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Exact Solutions of a Klein-Gordon System by (G'/G)-Expansion Method and Weierstrass Elliptic Function Method

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Abstract: This paper deals with the exact solutions of a Klein-Gordon system of equations. The (G'/G)-expansion method has been employed to derive kink solutions, solitary wave solutions and singular solutions. Solitary wave solutions have also been derived for the Klein-Gordon system using the Weierstrass elliptic function method.

Keywords: *(G'/G)-expansion method; Klein-Gordon equation; solitary wave solutions; Weierstrass elliptic function.*

Mathematics Subject Classification (2010): 74J35, 34G20, 93C10.

1 Introduction

The nonlinear evolution equations (NLEEs) are the most important fields of research in applied mathematics and theoretical physics. There are several forms of NLEEs that arise in various branches of science and engineering [1–5]. Exact solutions of NLEEs play an important role as they provide a better insight into the various aspects of the problem which leads to significant applications. Several methods such as the tanh method [6–11], exponential function method [12], Jacobi elliptic function (JEF) method [13–15], mapping methods [16–21] have been applied in the last few decades and the results have been reported. Also, many physical phenomena have been governed by systems of

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partial differential equations (PDEs) and there have been significant contributions in this area [22, 23].

In this paper, we use the (G'/G)-expansion method [24–28] to find some exact solutions for a coupled Klein-Gordon equation [29]. The paper is organized as follows. In Section 2, we give a mathematical analysis of the (G'/G)-expansion method, in Section 3, we derive solitary wave solutions (SWSs) and kink solutions to the nonlinear Klein-Gordon system, in Section 4, we use the Weierstrass elliptic function (WEF) method [30] to derive SWSs of the Klein-Gordon system of equations, in Section 5 we write down the conclusion.

2 (G'/G)-Expansion Method

Consider the nonlinear partial differential equation (PDE)

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{1}$$

where $u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives. The traveling wave variable $\xi = x - ct$ reduces the PDE (1) to the ordinary differential equation (ODE)

$$P(u, -cu', u', -c^2u'', -cu'', u'', ...) = o, \tag{2}$$

where $u = u(\xi)$ and $'$ denotes differentiation with respect to ξ.

We suppose that the solution of equation (2) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \;\; a_m \neq 0, \tag{3}$$

where $a_i (i = 0, 1, 2, ..)$ are constants. Here, G satisfies the second order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{4}$$

with λ and μ being constants. The positive integer m can be determined by a balance between the highest order derivative term and the nonlinear term appearing in equation (2). By substituting equation (3) into equation (2) and using equation (4), we get a polynomial in G'/G. The coefficients of various powers of G'/G give rise to a set of algebraic equations for $a_i\,(i = 0, 1, 2, ..., m)$, λ and μ.

The general solution of equation (4) is a linear combination of sinh and cosh or of sine and cosine functions if $\Delta = \lambda^2 - 4\mu > 0$ or $\Delta = \lambda^2 - 4\mu < 0$, respectively. In this paper we consider only the first case and so,

$$G(\xi) = e^{-\lambda\xi/2}\left(C_1 \sinh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\,\xi) + C_2 \cosh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\,\xi)\right), \tag{5}$$

where C_1 and C_2 are arbitrary constants.

3 Klein-Gordon System of Equations

Consider the Klein-Gordon system of equations

$$u_{xx} - u_{tt} - u - 2u^3 - 2uv = 0, \tag{6}$$

$$v_x - v_t - 4uu_t = 0. \tag{7}$$

We seek TWSs of equations (6) and (7) in the form $u = u(\xi), \quad v = v(\xi), \quad \xi = x - ct$. Then equations (6) and (7) give

$$(1 - c^2)u'' - u - 2u^3 - 2uv = 0, \tag{8}$$

$$v' + cv' + 4cuu' = 0. \tag{9}$$

Integrating equation (9) with respect to ξ and using the solitary wave boundary conditions, we get

$$v = -\frac{2c}{1+c}u^2. \tag{10}$$

Substituting for v into equation (8), we obtain

$$(1 - c)(1 + c)^2 u'' - (1 + c)u - 2(1 - c)u^3 = 0. \tag{11}$$

Assuming the expansion $u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \ a_m \neq 0$ in equation (11) and balancing the nonlinear term and the derivative term, we get $m + 2 = 3m$ so that $m = 1$.

So, we assume a solution of equation (11) in the form

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \ a_1 \neq 0. \tag{12}$$

So, we can obtain

$$u'(\xi) = -a_1 \left(\frac{G'}{G}\right)^2 - \lambda a_1 \left(\frac{G'}{G}\right) - \mu a_1, \tag{13}$$

$$u''(\xi) = 2a_1 \left(\frac{G'}{G}\right)^3 + 3a_1\lambda \left(\frac{G'}{G}\right)^2 + (a_1\lambda^2 + 2a_1\mu) \left(\frac{G'}{G}\right) + a_1\lambda\mu, \tag{14}$$

$$u^3(\xi) = a_1^3 \left(\frac{G'}{G}\right)^3 + 3a_0a_1^2 \left(\frac{G'}{G}\right)^2 + 3a_0^2a_1 \left(\frac{G'}{G}\right) + a_0^3. \tag{15}$$

Now, substituting equations (12), (14) and (15) into equation (11) and collecting the coefficients of $\left(\frac{G'}{G}\right)^i, i = 0, 1, 2, 3$, we get

$$(1 - c)(1 + c)^2 a_1\lambda\mu - (1 + c)a_0 - 2(1 - c)a_0^3 = 0, \tag{16}$$

$$a_1(1-c)(1+c)^2(\lambda^2+2\mu)-(1+c)a_1-6(1-c)a_0^2a_1=0, \tag{17}$$

$$3a_1\lambda(1-c)(1+c)^2-6a_0a_1^2(1-c)=0, \tag{18}$$

$$2(1-c)(1+c)^2a_1-2(1-c)a_1^3=0. \tag{19}$$

From equation (19), we get

$$a_1=\pm(1+c). \tag{20}$$

Equation (18) leads us to

$$a_0=\pm\frac{\lambda}{2}(1+c)=\frac{\lambda}{2}a_1. \tag{21}$$

When $\mu=0$ in equation (17), we get $\lambda=\pm\sqrt{\frac{2}{c^2-1}}$ and when $\lambda=0$, we get $\mu=\frac{1}{2(1-c^2)}$. In both cases, $\Delta=\lambda^2-4\mu=\frac{2}{c^2-1}$.

Equation (17) is identically satisfied in both cases without any constraints on the coefficients of the governing equation.

Case 1: $\mu=0,\ \lambda=\sqrt{\frac{2}{c^2-1}}$,

$$u_1(x,t)=\pm\sqrt{-\frac{1+c}{2(1-c)}}\left(1+\frac{(C_1-C_2)(1-\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct))}{C_1\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct)+C_2}\right), \tag{22}$$

$$v_1(x,t)=\frac{c}{1-c}\left(1+\frac{(C_1-C_2)(1-\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct))}{C_1\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct)+C_2}\right)^2. \tag{23}$$

Here, $C_1\neq\pm C_2$ and $c>1$.

Figure 1 and Figure 2 represent the solutions given by equations (22) and (23), respectively.

Case 2: $\mu=0,\ \lambda=-\sqrt{\frac{2}{c^2-1}}$,

$$u_2(x,t)=\pm\sqrt{-\frac{1+c}{2(1-c)}}\left(1+\frac{(C_1+C_2)(1-\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct))}{C_2-C_1\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct)}\right), \tag{24}$$

$$v_2(x,t)=\frac{c}{1-c}\left(1+\frac{(C_1+C_2)(1-\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct))}{C_2-C_1\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,(x-ct)}\right)^2. \tag{25}$$

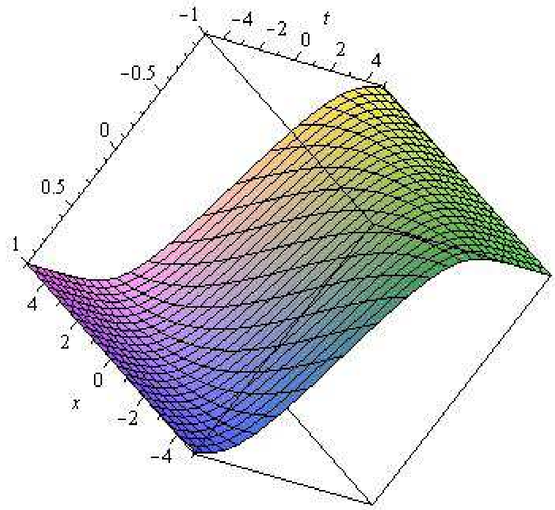

Figure 1: The solution for $u_1(x,t)$, $c = 3, C_1 = 0, C_2 = 1$.

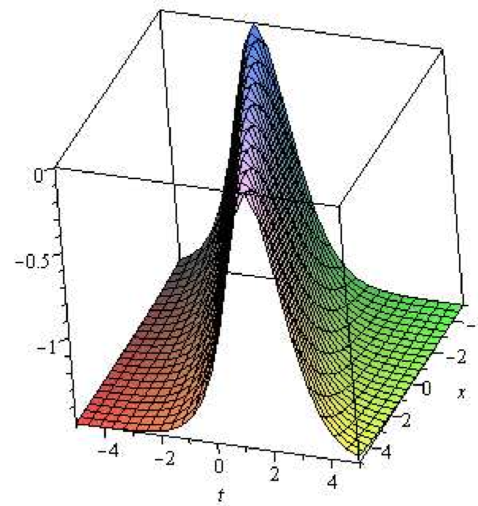

Figure 2: The solution for $v_1(x,t)$, $c = 3, C_1 = 0, C_2 = 1$.

Here also, $C_1 \neq \pm C_2$ and $c > 1$.

Case 3: $\lambda = 0,\ \mu = \dfrac{1}{2(1-c^2)}$,

$$u_3(x,t) = \pm\sqrt{-\frac{1+c}{2(1-c)}}\left(\frac{C_1 + C_2\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,\xi}{C_1\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,\xi + C_2}\right), \tag{26}$$

$$v_3(x,t) = \frac{c}{1-c}\left(\frac{C_1 + C_2\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,\xi}{C_1\tanh\frac{1}{2}\sqrt{\frac{2}{c^2-1}}\,\xi + C_2}\right)^2. \tag{27}$$

In this case also, we have the same restrictions on c, C_1 and C_2.

4 Weierstrass Elliptic Function Solutions of Klein-Gordon Equation

The Weierstrass elliptic function (WEF) $\wp(\xi; g_2, g_3)$ with invariants g_2 and g_3 satisfy

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{28}$$

where g_2 and g_3 are related by the inequality

$$g_2^3 - 27g_3^2 > 0. \tag{29}$$

The WEF $\wp(\xi)$ is related to the JEFs by the following relations:

$$\mathrm{sn}(\xi) = [\wp(\xi) - e_3]^{-1/2}, \tag{30}$$

$$\mathrm{cn}(\xi) = \left[\frac{\wp(\xi) - e_1}{\wp(\xi) - e_3}\right]^{1/2}, \tag{31}$$

$$\mathrm{dn}(\xi) = \left[\frac{\wp(\xi) - e_2}{\wp(\xi) - e_3}\right]^{1/2}, \tag{32}$$

where e_1, e_2, e_3 satisfy

$$4z^3 - g_2 z - g_3 = 0 \tag{33}$$

with

$$e_1 = \frac{1}{3}(2 - m^2),\ \ e_2 = \frac{1}{3}(2m^2 - 1),\ \ e_3 = -\frac{1}{3}(1 + m^2). \tag{34}$$

From equation (34), one can see that the modulus m of the JEF and the $e's$ of the WEF are related by

$$m^2 = \frac{e_2 - e_3}{e_1 - e_3}. \tag{35}$$

We consider the ODE of order $2k$ given by

$$\frac{d^{2k}\phi}{d\xi^{2k}} = f(\phi; r+1), \tag{36}$$

where $f(\phi; r+1)$ is an $(r+1)$ degree polynomial in ϕ. We assume that

$$\phi = \gamma Q^{2s}(\xi) + \mu \tag{37}$$

is a solution of equation (36), where γ and μ are arbitrary constants and $Q^{(2s)}(\xi)$ is the $(2s)^{\text{th}}$ derivative of the reciprocal Weierstrass elliptic function (RWEF) $Q(\xi) = \dfrac{1}{\wp(\xi)}$, $\wp(\xi)$ being the WEF.

It can be shown that the $(2s)^{\text{th}}$ derivative of the RWEF $Q(\xi)$ is a $(2s+1)$ degree polynomial in $Q(\xi)$ itself. Therefore, for ϕ to be a solution of equation (36), we should have the relation

$$2k - r = 2rs. \tag{38}$$

So, it is necessary that $2k \geq r$ for us to assume a solution in the form of equation (37). But this is in no way a sufficient condition for the existence of the PWS in the form of equation (37).

Now, we shall search for the WEF solutions of equation (11). For a solution in the form of equation (37), we should have $r = 2$ and $k = 1$ so that $s = 0$. So, our solution will be

$$u(\xi) = \frac{\gamma}{\wp(\xi)} + \mu. \tag{39}$$

Substituting equation (39) into equation (11) and equating the coefficients of like powers of $\wp(\xi)$ to zero, we obtain

$$\wp^3(\xi): \quad 2\gamma(1-c)(1+c)^2 - \mu(1+c) - 2\mu^3(1-c) = 0, \tag{40}$$

$$\wp^2(\xi): \quad -\gamma(1+c) - 6\gamma\mu^2(1-c) = 0, \tag{41}$$

$$\wp(\xi): \quad -\frac{3}{2}\gamma g_2(1-c)(1+c)^2 - 6\gamma^2\mu(1-c) = 0, \tag{42}$$

$$\wp^0(\xi): \quad -2\gamma g_3(1-c)(1+c)^2 - 2\gamma^3(1-c) = 0. \tag{43}$$

From equations (40)-(43), it can be found that

$$\gamma = \pm(1+c)\sqrt{-g_3}, \tag{44}$$

$$\mu = \pm\sqrt{-\frac{1+c}{6(1-c)}}, \tag{45}$$

$$g_2 = -\frac{4\gamma\mu}{(1+c)^2}. \tag{46}$$

From equations (44), (45) and (46), one can infer that $g_3 < 0$, $|c|$ should be greater than 1 and γ and μ are of opposite signs as g_2 should always be positive. Equation (40) leads us to the value of g_3 given by

$$g_3 = \frac{1}{54(1-c)^3(1+c)^3}, \tag{47}$$

which clearly indicates that $g_3 < 0$ when $|c| > 1$. The condition $g_2^3 - 27g_3^2 > 0$ gives the constraint relation

$$\gamma\mu < -\frac{1}{12(4)^{1/3}(1-c)^2}. \tag{48}$$

One may observe that both sides of the inequality (48) are always negative as γ and μ are of opposite signs.

The equations (30)–(32) will give rise to the same PWS of equation (11) which can be obtained using equation (39) with the help of equation (34). Thus, the PWS of equation (11) in terms of JEFs can be written as

$$u(\xi) = \frac{\gamma \text{sn}^2(\xi)}{1 - \frac{1}{3}(1+m^2)\text{sn}^2(\xi)} + \mu. \tag{49}$$

As $m \to 1$, the SWS of the Klein-Gordon system given by equations (6) and (7) are

$$u(x,t) = \frac{\gamma \tanh^2(x-ct)}{1 - \frac{2}{3}\tanh^2(x-ct)} + \mu \tag{50}$$

and

$$v(x,t) = -\frac{2c}{1+c}\left[\frac{\gamma \tanh^2(x-ct)}{1 - \frac{2}{3}\tanh^2(x-ct)} + \mu\right]^2, \tag{51}$$

where γ and μ are given by equations (44) and (45).

5 Conclusions

The (G'/G)-expansion method has been applied to a Klein-Gordon system of equations. The kink wave solutions and SWSs have been graphically illustrated. It was found that there are no restrictions on the coefficients in the governing equation for the solutions in terms of hyperbolic functions to exist. The WEF method has also been applied to the Klein-Gordon system to derive SWSs. We intend to apply the method for higher order and higher dimensional PDEs of physical interest.

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