integrable functions $L_i \in \mathrm{L}(J, \mathbb{R}^+)$ such that for any bounded sets $\{D_i\} \subset E$ and $t \in J$,*

$$\alpha(f(t, D_1, D_2, D_3)) \leq \sum_{i=1}^{3} L_i(t)\alpha(D_i). \tag{20}$$

Then (3) *has at least one global solution in* $\mathrm{C}^1(J, E)$. Comparing [21, Theorem 3.1] with Theorem 3.1 above, we can see that the only difference is between (13) and (20). For all bounded sets $\{D_i\} \subset E$, (20) should hold. For only those bounded and decreasing sequences $\{D_{in}\} \subset E$, $\alpha(D_{in}) \to 0$, $i = 1, 2, 3$, we need that (13) holds. So (13) is weaker than (20). Moreover, (20) is a special case of (13) when $D_{in} \equiv D_i$, $i = 1, 2, 3$, $n \in \mathbb{N}$. Thus, Theorem 3.1 includes and extends [21, Theorem 3.1], which extended and improved the main results of [11, 12, 20, 22]. Consequently, Theorem 3.1 in this paper extends and improves not only the main results of [21], but also related results of [11, 12, 20, 22].

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References

[1] Asadollah Aghajani and M. Aliaskari. Commuting mappings and generalization of darbo's fixed point theorem. *Math. Sci. Lett.* **4** (2) (2015) 187–190.

[2] Asadollah Aghajani, Reza Allahyari, and Mohammad Mursaleen. A generalization of Darbo's theorem with application to the solvability of systems of integral equations. *J. Comput. Appl. Math.* **260** (2014) 68–77.

[3] Asadollah Aghajani, Józef Banaś, and Navid Sabzali. Some generalizations of Darbo fixed point theorem and applications. *Bull. Belg. Math. Soc. Simon Stevin* **20** (2) (2013) 345–358.

[4] Theodore A. Burton. Existence and uniqueness results by progressive contractions for integro-differential equations. *Nonlinear Dyn. Syst. Theory* **16** (4) (2016) 366–371.

[5] Theodore A. Burton. Integro-differential equations, compact maps, positive kernels, and Schaefer's fixed point theorem. *Nonlinear Dyn. Syst. Theory* **17** (1) (2017) 19–28.

[6] Longsheng Cai and Jin Liang. New generalizations of Darbo's fixed point theorem. *Fixed Point Theory Appl.* **156** (2015) 8 pages.

[7] Alka Chadha. Mild solution for impulsive neutral integro-differential equation of Sobolev type with infinite delay. *Nonlinear Dyn. Syst. Theory* **15** (3) (2015) 272–289.

[8] Jaydev Dabas. Existence and uniqueness of solutions to quasilinear integro-differential equations by the method of lines. *Nonlinear Dyn. Syst. Theory* **11** (4) (2011) 397–410.

[9] Gabriele Darbo. Punti uniti in trasformazioni a codominio non compatto. *Rend. Sem. Mat. Univ. Padova* **24** (1955) 84–92.

[10] Zachary Denton and J. Diego Ramírez. Monotone method for finite systems of nonlinear Riemann-Liouville fractional integro-differential equations. *Nonlinear Dyn. Syst. Theory* **18** (2) (2018) 130–143.

[11] Dajun Guo. Solutions of nonlinear integrodifferential equations of mixed type in Banach spaces. *J. Appl. Math. Simulation* **2** (1) (1989) 1–11.

[12] Dajun Guo, V. Lakshmikantham, and Xinzhi Liu. *Nonlinear Integral Equations in Abstract Spaces*, vol. 373 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1996.

[13] Ahmed Hajji. A generalization of Darbo's fixed point and common solutions of equations in Banach spaces. *Fixed Point Theory Appl.* **62** (2013) 9 pages.

[14] Hossein Jafari and A. Azad. A computational method for solving a system of Volterra integro-differential equations. *Nonlinear Dyn. Syst. Theory* **12** (4) (2012) 389–396.

[15] Pradeep Kumar. On the new concepts of solutions and existence results for impulsive integro-differential equations with a deviating argument. *Nonlinear Dyn. Syst. Theory* **14** (1) (2014) 58–75.

[16] Pradeep Kumar, Rajib Haloi, Dhirendra Bahuguna, and Dwijendra N. Pandey. Existence of solutions to a new class of abstract non-instantaneous impulsive fractional integro-differential equations. *Nonlinear Dyn. Syst. Theory* **16** (1) (2016) 73–85.

[17] Kazimierz Kuratowski. Sur les espaces complets. *Fund. Math.* **15** (1) (1930) 301–309.

[18] V. Lakshmikantham. Some problems in integro-differential equations of Volterra type. *J. Integral Equations* **10** (1-3, suppl.) (1985) 137–146. Integro-differential evolution equations and applications (Trento, 1984).

[19] Lishan Liu. Iterative method for solutions and coupled quasi-solutions of nonlinear Fredholm integral equations in ordered Banach spaces. *Indian J. Pure Appl. Math.* **27** (10) (1996) 959–972.

[20] Lishan Liu. Existence of global solutions of initial value problems for nonlinear integro-differential equations of mixed type in Banach spaces. *J. Systems Sci. Math. Sci.* **20** (1) (2000) 112–116.

[21] Lishan Liu, Fei Guo, Congxin Wu, and Yonghong Wu. Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. *J. Math. Anal. Appl.* **309** (2) (2005) 638–649.

[22] Lishan Liu, Congxin Wu, and Fei Guo. Existence theorems of global solutions of initial value problems for nonlinear integrodifferential equations of mixed type in Banach spaces and applications. *Comput. Math. Appl.* **47** (1) (2004) 13–22.

[23] Ayub Samadi and Mohammad B. Ghaemi. An extension of Darbo's theorem and its application. *Abstr. Appl. Anal.* (2014), Art. ID 852324, 11 pages.