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Analysis of the Model Reduction Using Singular Perturbation Approximation on Unstable and Non-Minimal Discrete-Time Linear Systems and Its Applications

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Abstract: In the natural phenomena, many systems are unstable and non-minimal. Moreover, the systems in the universe often have large order. Therefore, we need to simplify the order of the system without any significant errors. Simplification of this system can be done using the reduction of the model. Model reduction can only be done on the stable and minimal system. Thus, we need a model reduction for unstable and non-minimal systems. There are many model reduction methods in the literature, for example, a balanced truncation method and a singular perturbation approximation. In this work, we propose a method to reduce unstable and non-minimal systems by using the singular perturbation approximation (SPA). First, we decompose the unstable system into a stable subsystem and an unstable subsystem. Then, if the stable subsystem is non-minimal, we apply the minimization process to obtain a stable subsystem that is minimal. Next, we apply the singular perturbation approximation to the stable and minimal subsystem to obtain a reduced subsystem. Finally, we obtain a total reduced model by combining the unstable subsystem and the reduced subsystem. Then we apply the method to shallow water equations. Based on the simulation results, frequency response of the original system and the reduced minimal system using the SPA method has similarity in low frequency, but in high frequency the value tends to be different. Furthermore, the error bound of the SPA and the balanced truncation method is almost the same.

Keywords: *model reduction; singular perturbation approximation; unstable systems; non-minimal systems.*

Mathematics Subject Classification (2010): 78M34.

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1 Introduction

Mathematics has an important role in solving the existing problems. One of them is using mathematical modeling. Mathematical modeling is used to represent the problem to be solved. The problems taken often adopt real natural phenomena. If we construct a model of the real natural phenomena as a mathematical model, it will contain many state variables. On the other hand, due to the huge number of variables in the system, the computational time to simulate the model is longer and it is not efficient. Not only that, the system that has larger order is more complicated than the system that has small order. Therefore, it should be simplified to smaller order without significant errors. Those simplifications of the system are called a model reduction [4].

The model reduction can be used if the system is stable and minimal. For an unstable system, we need to decompose the unstable system so that we obtain a stable subsystem that can be reduced. Then, we investigate whether the system is minimal. A system is called minimal if the system is controllable and observable. If we have an uncontrollable or unobservable system (commonly called a non-minimal system), we have to find the minimal representation of the system by applying the minimization process. After the decomposition process (for unstable systems) and minimization (for uncontrollable or unobservable systems) of the system, we obtain a system that is suitable for model reduction [2, 11].

There are many methods in the literature for the model order reduction, such as a balanced truncation, model analysis, Krylov method, Hankel norm approximation, singular value decomposition [6–8, 13, 14]. Ayadi and Benhadj Braiek [5] discuss a new method to reduce LTI and LTV systems by using orthogonal functions. Solikhatun et al. [16] discuss a procedure to choose the reduced order of bilinear time-invariant systems. Aleksandrov et al. [1] discuss the stability problem by using a nonlinear approach. Martynyuk [9] discusses a new approach for the stability analysis of dynamical systems defined over metric spaces. In this paper, we discuss a model reduction procedure for non-minimal systems by using a singular perturbation approximation (SPA) method. The procedure to reduce unstable and non-minimal systems using the singular perturbation approximation is as follows. First, we decompose the unstable system into a stable subsystem and an unstable subsystem. Then, for the stable subsystem, if the system is non-minimal, we apply the minimization process to obtain a stable subsystem that is minimal. Next, we apply the singular perturbation approximation to the stable and minimal subsystem to obtain a reduced subsystem. The total reduced system is obtained by combining the unstable subsystem and the reduced subsystem. Then, we apply the model reduction of non-minimal systems to shallow water equations. Shallow water equations are a system that may become stable, unstable or non-minimal system. Furthermore, shallow water equations can produce a high-order system. Thus, we are interested to apply the technique to this system. In the simulation, we compare the frequency response and infinity-norm error of the singular perturbation approximation method and the balanced truncation method.

2 Linear Systems

Give an n^{th} order discrete-time linear system as follows [12]:

$$\begin{cases} x(k+1) & = Ax(k) + Bu(k), \\ y(k) & = Cx(k) + Du(k). \end{cases} \tag{1}$$

Next, we refer to the system (1) as the original system. The system can be rewritten as

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

The transfer function of the system (A, B, C, D) is denoted as $G(z)$ and is defined as follows [12]:

$$G(z) = C(zI - A)^{-1}B + D. \tag{2}$$

We can analyze the properties of the system (1). We focus on the following properties: stability, controllability, and observability. To analyze the stability, we look at the eigenvalues (λ) [12]. If $|\lambda| < 1$, then the system is asymptotically stable. Otherwise, if the eigenvalue $|\lambda| > 1$, the system is unstable. To analyze the controllability of the system, we can use the controllability matrix M_c. If the controllability matrix has the rank equal to n, then the system is controllable. To analyze the observability of the system, we can use the observability matrix M_o. If the observability matrix has the rank equal to n, then the system is observable.

The relationship between stability, controllability, and observability of systems with the controllability Gramian W and the observability Gramian M is described by the following theorem.

Theorem 2.1 *Given a system (A, B, C, D) that is stable, controllable, and observable, the controllability Gramian W and the observability Gramian M, each is a single and positive definite solution from the Lyapunov equation [18]*

$$AWA^T + BB^T - W = 0, \tag{3}$$

$$A^TMA + C^TC - M = 0. \tag{4}$$

The model reduction can only be done on the stable and minimal system. If the system is unstable, we need to decompose the system into a stable subsystem and an unstable subsystem. In other words, the decomposition of unstable systems is a separation between the stable subsystem and the unstable subsystem. If the system is non-minimal, we apply the minimization process to the system to obtain minimal systems.

3 Decomposition Process of Unstable Systems

The decomposition process of unstable systems has two transformation steps. First, we apply a real Schur block transformation using the unitary matrix U_d in the Schur upper triangular form. Second, we solve the Lyapunov equation for transformation to obtain stable and unstable subsystems. We denote $(A_{ss}, B_{ss}, C_{ss}, D_{ss})$ for the stable subsystem and $(A_{us}, B_{us}, C_{us}, D_{us})$ for the unstable subsystem.

$$G_d = \left[\begin{array}{c|c} A_{ss} & B_{ss} \\ \hline C_{ss} & D_{ss} \end{array}\right] + \left[\begin{array}{c|r} A_{us} & B_{us} \\ \hline C_{us} & 0 \end{array}\right]$$

with

$$stable\ subsystem = \left[\begin{array}{c|c} A_{ss} & B_{ss} \\ \hline C_{ss} & D_{ss} \end{array}\right]$$

and

$$unstable\ subsystem = \left[\begin{array}{c|r} A_{us} & B_{us} \\ \hline C_{us} & 0 \end{array}\right].$$

So from (1), we obtain a stable subsystem as follows:

$$\begin{cases} x_{ss}(k+1) &= A_{ss}x_{ss}(k) + B_{ss}u_{ss}(k), \\ y_{ss}(k) &= C_{ss}x_{ss}(k) + D_{ss}u_{ss}(k). \end{cases} \tag{5}$$

4 Minimization Process of Non-Minimal Systems

If the system (5) is non-minimal, we can minimize the system to obtain a minimal system. The minimization process is done by removing the uncontrollable and unobservable states in the state-space model. After the minimization process is finished, we obtain a minimal system (observable and controllable) as follows:

$$\begin{cases} x_m(k+1) &= A_m x_m(k) + B_m u_m(k), \\ y_m(k) &= C_m x_m(k) + D_m u_m(k). \end{cases} \tag{6}$$

The transfer function of the minimal system is

$$G_m = \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & D_m \end{array}\right].$$

Because the model reduction needs a stable and minimal system, then we define a stable and minimal discrete-time linear system as follows:

$$\begin{cases} x_{sm}(k+1) &= A_{sm}x_{sm}(k) + B_{sm}u_{sm}(k), \\ y_{sm}(k) &= C_{sm}x_{sm}(k) + D_{sm}u_{sm}(k). \end{cases} \tag{7}$$

5 Balanced Systems

In order to use the SPA method, first the system has to be balanced. The balanced system is a system such that the controllability Gramian is the same as the observability Gramian. Thus, in a balanced system, the state variables are ordered based on their influence on the system. In general, the controllability Gramian and the observability Gramian are not the same. To construct a balanced system, we transform the original system using the transformation matrix T so that we obtain a balanced system.

The algorithm to compute matrix T is described as follows:

1 Find a matrix ϕ that satisfies $W = \phi^T\phi$.

2 Construct a matrix $\phi M\phi^T$ and diagonalize $\phi M\phi^T$ so that $\phi M\phi^T = U\Sigma^2U^T$.

3 The non-singular transformation matrix T is $T = \phi^T U\Sigma^{-\frac{1}{2}}$.

So we obtain a balanced system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, where

$$\tilde{A} = T^{-1}AT, \tilde{B} = T^{-1}B, \tilde{C} = CT, \tilde{D} = D.$$

The balanced system has the controllability Gramian $\tilde{W}$ and the observability Gramian $\tilde{M}$ that is a single solution from the Lyapunov equation:

$$\tilde{W} = T^{-1}WT^{-T} \tag{8}$$

and

$$\tilde{M} = T^T MT \tag{9}$$

such that

$$\tilde{W} = \tilde{M} = \sum = diag(\sigma_1, \sigma_2, \cdots, \sigma_r),\ \sigma_1 \geq \cdots \geq \sigma_r \geq \cdots \geq \sigma_n > 0,$$

where σ_i is the Hankel singular value from the balanced system that is defined as

$$\sigma_i = \sqrt{\lambda_i(WM)}, \quad i = 1, 2, ..., n,$$

where λ_i is the eigenvalue from the multiplication of the controllability Gramian W and the observability Gramian M.

6 Model Reduction Using Singular Perturbation Approximation

The singular perturbation approximation (SPA) method can be applied to stable and minimal systems. The reduction process in the SPA method is almost the same as in the balanced truncation (BT) method. In the BT method, the model reduction is performed by cutting the state variables that correspond to small Hankel singular values. In the reduced model using the SPA method, all state variables of the balanced system are partitioned into the fast and slow modes. The variable circumstances corresponding to the small Hankel singular value are defined as a fast mode, while the variable circumstances corresponding to the larger Hankel singular value are defined as a slow mode. Furthermore, the reduced model is obtained by assuming the speed of the fast mode equal to zero [3].

7 Model Reduction Procedure for Unstable and Non-Minimal Systems Using Singular Perturbation Approximation

In this section, we describe the procedure to reduce unstable and non-minimal systems using the singular perturbation approximation. The steps are as follows:

1. We apply the decomposition process to produce a stable and an unstable subsystem (see Section 3).
2. Because the system is non-minimal, then the stable subsystem is also non-minimal. Then we apply the minimization process to the stable subsystem to obtain a minimal stable subsystem (controllable and observable) (see Section 4).

   After step 1 and 2, we will obtain a stable and minimal subsystem.
3. The next step is applying the SPA method to the stable and minimal subsystem (see Section 6) to obtain a reduced stable subsystem.
4. We combine the reduced stable subsystem (from Step 3) and the unstable subsystem (from Step 1) to obtain a total reduced system.
5. The last step is to check or re-analyze the properties of the total reduced system. The properties are stability, controllability and observability.

8 Numerical Example

In this section, we introduce a numerical example as an application of the model reduction using the SPA to the shallow water equation. In the simulation results, we compare the errors of the SPA method and the BT method.

8.1 Shallow water equations

We discuss the shallow water equation that describes the flow of water in rivers [17]:

$$\frac{\partial h}{\partial t} + D\frac{\partial v}{\partial x} = 0, \tag{10}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial x} + C_f u = 0, \tag{11}$$

with the initial and boundary conditions

$$h(x,0) = 1, v(x,0) = 0, h(0,t) = \psi_b(t), v(L,t) = v_N(t), \tag{12}$$

where $h(x,t)$ is the water level above the reference plane at position x and time t, $v(x,t)$ is the average current velocity at position x and time t, t is the time variable, x is the position along the river, D is the water depth, g is the gravitational acceleration and c is a friction constant.

Using the initial and boundary conditions (12) for $i = 14$ to $n = 30$, then, the following values for the parameters are assumed:

$$D = 10m, C_f = 0.0002, \Delta x = 60000/N\Delta t = 360, g = 9.8m/s^2.$$

Thus, we can write the equations (10) and (11) in matrix notation as follows:

$$A_1 x(k+1) = A_2 x(k) + Bu(k), \tag{13}$$

$$x(k+1) = A_1^{-1} A_2 x(k) + A_1^{-1} B_1 u(k). \tag{14}$$

The above system is a discrete-time linear-time-invariant system

$$x(k+1) = Ax(k) + Bu(k),$$

where $A = A_1^{-1} A_2$ and $B = A_1^{-1} B_1$. By using the parameters described above, those matrices are given by

$$A = \left(\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0.3468 & 0.6851 & -0.6054 & 0.2142 & -0.0770 & 0.0272 & -0.0098 & 0.0035 & \cdots \\
0.1271 & 0.6178 & 0.5238 & -0.5392 & 0.1937 & -0.0686 & 0.0246 & -0.0087 & \cdots \\
0.0441 & 0.2142 & 0.5284 & 0.4981 & -0.5382 & 0.1905 & -0.0684 & 0.0242 & \cdots \\
0.0162 & 0.0785 & 0.1937 & 0.5492 & 0.5484 & -0.5479 & 0.1969 & -0.0697 & \cdots \\
0.0056 & 0.0272 & 0.0672 & 0.1905 & 0.5370 & 0.4951 & -0.5371 & 0.1901 & \cdots \\
0.0021 & 0.0100 & 0.0246 & 0.0698 & 0.1969 & 0.5481 & 0.5488 & -0.5481 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & \cdots \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & \cdots \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & \cdots \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & \cdots \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & \cdots \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{matrix}\right.$$

$$\left.\begin{matrix}
\cdots & 0 & 0 & 0 & 0 \\
\cdots & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\
\cdots & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\
\cdots & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\
\cdots & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\
\cdots & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\
\cdots & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\
\ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & -0.0684 & 0.0238 & -0.0098 & 0.0021 \\
\cdots & 0.1973 & -0.0686 & 0.0282 & -0.0059 \\
\cdots & -0.5382 & 0.1870 & -0.0770 & 0.0162 \\
\cdots & 0.5520 & -0.5392 & 0.2219 & -0.0466 \\
\cdots & 0.5284 & 0.4708 & -0.6054 & 0.1271 \\
\cdots & 0.2219 & 0.6178 & 0.7457 & -0.3666 \\
\cdots & 0 & 0 & 0 & 0
\end{matrix}\right), \tag{15}$$

$$B=\begin{pmatrix}1.0000\\0.3468\\0.1271\\0.0441\\0.0162\\0.0056\\0.0021\\0.0007\\ \vdots\\0.0000\\0.0000\\0.0000\\0.0000\\0.0000\\0\end{pmatrix},\ x_{k+1}=\begin{pmatrix}h_0^{k+1}\\u_1^{k+1}\\h_1^{k+1}\\u_2^{k+1}\\h_2^{k+1}\\u_3^{k+1}\\h_3^{k+1}\\u_4^{k+1}\\ \vdots\\h_N^{k+1}\\u_{N+\frac{1}{2}}^{k+1}\\h_{N+1}^{k+1}\\u_{N+1}^{k+1}\\h_{N+2}^{k+1}\\u_{N+\frac{3}{2}}^{k+1}\end{pmatrix},\ x_k=\begin{pmatrix}h_0^k\\u_{\frac{1}{2}}^k\\h_1^k\\u_{\frac{3}{2}}^k\\h_2^k\\u_{\frac{5}{2}}^k\\h_3^k\\u_{\frac{7}{2}}^k\\ \vdots\\h_{N-1}^k\\u_{N-\frac{1}{2}}^k\\h_N^k\\u_{N+\frac{1}{2}}^k\\h_{N+1}^k\\u_{N+1}^k\end{pmatrix},\ u_k=\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\ \vdots\\0\\0\\0\\0\\0\\0\end{pmatrix}, \tag{16}$$

then, we construct a measurement equation at time k as follows:

$$y(k)=Cx(k)+Du(k), \tag{17}$$

where

$$C=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&\cdots&0\end{pmatrix},\quad D=0. \tag{18}$$

Based on the eigenvalues of the original system (A,B,C,D), the system is asymptotically stable, while based on the rank of the controllability and observability matrix, the system is uncontrollable (rank=29) and unobservable (rank=29). Thus, the original system is non-minimal.

Because the original system is asymptotically stable and non-minimal, then we go to step 2 (minimization, see Section 4). To produce a minimal system, we use command "minreal.m" in MATLAB [10]. The order of minimal system G_m is 29. By using the parameters defined in this section, the asymptotically stable and minimal system (controllable and observable) is given by

$$G_m=\left(\begin{array}{cccccccccccc}
0.9614 & -0.1228 & -0.0598 & -0.0799 & -0.0213 & \cdots & -0.0476 & 0.0975 & 0.2965 & 0.0000 & | & 0.1482\\
0.0515 & 0.9614 & 0.0318 & 0.1105 & -0.0233 & \cdots & 0.0671 & 0.0223 & 0.2424 & -0.0000 & | & 0.1212\\
0 & 0 & 0.9315 & -0.3160 & 0.0245 & \cdots & 0.0520 & 0.0104 & 0.1117 & -0.0000 & | & 0.0559\\
0 & 0 & 0.2025 & 0.9315 & 0.0656 & \cdots & 0.1113 & 0.1192 & 0.3712 & 0.0000 & | & 0.1856\\
0 & 0 & 0 & 0 & 0.8756 & \cdots & 0.2364 & -0.0964 & 0.1617 & 0.0000 & | & 0.0809\\
0 & 0 & 0 & 0 & 0.3766 & \cdots & -0.1058 & 0.1330 & 0.2081 & -0.0000 & | & 0.1041\\
0 & 0 & 0 & 0 & 0 & \cdots & -0.0591 & -0.0082 & 0.1998 & -0.0000 & | & 0.0999\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & | & \vdots\\
0 & 0 & 0 & 0 & 0 & \cdots & -0.0042 & -0.0460 & 0.0093 & -0.0000 & | & 0.0047\\
0 & 0 & 0 & 0 & 0 & \cdots & -0.0274 & -0.1478 & 0.0198 & 0.0000 & | & 0.0099\\
0 & 0 & 0 & 0 & 0 & \cdots & 0.1931 & -0.0271 & -0.0212 & 0.0000 & | & -0.0106\\
0 & 0 & 0 & 0 & 0 & \cdots & 0.2287 & -0.8517 & -0.0339 & 0.0000 & | & -0.0169\\
0 & 0 & 0 & 0 & 0 & \cdots & 1.0617 & 0.2287 & -0.0240 & -0.0000 & | & -0.0120\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & | & 0.5000\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & | & 0\\
---- & ---- & ---- & ---- & ---- & ---- & ---- & ---- & ---- & ---- & | & ----\\
-0.2529 & 0.3194 & -0.1701 & 0.0569 & 0.3133 & \cdots & 0.0223 & 0.2130 & 0 & 0 & | & 0
\end{array}\right). \tag{19}$$

Then we construct a balanced system of shallow water equations. In our case, the balanced system ($\tilde{A}$,$\tilde{B}$,$\tilde{C}$,$\tilde{D}$) has the Hankel singular value shown in Figure 1.

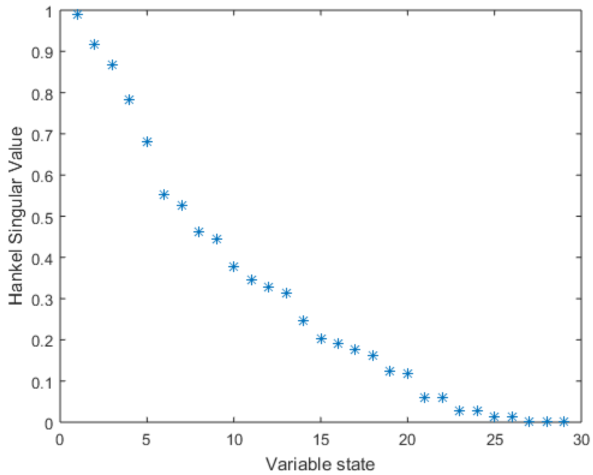

Figure 1: The Hankel singular value of shallow water equation.

After we get a balanced system, we go to the model reduction process. In this case, we reduce the model from order 29 to order 15 ($\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r$) by using the SPA. Based on the eigenvalues of the reduced system, we conclude that the reduced system is asymptotically stable, while based on the rank of the controllability matrix M_c and the observability matrix M_o, the system is controllable (rank=15) and observable (rank=15). Thus, the reduced system is minimal.

After that we compare the frequency response of the total reduced model of order 15 and the original system (minimal system) of order 29 to the original system in Figure 2 as follows. Here, we also show the model reduction using the balanced truncation (BT).

From Figure 2, we can analyze the frequency response of the original system and the total reduced system of order 15 using the SPA method. We can see that the frequency response has similarity in low frequency, but in high frequency, the frequency response tends to be different. Furthermore, we can see that by using the BT method, the frequency response has similarity in high frequency, but in low frequency, the frequency response tends to be different. Thus, the SPA method is better in low frequency, whereas the BT is better in high frequency.

From Figure 2, we can also determine the infinity norm of the error. The results are given in Table 1. Based on Table 1, we know that both model reduction methods (SPA and BT) have the same error bound.

9 Conclusions

The model reduction of unstable and non-minimal systems is conducted by applying the decomposition and minimization to the system. From the results of analysis, the properties of the reduced system are the same as the properties of the original system. From the simulation results, we obtained that the SPA method is better in low frequency

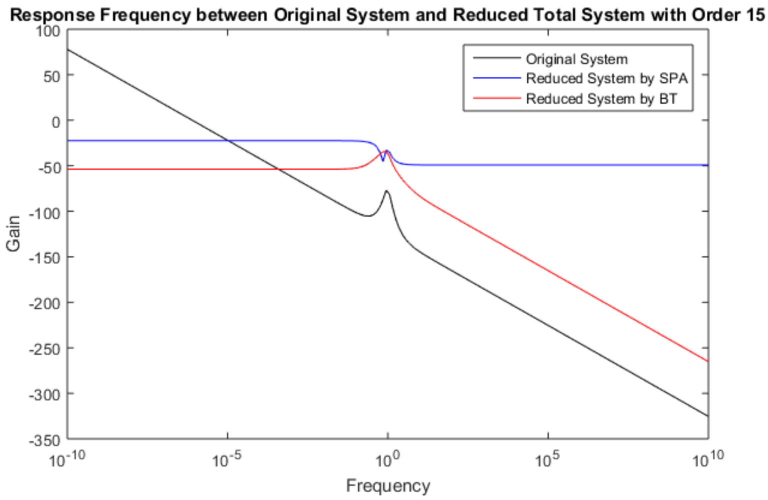

Figure 2: Frequency response of the reduced system with order 15 using SPA and BT to the original system.

Order of Result Reduced Model (r)	$\parallel G - G_r \parallel_\infty$ by SPA	$\parallel G - G_r \parallel_\infty$ by BT
6	1.0199	1.0644
9	0.7101	1.1475
15	0.4047	0.2775
18	0.0443	0.0405
24	0.0061	0.0038
27	8.7333×10^{-4}	8.5926×10^{-4}

Table 1: The ∞ norm of $(G - G_r)$ and the error bound.

and the BT method is better in high frequency. In terms of the error bound, the SPA method and the BT method are almost the same.

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