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Stability Analysis for Stochastic Neural Networks with Markovian Switching and Infinite Delay in a Phase Space

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Abstract: This paper focuses on global stochastic (asymptotic) stability for a kind of stochastic neutral networks with infinite delay and Markovian switching in a fading memory phase space. Our approach is based on the Lyapunov method, stochastic analysis technique and M-matrix theory. The results complete some existing ones. Two examples are illustrated for demonstration of applicability and effectiveness of the proved theoretical theorems.

Keywords: *Markov chain; stochastic stability; neural networks; infinite delay; Lyapunov function.*

Mathematics Subject Classification (2010): 37Hxx, 37B25, 37C75.

1 Introduction

Recently, some interesting studies in the literature have been reported, such as stability of dynamical systems, especially stability of neural networks with Markovian switching and time delay [1–3, 6, 11, 18].

Thanks to the advantages given by neural networks (NNs) they have attracted much attention in these few recent years, and we have noted that the number of studies in that field has rised. NNs systems have witnessed successful applications in many areas such as securing communication systems, pattern recognition, signal processing, population dynamics systems, chemical process control and especially, in processing static images and combinatorial optimization [4]. All these applications are mainly related to the

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dynamical behaviors of the considered systems and their NNs equilibrium points [5,6,12, 14].

Since time-delay assignment in neural networks can cause oscillation and instability behavior, so many researchers have been interested in the study of delay neural networks [15–17].

It is worth noting that NNs are often perturbed by various kinds of environmental noises, under which some properties of NNs may change. As mentioned in [12], environmental noises can turn a given stable system into an unstable, that is why, one can find many works that deal with stability of NNs disturbed by white noise. For example, in [25], the authors have discussed the exponential stability of a kind of NNs with white noise. They have set the sufficient conditions to guarantee stability of the considered system. Also, in [27], the authors have studied a stochastic NNs system with infinite delay, by means of Lyapunov method and It's formula. They have derived some sufficient conditions to ensure three types of stability. They have also shown that stochastic stability of the considered system with small noise is maintained if the NNs with infinite delay, is stable under some conditions. Recently, NNs with Markovian switching have been considered, because NNs with Markovian switching comprise general NNs as a special case [20–24]. For example, in [20], the authors have studied stability of a class of delayed NNs with Markovian switching in which the jumping parameters are determined by a continuous-time, and under some conditions, the pth moment exponential stability is ensured. They have provided a numerical example to validate the theoretical results. On the other hand, the work in [22] has dealt with stability of delayed stochastic NNs with Markovian switching. The authors have shown stability of the considered system and they have verified the founded results on three numerical examples.

To our best of knowledge, stability of stochastic Markovian switched NNs with infinite delay in a fading memory phase space is not fully investigated in the literature, which is the subject of our article.

Stimulated by the discussion of the studies mentioned above, our aim in this paper is to study a kind of stochastic Markovian switched NNs system with infinite delay in a fading memory phase space, considering white noise, infinite delay, and Markovian switching in such model. Firstly, the existence and uniqueness of solutions are shown. Secondly, by defining a Lyapunov Krasovskii functional, and using stochastic analysis technique and M-matrix theory, we give sufficient conditions to ensure three easily verified kinds of stochastic stability. These conditions are in terms of the coefficients of the system. Finally, two numerical examples are provided to test the proposed conditions and results.

This paper is organized as follows: we start by this introduction, then, in Section 2, we give the used notations in this paper and we define the model to study. After that, we introduce the definitions of three types of stochastic stability. In Section 3, existence and uniqueness of solutions for the studied system are established. Then, the three types of stochastic stability are discussed after styding existence and uniqueness. In Section 4, two numerical examples are given.

2 Preliminaries

For the sake of simplicity, we give the following notations of this paper. Write $\mathbb{R}$ for the set of real numbers and $\mathbb{R}^n$ for n dimensional Euclidean space. Denote $a \wedge b$ ($a \vee b$) be the minimum (maximum) for $(a, b) \in \mathbb{R}^2$. For matrix A, its trace norm is defined by $|A| = \big(Trace(A^TA)\big)^{\frac{1}{2}}$, with A^T its transpose. Let $C^\mu = \big\{\phi \in C((-\infty, 0]; \mathbb{R}^n) : \lim_{\theta\to-\infty} e^{\mu\theta}\phi(\theta)$ exists in $\mathbb{R}^n\big\}$, with $\mu > 0$, denote the family of continuous functions ϕ defined on $(-\infty, 0]$ with norm $|\phi|_\mu = \sup_{\theta\le 0} e^{\mu\theta}|\phi(\theta)|$.

The process $x_t : (-\infty, 0] \longrightarrow \mathbb{R}^n; \theta \longmapsto x_t(\theta) = x(t+\theta); -\infty < \theta \le 0$ can be regarded as a C^μ-value stochastic process, where $x_t(\theta) = (x_t^1(\theta), x_t^2(\theta), ..., x_t^n(\theta))^T$. The initial data of the stochastic process is defined on $(-\infty, 0]$. That is, the initial value is $x_0(\theta) = \xi(\theta)$ for $-\infty < \theta \le 0$.

We define $C_\alpha^\mu \triangleq \{\phi \in C^\mu; |\phi|_\mu < \alpha\}$. Let G be a vector or matrix. By $G \ge 0$ we mean each element of G is non-negative. By $G > 0$ we mean $G \ge 0$ and at least one element of G is positive. By $G \gg 0$ we mean all elements of G are positive.

Let $(\Omega, \mathcal{F}, \mathfrak{F}, P)$ be a complete probability space with a filtration $\mathfrak{F} = \{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, and $W(.)$ be a Brownian motion defined on the space. The mathematical expectation with respect to the given probability measure P is denoted by $E(.)$. Let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathcal{M} = \{1, 2, ..., N\}$ with the generator $\Gamma = (\gamma_{k\ell})_{N\times N}$ given by

$$P(r(t+\Delta t) = \ell / r(t) = k) = \begin{cases} \gamma_{k\ell}\Delta t + o(\Delta t), & k \ne \ell \\ 1 + \gamma_{kk}\Delta t + o(\Delta t), & \text{otherwise}, \end{cases}$$

where $\Delta t > 0$ and $\gamma_{k\ell} > 0$ is the transition rate from k to ℓ. If $k = \ell$, it follows $\gamma_{kk} = -\sum_{\ell=1,\ell\ne k}^{N} \gamma_{k\ell}$. We also assume that Markov chain $r(t)$ is independent of Brownian motion $W(t)$, and it is irreducible in the sense that the system of equations

$$\begin{cases} \pi\Gamma = 0, \\ \pi\mathbb{1} = 1, \end{cases} \tag{1}$$

has a unique positive solution, where $\mathbb{1}$ is a column vector with all component being 1. The positive solution is termed a stationary distribution.

For any $M > 0$, define two random variables τ_M^y and τ_y^M as follows:

$$\tau_M^y = \inf\{t \ge t_0 : |y(t)| \ge M, |\xi|_\mu < M, a.s.\},$$

$$\tau_y^M = \inf\{t \ge t_0 : |y(t)| \le M, |\xi|_\mu > M, a.s.\},$$

where $y : [0, +\infty) \times \Omega \longrightarrow \mathbb{R}$ is a continuous stochastic process. The general NNs with infinite delay can be described as Volterra integro-differential equations as follows:

$$\dot{u}(t) = -Du(t) + Ag(u(t)) + \int_{-\infty}^{t} CK^T(t-s)g(u(s))ds + J, \tag{2}$$

$$\dot{u}_i(t) = -d_iu_i(t) + \sum_{j=1}^{n} a_{ij}g_j(u_j(t)) + \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{t} K_{ij}(t-s)g_j(u_j(s))ds + J_i, \quad i = 1, 2, ..., n, \tag{3}$$

where $u(t) = (u_1(t), u_2(t), ..., u_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons, $D = diag(d_1, d_2, ..., d_n) \gg 0$ is the firing rate of the neurons, $A = (a_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$ are connection weight matrices $J = (J_1, J_2, ..., J_n)^T$ is the constant external input vector, $g(u) = (g_1(u_1), g_2(u_2), ..., g_n(u_n)^T$ is the neuron activation function vector, and $K_{ij} : [0, +\infty) \longrightarrow [0, +\infty)$ $(i, j = 1, 2, ..., n)$ are piecewise continuous on $[0, +\infty)$ satisfying

$$\int_0^{+\infty} K_{ij}(s) e^{\mu s} ds = \bar{K}. \qquad i, j = 1, 2, ..., n. \tag{4}$$

where $\bar{K}$ is a positive constant depending on μ.

As mentioned in Section 1, it is assumed that system (2) has an equilibrium point $u^* = (u_1^*, u_2^*, ..., u_n^*)$. The conditions, which guarantee that system (2) has a unique equilibrium point, can be found in [26]. By making a transformation $x(t) = u(t) - u^*$, system (2) can be rewritten as

$$\dot{x}(t) = -Dx(t) + AF(x(t)) + \int_{-\infty}^{t} CK^T(t-s)F(x(s))ds, \tag{5}$$

where $F(x(t)) = \big(g_1(x_1(t) + u_1^*), g_2(x_2(t) + u_2^*), ..., g_n(x_n(t) + u_n^*)\big)^T \triangleq \big(f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t))\big)^t$. The main purpose of this paper is to study system (5) disturbed by white noise and Markovian switching, which, naturally, could be generalized into stochastic NNs with infinite delay and Markovian switching as follows:

$$\begin{aligned} dx(t) =& \Big[- Dx(t) + A(r(t))F(x(t)) + \int_{-\infty}^{t} C(r(t))K^T(t-s)F(x(s))ds \Big] dt \\ &+ B(r(t))Q(x(t))dW(t), \end{aligned} \tag{6}$$

where $A(r(t)) = \big(a_{ij}(r(t))\big)_{n\times n}$, $C(r(t)) = \big(c_{ij}(r(t))\big)_{n\times n}$, $B(r(t)) = \big(b_{ij}(r(t))\big)_{n\times n}$ and $Q(x) = \big(q_1(x_1(t)), q_2(x_2(t)), ..., q_n(x_n(t))\big)^T$ represents the disturbance intensity of white noise satisfying $Q(0) = 0$.

For any $k \in \mathcal{M}$, system (6) can be regarded as the result of the n stochastic NNs with infinite delay

$$dx(t) = \Big[-Dx(t) + A(k)F(x(t)) + \int_{-\infty}^{t} C(k)K^T(t-s)F(x(s))ds \Big] dt + B(k)Q(x(t))dW(t), \tag{7}$$

switching from one to the others according to the movement of the Markov chain.

For any $(\phi, k) \in C^\mu \times \mathcal{M}$, we denote

$$\begin{cases} \mathbb{E}\big(\phi, k\big) = -D\phi(0) + A(k)F(\phi(0)) + \int_{-\infty}^{t} C(k)K^T(t-s)F(\phi(s-t))ds, \\ \mathbb{H}\big(\phi, k\big) = B(k)Q(\phi(0)). \end{cases}$$

Let $\mathcal{C}^{2,1}\big(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{M}; \mathbb{R}_+\big)$ denote the family of all nonnegative functions $V(x, t, k)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{M}$ which are continuously twice differentiable in x and one differentiable in t. If $V \in \mathcal{C}^{2,1}\big(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{M}; \mathbb{R}_+\big)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{M}$ to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(x, t, k) =& V_t(x, t, k) + V_x(x, t, k)\mathbb{E}(x_t, k) + \frac{1}{2} Trace\big[\mathbb{H}^T(x_t, k)V_{xx}(x, t, k)\mathbb{H}(x_t, k)\big] \\ &+ \sum_{\ell=1}^{N} \gamma_{k\ell} V(x, t, \ell), \end{aligned}$$

where

$$V_t(x, t, k) = \frac{\partial V(x, t, k)}{\partial t}, \quad V_x(x, t, k) = \Big(\frac{\partial V(x, t, k)}{\partial x_1}, ..., \frac{\partial V(x, t, k)}{\partial x_n}\Big)$$

and

$$V_{xx}(x,t,k) = \Big(\frac{\partial^2 V(x,t,k)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

In the sequel, we introduce the following concepts of stochastic stability.

Definition 2.1 [24]. The trivial solution of system (6) with initial data $x_{t_0} = \xi$ is said to be stochastically stable if for every pair $\varepsilon \in (0,1)$ and $\alpha > 0$, there exists a $\delta = \delta(\varepsilon, \alpha) > 0$ such that

$$P\big\{|x(t,t_0,\xi)| < \alpha, t \geq t_0\big\} \geq 1 - \varepsilon,$$

whenever $(\xi, k) \in C^\mu_\delta \times \mathcal{M}$.

Definition 2.2 [24]. The trivial solution of system (6) with initial data $x_{t_0} = \xi$ is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$, there exist $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$P\big\{\lim_{t\to\infty} x(t,t_0,\xi) = 0\big\} \geq 1 - \varepsilon,$$

whenever $(\xi, k) \in C^\mu_{\delta_0} \times \mathcal{M}$.

Definition 2.3 [24]. The trivial solution of system (6) with initial data $x_{t_0} = \xi$ is said to be globally stochastically asymptotically stable if it is stochastically stable and, moreover, for any $(\xi, k) \in C^\mu \times \mathcal{M}$,

$$P\big\{\lim_{t\to\infty} x(t,t_0,\xi) = 0\big\} = 1.$$

3 Main Results

In this section, we derive the criteria which are concerned with the three kinds of stochastic stability defined in Section 2 for the solution to system (6). The proof is based on the Lyapunov method, generalized It's formula, some inequalities, and M-matrix technique. Let us introduce first the following assumption.

Assumption 3.1 For each $j \in \{1, 2, ..., n\}$, functions $g_j : \mathbb{R} \longrightarrow \mathbb{R}$ and $q_j : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy global Lipschitz conditions

$$|g_j(x) - g_j(y)| \vee |q_j(x) - q_j(y)| \leq L_j|x - y|, \quad \text{for} \quad x, y \in \mathbb{R}, \tag{8}$$

that is,

$$|F(x)| \vee |Q(x)| \leq L|x|, \tag{9}$$

where $L = \max\{L_1, L_2, ..., L_n\}$. In addition, the initial data $x_{t_0} = \xi$ satisfies $|\xi| := \sup\limits_{\theta\leq 0} |\xi(\theta)| < \infty$.

Let us denote

$$\beta_k := -d + L|A(k)| + \frac{1}{2}L^2|B(k)|^2 + n^2\bar{K}L|C(k)|, \quad k \in \mathcal{M},$$

and $\mathcal{A} := -diag(2\beta_1, 2\beta_2, ..., 2\beta_n) - \Gamma$, where $d = \min\{d_1, d_2, ..., d_n\}$.

Now, we introduce an existence and uniqueness result for the solution of system (6). The steps of proof for this result are similar to the proof of Theorem 1 in [13].

Theorem 3.1 *Suppose that Assumption 3.1 holds. Then system (6) has a unique global solution on* $(-\infty, \infty)$ *with initial data* $\xi \in C^\mu$ *and* $r(t_0) = r_0$.

Proof. By definition of the right continuous Markov jump $r(.)$, there is a sequence $\{\tau_k\}_{k\geq 0}$ of stoping times such that $r(.)$ is a random constant on every interval $[\tau_k, \tau_{k+1})$, that is $r(t) = r(\tau_k)$ on $\tau_k \leq t < \tau_{k+1}$, for any $k \geq 0$. We proceed by induction. We consider first system (6) for $t \in [\tau_0, \tau_1)$

$$\begin{aligned} dx(t) &= \Big[- Dx(t) + A(r_0)F(x(t)) + \int_{-\infty}^{t} C(r_0)K^T(t-s)F(x(s))ds \Big] dt \\ &+ B(r_0)Q(x(t))dW(t) \\ &= \Big[- Dx_t(0) + A(r_0)F(x_t(0)) + \int_{-\infty}^{t} C(r_0)K^T(t-s)F(x_t(s-t))ds \Big] dt \\ &+ B(r_0)Q(x_t(0))dW(t). \end{aligned} \tag{10}$$

By change of variable $v = t - s$, we get

$$\begin{aligned} dx(t) =& \Big[- Dx_t(0) + A(r_0)F(x_t(0)) + \int_{0}^{+\infty} C(r_0)K^T(v)F(x_t(-v))dv \Big] dt \\ &+ B(r_0)Q(x_t(0))dW(t). \end{aligned}$$

For any $\xi \in C^\mu$, let

$$\begin{cases} \mathbb{E}(\xi, r_0) = -D\xi(0) + A(r_0)F(\xi(0)) + \int_0^{+\infty} C(r_0)K^T(v)F(\xi(-v))dv, \\ \mathbb{H}(\xi, r_0) = B(r_0)Q(\xi(0)), \end{cases} \tag{11}$$

then system (10) for $t \in [\tau_0, \tau_1)$ can be rewrite as

$$dx(t) = \mathbb{E}(x_t, r_0)dt + \mathbb{H}(x_t, r_0)dW(t). \tag{12}$$

From (4), (9) and (11), we have

$$\begin{aligned} |\mathbb{E}(\xi, r_0) - \mathbb{E}(\zeta, r_0)| &\leq |D||\xi(0) - \zeta(0)| + |A(r_0)||F(\xi(0)) - F(\zeta(0))| \\ &+ \int_0^{+\infty} |C(r_0)K^T(v)||F(\xi(-v)) - F(\zeta(-v))|dv \\ &\leq |D||\xi(0) - \zeta(0)| + L|A(r_0)||\xi(0) - \zeta(0)| \\ &+ L\int_0^{+\infty} |C(r_0)||K^T(v)||\xi(-v) - \zeta(-v)|dv \\ &\leq |D||\xi(0) - \zeta(0)| + L|A(r_0)||\xi(0) - \zeta(0)| \\ &+ L\int_0^{+\infty} |C(r_0)||K^T(v)|e^{\mu v}e^{-\mu v}|\xi(-v) - \zeta(-v)|dv \\ &\leq \Big(|D| + L|A(r_0)| + n^2 L|C(r_0)|\bar{K} \Big) |\xi - \zeta|_\mu \end{aligned}$$

and

$$|\mathbb{H}(\xi, r_0) - \mathbb{H}(\zeta, r_0)| \leq L|B(r_0)||\xi - \zeta|_\mu.$$

By the main theorem in [7], system (10) with initial condition $\xi \in C^\mu$ and $r(t_0) = r_0$ has a unique solution on $[\tau_0, \tau_1)$.

If we consider system (10) for $t \in [\tau_1, \tau_2)$, then, (12) becomes

$$dx(t) = \mathbb{E}(x_t, r_1)dt + \mathbb{H}(x_t, r_1)dW(t). \tag{13}$$

By the same argument of existence and uniqueness as the first step above, system (6) with initial condition $x_{\tau_1} \in C^\mu$ and $r(\tau_1) = r_1$ has a unique solution on $[\tau_1, \tau_2)$.

By induction, system (6) with initial condition $\xi \in C^\mu$ and $r(0) = r_0$ has a unique solution on $(-\infty, \infty)$. Next, to show stochastic stability, we need to following assumption.

Assumption 3.2 There is a $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)^T \geq 0$ in $\mathbb{R}^N$ such that $P = \mathcal{A}\lambda \geq 0$.

Theorem 3.2 *Suppose that Assumptions 3.1 and 3.2 hold. Then the trivial solution to system (6) is stochastically stable.*

Proof. For any $\varepsilon \in (0, 1)$ and $\alpha > 0$, we choose a sufficiently small $\delta = \delta(\varepsilon, \alpha) < \alpha$, such that for any $\xi \in C^\mu_{\delta(\varepsilon,\alpha)}$,

$$\lambda_k|\xi|^2_\mu + 2n^2\bar{K}L|\xi|_\mu < \lambda_k\varepsilon\alpha^2 \text{ for any } k \in \mathcal{M}.$$

The choice of δ above is guaranteed by taking

$$\lambda_k\delta^2 + 2n^2\bar{K}L\delta < \lambda_k\varepsilon\alpha^2 \text{ for any } k \in \mathcal{M}.$$

Fix any $\xi \in C^\mu_\delta$ and write $x(t) \triangleq x(t, t_0, \xi)$. For $t \geq t_0$, $\quad k = 1, 2, ..., N$, let

$$V(x, t, k) = \frac{1}{2}\lambda_k|x|^2 + \int_t^{+\infty} \sum_{i=1}^n \sum_{j=1}^n K_{ij}(s-t)|f_j(x_j(2t-s))|ds. \tag{14}$$

From Assumptions 3.1 and 3.2, we infer

$$\begin{aligned}
\mathcal{L}V(x(t), t, k) \triangleq{}& V_x(x, t, k)\Big[-Dx(t) + A(k)F(x(t)) + \int_{-\infty}^t C(k)K^T(t-s)F(x(s))ds\Big] \\
&+ V_t(x, t, k) + \frac{1}{2}Trace\Big[\big(B(k)Q(x(t))\big)^T V_{xx}(x, t, k)\big(B(k)Q(x(t))\big)\Big] \\
&+ \sum_{\ell=1}^N \gamma_{k\ell}V(x(t), t, \ell) \\
={}& \lambda_k x^T(t)\Big[-Dx(t) + A(k)F(x(t)) + \int_{-\infty}^t C(k)K^T(t-s)F(x(s))ds\Big] \\
&- \sum_{i=1}^n \sum_{j=1}^n K_{ij}(0)\big|f_j(x_j(t))\big| + \frac{\lambda_k}{2}Trace\Big[\big(B(k)Q(x(t))\big)^T\big(B(k)Q(x(t))\big)\Big] \\
&+ \sum_{\ell=1}^N \gamma_{k\ell}\Big[\frac{\lambda_\ell}{2}|x(t)|^2 + \int_t^{+\infty} \sum_{i=1}^n \sum_{j=1}^n K_{ij}(s-t)\big|f_j(x_j(2t-s))\big|ds\Big] \\
={}& -\lambda_k \sum_{i=1}^n d_i x_i^2(t) + \lambda_k \sum_{i=1}^n x_i(t) \sum_{j=1}^n a_{ij}(k)f_j(x_j(t))
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_k \int_{-\infty}^{t} \sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_{i\ell}(k) K_{\ell j}(t-s) f_j(x_j(s)) ds \\
&- \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij}(0) \big| f_j(x_j(t)) \big| + \frac{\lambda_k}{2} \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} b_{ij}(k) q_j(x_j(t)) \Big)^2 \\
&+ \sum_{\ell=1}^{N} \gamma_{k\ell} \frac{\lambda_\ell}{2} |x(t)|^2 \\
&\leq -\lambda_k \sum_{i=1}^{n} d_i x_i^2(t) + \lambda_k \sum_{i=1}^{n} |x_i(t)| \sum_{j=1}^{n} L_j |a_{ij}(k)| |x_j(t)| \\
&+ \frac{\lambda_k}{2} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2(k) \Big) \sum_{j=1}^{n} q_j^2(x_j(t)) \\
&+ \lambda_k \int_{-\infty}^{t} \sum_{i=1}^{n} |x_i(t)| \sum_{j=1}^{n} \sum_{\ell=1}^{n} L_j K_{ij}(t-s) |c_{i\ell}(k)| |x_j(s)| ds + \sum_{\ell=1}^{N} \gamma_{k\ell} \frac{\lambda_\ell}{2} |x(t)|^2.
\end{aligned}
$$

By using the fact that $x(t) = x(t+0) = x_t(0)$ and the transformation $v = t - s$, one has

$$
\begin{aligned}
\mathcal{L}V(x(t), t, k) &\leq -\lambda_k \sum_{i=1}^{n} d_i x_i^2(t) + \lambda_k L \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2(k) \Big)^{\frac{1}{2}} \sum_{i=1}^{n} x_i^2(t) \\
&+ \frac{\lambda_k}{2} L^2 \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2(k) \Big) \sum_{i=1}^{n} x_i^2(t) \\
&+ \lambda_k \int_{0}^{+\infty} \sum_{i=1}^{n} |x_t^i(0)| \sum_{j=1}^{n} \sum_{\ell=1}^{n} L_j K_{ij}(v) |c_{i\ell}(k)| |x_t^j(-v)| dv + \sum_{\ell=1}^{N} \gamma_{k\ell} \frac{\lambda_\ell}{2} |x(t)|^2 \\
&\leq \lambda_k \Big[-\sum_{i=1}^{n} d_i |x_t^i(0)|^2 + L \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2(k) \Big)^{\frac{1}{2}} \sum_{i=1}^{n} |x_t^i(0)|^2 \\
&+ \frac{L^2}{2} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2(k) \Big) \sum_{i=1}^{n} |x_t^i(0)|^2 \\
&+ |x_t|_\mu |C(k)| L \int_{0}^{+\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij}(v) e^{\mu v} e^{-\mu v} |x_t^j(-v)| dv \Big] + \sum_{\ell=1}^{N} \gamma_{k\ell} \frac{\lambda_\ell}{2} |x_t(0)|^2 \\
&\leq \lambda_k \Big[-d + L|A(k)| + \frac{1}{2} L^2 |B(k)|^2 + n^2 \bar{K} L |C(k)| \Big] |x_t|_\mu^2 + \sum_{\ell=1}^{N} \gamma_{k\ell} \frac{\lambda_\ell}{2} |x_t|_\mu^2 \\
&\leq \frac{1}{2} \Big(2\lambda_k \beta_k + \sum_{\ell=1}^{N} \gamma_{k\ell} \lambda_\ell \Big) |x_t|_\mu^2 \\
&\leq -\frac{1}{2} p_k |x_t|_\mu^2.
\end{aligned}
$$

Thus, by use of It's generalized formula, for any $t \geq t_0$,

$$
EV(x(t \wedge \tau_x^\alpha), t \wedge \tau_x^\alpha, k) = EV(x(t_0), t_0, k) + E \int_{t_0}^{t \wedge \tau_x^\alpha} \mathcal{L}V(x(s), s, r(s)) ds
$$

$$
\begin{aligned}
&= EV(\xi(0), t_0, k) + E\int_{t_0}^{t\wedge\tau_x^\alpha} \mathcal{L}V(x(s), s, r(s))ds \\
&\leq EV(\xi(0), t_0, k).
\end{aligned}
$$

Besides, by use of Assumption 3.1 and Eq. (4) we obtain

$$
\begin{aligned}
EV(x(t_0), t_0, k) &= E\big(\frac{\lambda_k}{2}\sum_{i=1}^n x_i^2(t_0)\big) + E\big(\int_{t_0}^{+\infty}\sum_{i=1}^n\sum_{j=1}^n K_{ij}(s-t_0)|f_j(x_j(2t_0-s))|ds\big) \\
&\leq \frac{\lambda_k}{2}E|\xi|_\mu^2 + E\int_0^{+\infty}\sum_{i=1}^n\sum_{j=1}^n K_{ij}(v)|f_j(x_{t_0 j}(-v))|dv \\
&\leq \frac{\lambda_k}{2}E|\xi|_\mu^2 + n^2\bar{K}LE|\xi|_\mu.
\end{aligned}
$$

We also have

$$
\begin{aligned}
V(x(t\wedge\tau_x^\alpha), t\wedge\tau_x^\alpha, k) &\geq \frac{\lambda_k}{2}|x(t\wedge\tau_x^\alpha)|^2 \\
E\big[\mathbb{1}_{\{\tau_x^\alpha<t\}}V(x(t\wedge\tau_x^\alpha), t\wedge\tau_x^\alpha, k)\big] &\geq E\big[\mathbb{1}_{\{\tau_x^\alpha<t\}}V(x(\tau_x^\alpha), \tau_x^\alpha, k)\big] \\
&\geq \frac{\lambda_k}{2}E\big[\mathbb{1}_{\{\tau_x^\alpha<t\}}|x(\tau_x^\alpha)|^2\big] \\
&\geq \frac{\lambda_k}{2}\alpha^2E\big[\mathbb{1}_{\{\tau_x^\alpha<t\}}\big] = \frac{\lambda_k}{2}\alpha^2P(\tau_x^\alpha<t).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\frac{\lambda_k\alpha^2}{2}P\{\tau_x^\alpha<t\} &\leq E\big(\mathbb{1}_{\{\tau_x^\alpha<t\}}V(x(\tau_x^\alpha), \tau_x^\alpha, k)\big) \\
&\leq EV(x(t\wedge\tau_x^\alpha), t\wedge\tau_x^\alpha, k) \\
&\leq \frac{\lambda_k}{2}|\xi|_\mu^2 + n^2\bar{K}L|\xi|_\mu \\
&< \frac{\lambda_k}{2}\varepsilon\alpha^2,
\end{aligned}
$$

gives

$$P\{\tau_x^\alpha<t\}<\varepsilon.$$

Letting $t\longrightarrow\infty$ we have $P\{\tau_x^\alpha<\infty\}<\varepsilon$, which is equivalent to

$$P\{|x(t,t_0,\xi)|\leq\alpha, t\geq t_0\}\geq 1-\varepsilon. \tag{15}$$

This completes the proof.

For stochastic asymptotic stability and global stochastic asymptotic stability, we add the following assumption.

Assumption 3.3 If $\mathcal{A}$ is a nonsingular M-matrix, there is a $\lambda=(\lambda_1,\lambda_2,...,\lambda_N)^T\gg 0$ in $\mathbb{R}^N$ such that $P=\mathcal{A}\lambda\gg 0$.

For further properties on M-matrices, we refer the readers to Chapter 2 of [24].

Theorem 3.3 *Suppose that Assumptions 3.1 and 3.3 hold. Then the trivial solution of system (6) is stochastically asymptotically stable.*

Idea of proof:
The proof is similar to the one in [13]. The only differences are

- Inequality (17) in [13] becomes: For δ_1 and any $\varepsilon_1 \in (0,1)$, there exists a $H(\varepsilon_1, \delta)$ sufficiently large such that

$$P\{|x(t,\theta^*,\xi_{\theta^*})| \leq H, t \geq \theta^*\} \geq 1 - \frac{\varepsilon_1}{4}, \quad \text{and } P\{|x_{\theta^*}|_\mu < H, \theta^* \leq t\} = 1.$$

- Inequality before (20) in [13] becomes

$$\bar{K} < \frac{(\lambda_k \alpha)^2}{2n^2 LH}.$$

- In a slightly different way as in [13], we define the Lyapunov function by

$$V(x,t,k) = \frac{\lambda_k}{2}\sum_{i=1}^{n} x_i^2 + \int_t^{+\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}(s-t)|f_j(x_j(2t-s))|ds.$$

Theorem 3.4 *Suppose that Assumptions 3.1 and 3.3 hold. Then the trivial solution of system (6) is globally stochastically asymptotically stable.*

We omit this proof because it is very similar to the equivalent one in [13].

4 Applications

Two examples are given. In the first example, we consider system (6) on $\mathbb{R}^3$ and the Markov process $r(t)$ is switching between two subsystems. In the second example, we define system (6) on $\mathbb{R}^2$ and the Markov process $r(t)$ switches between tree subsystems.

Example 4.1 Let $r(t)$ be a right-continuous Markovian chain taking values in $\mathcal{M} = \{1,2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}.$$

Consider a three dimensional system of type (6) with the following specifications

$$D = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}, \quad A(1) = \begin{pmatrix} 1.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0.5 \end{pmatrix},$$

$$B(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C(1) = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix},$$

$$A(2) = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}, \quad B(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C(2) = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}.$$

We rewrite System (6) in the following detailed form

$$\begin{cases} dx(t) &= \big[-12x(t) + a_{11}(r(t))f(x(t)) + a_{12}(r(t))f(y(t)) + a_{13}(r(t))f(z(t)) \\ &\quad + \int_{-\infty}^{t} c_{11}(r(t))e^{s-t}(f(x(s))+f(y(s))+f(z(s)))ds\big]dt + b_{11}(r(t))\sin x(t)dW(t), \\ dy(t) &= big[-12y(t) + a_{21}(r(t))f(x(t)) + a_{22}(r(t))f(y(t)) + a_{23}(r(t))f(z(t)) \\ &\quad + \int_{-\infty}^{t} c_{22}(r(t))e^{s-2t}(f(x(s))+f(y(s))+f(z(s)))ds\big]dt + b_{22}(r(t))\sin y(t)dW(t), \\ dz(t) &= \big[-12z(t) + a_{31}(r(t))f(x(t)) + a_{32}(r(t))f(y(t)) + a_{33}(r(t))f(z(t)) \\ &\quad + \int_{-\infty}^{t} c_{33}(r(t))e^{s-t}(f(x(s))+f(y(s))+f(z(s)))ds\big]dt + b_{33}(r(t))\sin z(t)dW(t), \end{cases} \tag{16}$$

where $f(x)$ satisfies a global Lipschitz condition with a Lipschitz constant $L = 1$. We choose $f(x) = x$.

In order to get the conditions of Theorem 3.4

a) We take $q_j(x) = \sin x, j = 1, 2, 3$. Then Assumption 3.1 holds.

b) We choose $K(t-s) = e^{s-t}$. Then, $\bar{K} = \frac{1}{1-\mu}$ for $0 < \mu < 1$.

c) For $\mu = 0.5$, we can see that $\beta_1 = -0.5870$ and $\beta_2 = -0.1768$, so $\mathcal{A} = \begin{pmatrix} 2.1739 & -1.000 \\ -3.000 & 3.3537 \end{pmatrix}$,

which implies immediately that Assumption 3.3 is satisfied. By Theorem 3.4, the trivial solution to System (16) is globally stochastically asymptotically stable.

Figure 1 shows the way of randomly switching between the two subsystems with initial condition $r(0) = 1$. Figure 2 shows trajectory of the stochastic approximate solution for system (16) with initial condition $x(t) = \sin^2(t), y(t) = 0.5\cos^2(t), z(t) = 0$.

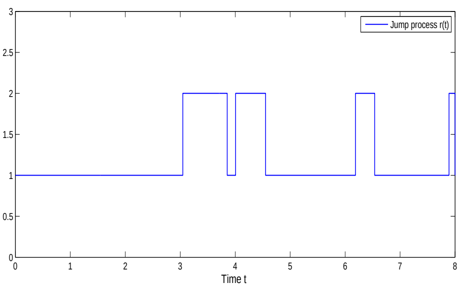

Figure 1: Jump process r(t) with initial condition r(0)=1.

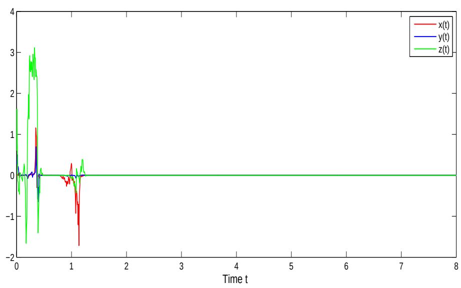

Figure 2: Approximate solution of system (16).

Example 4.2 Let $r(t)$ be a right-continuous Markovian chain taking values in $\mathcal{M} = \{1, 2, 3\}$ with generator

$$\Gamma = (\gamma_{ij})_{3\times 3} = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 2 & -5 \end{pmatrix}.$$

Consider a two-dimensional System (6) with the following specification

$$D = \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}, \quad A(1) = \begin{pmatrix} 2 & 1 \\ 1 & 1.5 \end{pmatrix}, \quad B(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C(1) = \begin{pmatrix} 0.5 & 0 \\ 0 & \sqrt{2}, \end{pmatrix},$$

$$A(2) = \begin{pmatrix} 2 & 0.5 \\ 0.3 & 0.8 \end{pmatrix} \quad B(2) = \begin{pmatrix} \sqrt{0.2} & 0 \\ 0 & \sqrt{0.2} \end{pmatrix}, \quad C(2) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2}\sqrt{2} \end{pmatrix}.$$

$$A(3) = \begin{pmatrix} 2 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}, \quad B(3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C(3) = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.5, \end{pmatrix}.$$

We rewrite system (6) in the following detailed form

$$\begin{cases} dx(t) & = \big[-15x(t) + a_{11}(r(t))h(x(t)) + a_{12}(r(t))y(t) \\ & + \int_{-\infty}^{t} c_{11}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \big] dt + b_{11}(r(t))q_1(x(t))dW(t), \\ dy(t) & = \big[-15x(t) + a_{21}(r(t))h(x(t)) + a_{22}(r(t))y(t) \\ & + \int_{-\infty}^{t} c_{22}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \big] dt + b_{22}(r(t))q_2(x(t))dW(t), \end{cases} \tag{17}$$

where $q_1(x) = q_2(x) = \sin x$ satisfy a global Lipschitz condition with Lipschitz constant $L = 1, h(x) = \sin x$. This means that Assumption 3.1 is verified. To assure Assumption 3.3, let $\mu = 0.4$. Then, we can see that $\beta_1 = -1.1277, \beta_2 = -1.0214$ and $\beta_3 = -8.0210$. So

$$\mathcal{A} = \begin{pmatrix} +4.2554 & -1.000 & -1.0000 \\ -2.0000 & +6.0428 & -2.0000 \\ -3.0000 & -2.0000 & 21.0421 \end{pmatrix}.$$

Hence, it is guaranteed that $\mathcal{A}$ is a nonsingular M-matrix. By Theorem 3.4, System (17) is globally asymptotically stochastic stable.

Figure 3 shows a way of random switching between the three subsystems with initial condition $r(0) = 1$. Figure 4 depicts the stochastic approximate solution for System (17) with initial condition $x(t) = \sin^2(t), y(t) = 0.6$.

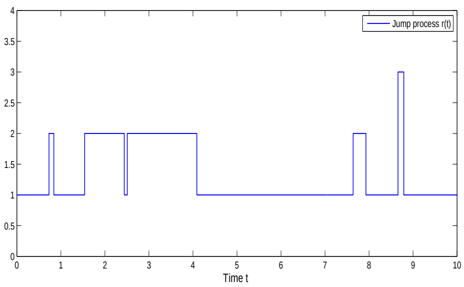

Figure 3: Jump process r(t) with initial condition r(0)=1.

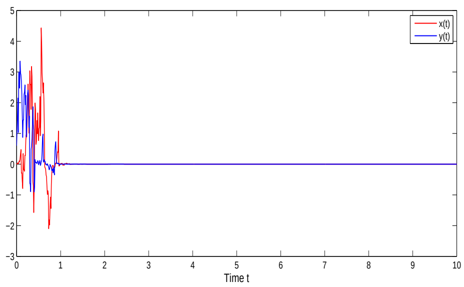

Figure 4: Approximate solution of System (17).

5 Conclusion

We have provided the existence and uniqueness of solutions for a kind of NNs with Markovian switching. Basing on the Lyapunov method and stochastic analysis and M-matrix theory, we have given the new conditions that ensure stochastic stability, stochastic asymptotic stability and global stochastic asymptotic stability of neural networks with Markovian switching and infinite time delay in a phase space. Two simulated numerical examples under Matlab have been presented to validate the proposed conditions.

We notice that the stability of the system depends also on the positive number μ associated to the phase space. Also the theoretical outcome in this paper can be applied to many complex systems and other NNs, such as the processing of motion related phenomena.

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