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Sumudu Decomposition Method for Solving Higher-Order Nonlinear Volterra-Fredholm Fractional Integro-Differential Equations

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Abstract: In this paper, the Sumudu decomposition method is developed to solve the general form of the fractional nonlinear Volterra-Fredholm integro-differential equation. The fractional derivative is described in the Caputo sense. The proposed method is based on the application of the Sumudu transform to the fractional nonlinear Volterra-Fredholm integro-differential equation. The nonlinear term can easily be handled with the help of Adomian polynomials. Illustrative examples are given, and numerical results are provided to demonstrate the efficiency of the proposed method.

Keywords: *approximate Solutions; fractional integro-differential equation; Adomian decomposition; Sumudu transform.*

Mathematics Subject Classification (2010): 45J05, 45G10, 65R20, 65D20.

1 Introduction

Many problems in mathematical physics, theory of elasticity, visco-dynamics fluid and mixed problems of mechanics of continuous media can be reduced to the integral equation (Volterra or Fredholm) of the first or second kind. In [1, 2], the Adomian decomposition method was used to solve a higher-order nonlinear Volterra-Fredholm integro-differential equation of the form

$$\sum_{k=0}^{m} p_k(x)u^{(k)}(x) = f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(u(t))dt + \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(u(t))dt \tag{1}$$

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subject to the initial conditions

$$u^{(\ell)}(0) = \alpha_\ell, \ \ \ell = 0, 1, 2, ..., k-1, \tag{2}$$

where $p_k(x)$ $(k = 0, 1, ..., m)$, $A_i(x,t)$ $(i = 0, 1, ..., r)$, $B_j(x,t)$ $(j = 0, 1, ..., s)$ and $f(x)$ all are given functions. $u^{(k)}$ indicates the $k-$th derivative of $u(x)$, $F(u(x))$ are non-linear functions. It is to be pointed out that $u(x), f(x)$ are assumed to be real, and $\lambda_1, \lambda_2,\ \alpha_\ell, \ell = 0, 1, ..., k-1$ are all real finite constants.

It has turned out that many phenomena in engineering, physics and other sciences can be described very successfully by models using mathematical tools from fractional calculus. Integro-differential equations model many situations from science and engineering, for example, in circuit analysis. The activity of interacting inhibitory and excitatory neurons can be described by a system of integro-differential equations. For a better understanding of the phenomena, fractional derivatives provide more accurate models of real world problems than integer order derivatives do. Because of their many applications in scientific fields, fractional integro-differential equations are found to be an effective tool to describe certain physical phenomena. The most important advantage of using the fractional derivatives in mathematical modeling is due to the non-local property. It is well known that the integer-order differential operator possesses a local operator whereas the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states [3]. In recent times, the fractional calculus is used in different physical and biological problems, see [4–6] and the references therein. Oldham and Spanier [4], Miller and Ross [7], Momani [8] and Podlubny [9] provide the history and a comprehensive treatment of this subject. To solve integro-differential equations, approximate solution and numerical solution methods are being used.

In this paper, we apply the Sumudu transform to solve the general form of the nonlinear Volterra-Fredholm integro-differential equation

$$\begin{aligned} u^{(\alpha)}(x) = \tfrac{-1}{p_m(x)} \sum_{k=0}^{m-1} p_k(x) u^{(k)}(x) + f(x) + \lambda_1 \textstyle\int_a^x \sum_{i=0}^{r} A_i(x,t) F_i(u(t)) dt \\ +\lambda_2 \textstyle\int_a^b \sum_{j=0}^{s} B_j(x,t) G_j(u(t)) dt, \ \ m-1 < \alpha \le m. \end{aligned} \tag{3}$$

The Sumudu transform was first proposed by Watugala [10, 11]. In [12, 13] some fundamental properties of the Sumudu transform were established in light of which the authors developed efficient and straightforward methodologies for treating differential equations. The Sumudu transform method is one of the most important transform methods, it is a powerful tool for solving many kinds of PDEs in various fields of science and engineering [12]. In [14], the authors start from the definition of the Sumudu transform on a general time scales to define the discrete Sumudu transform, and present its basic properties.

In [15], we used a reliable strategy, based on using the Adomian decomposition method (ADM), for solving the same system as in (3). Saadatmandi and Dehghan [17] applied the Legendre collocation method to find numerical solutions of a nonlinear fractional integro-differential equation of only Volterra type. For this current work, we implement the Adomian-Sumudu decomposition method (ASDM) for solving higher-order non-linear fractional Volterra-Fredholm integro-differential equations. The ASDM is an elegant combination of the Sumudu transform method and the ADM. This technique is more powerful because we can combine the Sumudu method and the ADM to obtain the ASDM and it will provide exact and approximate analytical solutions for fractional nonlinear equations. We would like to mention that the ASDM can provide high accuracy of numerical results, reduce the computational time and volume of the work. We would also like to point out that for obtaining the solution by using other methods, we need to solve the equation at other values of the parameter α, and we shall have to compute again for new α. In our method there is no need to perform such repetitive calculation. Against this backdrop, we would like to extend the previous results [1, 18], and also to generalize the results obtained in [15, 16] and to solve the fractional Volterra-Fredholm integro-differential equations (3).

The fractional differential operator $u^{(\alpha)}(x)$ describes the fractional derivatives of order α of equation (3). When $\alpha \in \mathbf{N}$, the equation (3) reduces to a linear integro-differential equation, while if $\lambda_1 = \lambda_2 = 0$, the equation reduces to linear fractional differential equations. Such kind of integro-differential equations is considered for generalizations of the work in [19]. The main objective of this paper is to study the behavior of the solution for equation (1) using the Sumudu decomposition method.

The layout of the paper is as follows. In Section 2, we briefly review some general concepts of the fractional theory and the Sumudu transform required for our subsequent development. In Section 3, we extend the application of the Sumudu-Adomian decomposition to construct our analytical approximate solutions for the general integro-differential equation (3). Finally, numerical experiments are presented and some comparisons are made in Section 4. The paper ends with some concluding remarks.

2 Basics of Fractional Calculus

This section is devoted to the description of the operational properties with the purpose of acquainting with sufficient fractional calculus theory. Many definitions and studies of fractional calculus have been proposed in the last two centuries. These definitions include the Riemman-Liouville, Weyl, Reize, Campos, Caputa, and Nishimoto fractional operators. Mainly, in this paper, we will re-introduce Section 2 of [20]. The Riemann-Liouville definition of fractional derivative operator J_a^α is stated as follows.

Definition 2.1 Let $\alpha \in \mathbf{R}_+$. The operator J^α, defined on the usual Lebesgue space $L_1[a,b]$ by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad J_a^0 f(x) = f(x)$$

for $a \le x \le b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J^α can be found in [9], we mention the following: For $f \in L_1[a,b], \alpha, \beta \ge 0$ and $\gamma > -1$:

1. $J_a^\alpha f(x)$ exists for almost every $x \in [a,b]$.
2. $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$.
3. $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$.
4. $J_a^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma}$.

As mentioned in [8], the Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differentiation operator D^α proposed by Caputo in his work on the theory of visco-elasticity [21].

Definition 2.2 The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{4}$$

$m-1 < \alpha \le m, m \in \mathbf{N}, x > 0.$

Also, we need here two of its basic properties.

Lemma 2.1 *If $m-1 < \alpha \le m$, and $f \in L_1[a,b]$, then $D_a^\alpha J_a^\alpha f(x) = f(x)$, and*

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{(x-a)^k}{k!}, \ x > 0.$$

The Caputo fractional derivative is considered in the Caputo sense. The reason for adopting the Caputo definition is as follows. To solve differential equations, we need to specify additional conditions in order to produce a unique solution. For the case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are taken to the classical differential equations, and are therefore familiar to us. In contrast, for the Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives of the unknown solution at the initial point $x = 0$, which are functions of x. The initial conditions are not physical; furthermore, it is not clear how many quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types, see [8, 21].

Definition 2.3 For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ are defined as

$$D^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N. \end{cases}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

In the early 90s, Watugala [10, 11] introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems.

Definition 2.4 The Sumudu transform over the following set of functions

$$\mathbb{A} = \left\{ f(t) \mid \exists M, \ \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{t_j}}, \ \text{if} \ t \in (-1)^j \times [0, \infty) \right\} \tag{5}$$

is defined for $u \in (\tau_1, \tau_2)$ as

$$\mathbb{S}[f(t)] = F(u) = \int_0^\infty f(ut)e^{-t}dt = \int_0^\infty \frac{1}{u} f(t) e^{-t/u} dt, \tag{6}$$

where u is a parameter and it may be real or complex, that is, independent of t. The inversion formula for the Sumudu transform is given by

$$\mathbb{S}[G(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}G(\frac{1}{s})\frac{ds}{s}.$$

In Belgacem et al. [22], the Sumudu transform was shown to be the theoretical dual of the Laplace transform. Hence, one should be able to rival it to a great extent in problem solving. Given an initial $f(t)$, its Laplace transform $F(s)$ can be transformed into the Sumudu transform $F_s(u)$ of f by means of

$$\mathbb{S}(u) = \frac{F(\frac{1}{u})}{u}.$$

And its inverse is

$$F(s) = \frac{\mathbb{S}(\frac{1}{s})}{s}.$$

Every proven property of the Laplace transform may routinely be turned into a corresponding property of the Sumudu transform. Many of special properties of the Sumudu transform are mentioned and tabulated in [13,22]. Some special properties of the Sumudu transform are as follows:

1. $\mathbb{S}[1] = 1$.
2. $\mathbb{S}\Big[\frac{t^n}{\Gamma(n+1)}\Big] = u^n,\ n > 0$.
3. $\mathbb{S}[f(x) \mp g(x)] = \mathbb{S}[f(x)] \mp \mathbb{S}[g(x)]$.

Theorem 2.1 *[22] Let $G(u)$ be the Sumudu transform of $f(t)$ such that*

1. *$G(1/s)/s$ is a meromorphic function, with singularities having $Re(s) < \gamma$, and*
2. *there exists a circular region Γ with radius R and positive constants M and k with*

$$\Big|\frac{G(1/s)}{s}\Big| < MR^{-k},$$

then the function $f(t)$ is given by

$$f(t) = \mathbb{S}^{-1}[G(t)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+\infty} e^{st}G(\frac{1}{s})\frac{ds}{s} = \sum residuse\Big[e^{st}\frac{G(1/s)}{s}\Big].$$

To solve fractional differential equations, the following lemma of the Sumudu transform will be needed.

Lemma 2.2 *[22] The Sumudu transform $\mathbb{S}[f(t)]$ of the fractional derivative introduced by Caputo is given by*

$$\mathbb{S}[D_t^{\alpha} f(t)] = \frac{G(u)}{u^{\alpha}} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{\alpha-k}}, \quad \text{where } G(u) = \mathbb{S}[f(t)]. \tag{7}$$

3 Implementation of Sumudu Decomposition Method

In the analysis of the numerical method that follows, we will assume that problem (1)-(2) has a unique and sufficiently smooth solution. We apply the Sumudu-Adomian decomposition to find an approximate solution for the fractional integro-differential equations (3). We assume that $u(x)$ is sufficiently differentiable and that a unique solution of (3) exists. Take the Sumudu transform of both sides of equation (3)

$$\begin{aligned}\mathbb{S}[u^{(\alpha)}(x)] &= \mathbb{S}\Big[\frac{-1}{p_m(x)}\sum_{k=0}^{m-1} p_k u^{(k)}(x)\Big] + \mathbb{S}[f(x)] + \mathbb{S}\Big[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(u(t))dt\Big] \\ &+ \mathbb{S}\Big[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(u(t))dt\Big], \quad m-1<\alpha\leq m.\end{aligned}$$

Using the result of equation (7) on the left-hand side of the above equation we arrive at

$$u^{-\alpha}\mathbb{S}[u(t)] - \sum_{k=0}^{n-1} u^{-(\alpha-k)}u^{(k)}(0) = \mathbb{S}\Big[\frac{-1}{p_m(x)}\sum_{k=0}^{m-1} p_k u^{(k)}(x)\Big] + \mathbb{S}[f(x)]$$

$$+ \quad \mathbb{S}\Big[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(u(t))dt\Big] + \mathbb{S}\Big[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(u(t))dt\Big], m-1<\alpha\leq m.$$

Solving for $\mathbb{S}[u(t)]$, we get

$$\mathbb{S}[u(t)] = u^{\alpha}\sum_{k=0}^{n-1} s^{\alpha-k-1}u^{(k)}(0) + u^{\alpha}\mathbb{S}\Big[\frac{-1}{p_m(x)}\sum_{k=0}^{m-1} p_k(x) u^{(k)}(x)\Big] + u^{\alpha}\mathbb{S}[f(x)]$$

$$+u^{\alpha}\mathbb{S}\Big[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(u(t))dt\Big] + u^{\alpha}\mathbb{S}\Big[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(u(t))dt\Big], m-1<\alpha\leq m.$$

Now, following [23, 24], the Sumudu decomposition method introduces the following expressions:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{8}$$

for the solution of our problem, where the components $\mathbb{S}[u_n(t)]$ will be determined recurrently according to a recursive relation. Moreover, the method defines the nonlinear functions $F_i(u(x))$, $(i = 0, 1, ..., r)$, $G_j(u(x))$, $(j = 0, 1, ..., s)$ by the infinite series of polynomials

$$F_i(u(x)) = \sum_{n=0}^{\infty}(C_i)_n, \quad G_j(u(x)) = \sum_{n=0}^{\infty}(D_i)_n \quad \text{and} \quad u^{(k)}(x) = \sum_{n=0}^{\infty} E_i, \tag{9}$$

where the $(C_i)_n, (D_j)_n, E_n$ are the Adomian polynomials which are generated according to specific algorithms set by Adomian [23,24], or by Wazwaz [25]. Substituting equations (8)-(9), yields

$$\mathbb{S}[\sum_{n=0}^{\infty} u_n(x)] = u^{\alpha}\sum_{k=0}^{n-1} u^{-(\alpha-k)}u^{(k)}(0) + u^{\alpha}\mathbb{S}\Big[\frac{-1}{p_m(x)}\sum_{k=0}^{m-1} p_k(x)\sum_{n=0}^{\infty} E_n\Big] + u^{\alpha}\mathbb{S}[f(x)]$$

$$+u^{\alpha}\mathbb{S}\Big[\lambda_1\int_a^x\sum_{i=0}^{r}A_i(x,t)\sum_{n=0}^{\infty}(C_i)_n dt\Big]+u^{\alpha}\mathbb{S}\Big[\lambda_2\int_a^b\sum_{j=0}^{s}B_j(x,t)\sum_{n=0}^{\infty}(D_i)_n dt\Big], m-1<\alpha\leq m.$$

It is useful to note that the recursive relation is constructed on the basis that the zeroth component $\mathbb{S}[u_0]$ is defined by all terms that arise from the initial conditions and from the source term $f(x)$, i.e.,

$$\mathbb{S}[u_0(x)] = u^{\alpha}\sum_{k=0}^{n-1}u^{-(\alpha-k)}u^{(k)}(0) + u^{\alpha}\mathbb{S}[f(x)]. \tag{10}$$

The remaining components of $\mathbb{S}[u(x)]$ can be completely determined so that each term is computed by using the previous terms as

$$\begin{aligned}\mathbb{S}[u_{k+1}(x)] &= u^{\alpha}\mathbb{S}\Big[\frac{-1}{p_m(x)}\sum_{k=0}^{m-1}p_k(x)\sum_{k=0}^{\infty}E_k\Big]\\ &+ u^{\alpha}\mathbb{S}\Big[\lambda_1\int_a^x\sum_{i=0}^{r}A_i(x,t)\sum_{n=0}^{\infty}(C_i)_k dt\Big]\\ &+ u^{\alpha}\mathbb{S}\Big[\lambda_2\int_a^b\sum_{j=0}^{s}B_j(x,t)\sum_{n=0}^{\infty}(D_i)_k dt\Big],\ k\geq 1.\end{aligned}$$

As a result, the components $u_0, u_1, u_2, ...$ are identified by applying the inverse Sumudu transform of the above equations to obtain

$$u_0(x) = \mathbb{S}^{-1}\Big[u^{\alpha}\sum_{k=0}^{n-1}u^{-(\alpha-k)}u^{(k)}(0)\Big] + \mathbb{S}^{-1}\Big[u^{\alpha}\mathbb{S}[f(x)]\Big] \tag{11}$$

and

$$\begin{aligned}u_{k+1}(x) &= \mathbb{S}^{-1}\Big(u^{\alpha}\mathbb{S}\Big[\frac{-1}{p_m(x)}\sum_{k=0}^{m-1}p_k(x)\sum_{k=0}^{\infty}E_k\Big]\Big)\\ &+ \mathbb{S}^{-1}\Big(u^{\alpha}\mathbb{S}\Big[\lambda_1\int_a^x\sum_{i=0}^{r}A_i(x,t)\sum_{n=0}^{\infty}(C_i)_k dt\Big]\Big)\\ &+ \mathbb{S}^{-1}\Big(u^{\alpha}\mathbb{S}\Big[\lambda_2\int_a^b\sum_{j=0}^{s}B_j(x,t)\sum_{n=0}^{\infty}(D_i)_k dt\Big]\Big),\ k\geq 1.\end{aligned}$$

Thus, the series solutions are entirely determined. However in many cases (when α is an integer) the exact solution in a closed form may be obtained [1]. The $n-th$ term approximation $\Phi_n = \sum_{k=0}^{n-1}u_k$ can be used to approximate the solution. The choice of (11) as the initial solution always leads to noise oscillation during the iteration procedure [19]. Also, the choice of $\mathbb{S}[u_0(x)]$ to contain minimal number of terms is giving more flexibility to solve complicated non-linear equations, especially in calculation of inverse Sumudu transform. A reliable modified form of the decomposition method has been introduced by Wazwaz [25]. The construction of the zeroth component of the decomposition series can be defined in a slightly different way. Wazwaz [25] assumed that if the zeroth component $\mathbb{S}[u_0(x)]$ depicted in (10) can be divided into two parts, then one part will be assigned to $\mathbb{S}[u_0(x)]$, while the second part of $\mathbb{S}[u_0(x)]$ can be included in the component of $\mathbb{S}[u_1(x)]$ among other terms.

4 Numerical Examples

In order to assess the advantages of the proposed method (the Sumudu-Adomian method) over the Adomian decomposition method [15] in terms of accuracy and efficiency for solving fractional integro-differential equations, we have applied the method to two different examples with known exact solutions at some values of α. The computations associated with the examples were performed using mathematica.

Example 4.1 Consider the following nonlinear fractional integro-differential equation [15]

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1/2)}\Big(\frac{8}{3}t^{3/2} - 2t^{1/2}\Big) - \frac{t}{1260} + \int_0^1 xtu^4(x)dx,\ 0 \le t \le 1, \tag{12}$$

where $u(0) = 0$, and $\alpha \in (0, 1]$.

Apply the Sumudu transform to both sides of equation (12). For the left-hand side $D^{\alpha}u(t)$ we use the initial condition together with equation (7), while for the first three terms on the right-hand side we use the fact that

$$\mathbb{S}\left(\frac{t^{a-1}}{\Gamma(a)}\right) = u^{a-1}, a > 0.$$

Upon passing simple calculations, we arrive at

$$\mathbb{S}[u(t)] = 2u^{\alpha+\frac{3}{2}} - 2u^{\alpha+\frac{1}{2}} - \frac{1}{1260}u^{\alpha+1} + u^{\alpha}\mathbb{S}\Big[\int_0^1 xtu^4(x)dx\Big].$$

Substituting the decomposition series (8) for $u(t)$, and the series $\sum_{n=0}^{\infty} A_n(t)$ for the nonlinear term $u^4(t)$, we have

$$\mathbb{S}[\sum_{n=0}^{\infty} u_n(t)] = 2u^{\alpha+\frac{3}{2}} - 2u^{\alpha+\frac{1}{2}} - \frac{1}{1260}u^{\alpha+1} + u^{\alpha}\mathbb{S}\Big[\int_0^1 xt\sum_{n=0}^{\infty} A_{n-1}(t)dx\Big],$$

where the first few Adomian polynomials are given by $A_0(t) = u_0^4(t)$, $A_1(t) = 4u_0^3(t)u_1(t)$, $A_2(t) = 6u_0^2(t)u_1^2(t) + 4u_0^3(t)u_2(t)$. The modified decomposition technique introduces the use of the recursive relation

$$\mathbb{S}[u_0(x)] = 2u^{\alpha+\frac{3}{2}} - 2u^{\alpha+\frac{1}{2}}, \tag{13}$$

$$\mathbb{S}[u_1(t)] = -\frac{1}{1260}u^{\alpha+1} + u^{\alpha}\mathbb{S}\Big[\int_0^1 xtA_0(t)dx\Big] \tag{14}$$

and

$$\mathbb{S}[u_2(t)] = u^{\alpha}\mathbb{S}\Big[\int_0^1 xtA_1(t)dx\Big]. \tag{15}$$

In general, we take the n-th term to be

$$\mathbb{S}[u_n(t)] = u^{\alpha}\mathbb{S}\Big[\int_0^1 xtA_{n-1}(t)dx\Big], n \ge 3. \tag{16}$$

Taking the inverse Sumudu transform of both sides of $\mathbb{S}[u_0(t)]$ yields

$$u_0(t) = \frac{2}{\Gamma[\alpha+\frac{3}{2}]}\Big[\frac{t^{\alpha+\frac{3}{2}}}{\alpha+\frac{3}{2}} - t^{\alpha+\frac{1}{2}}\Big].$$

So, we can simplify $\mathbb{S}[u_1(t)]$ appeared in equation (14) as

$$\begin{aligned}\mathbb{S}[u_1(t)] &= -\frac{1}{1260}u^{\alpha+1} + u^{\alpha}\mathbb{S}\Big[\int_0^1 xtA_0(t)dx\Big] \\ &= -\frac{1}{1260}u^{\alpha+1} + u^{\alpha}\mathbb{S}\Big[\frac{t^{3+4\alpha}(3-2t+2\alpha)^4}{2(\Gamma[\alpha+\frac{5}{2}])^4}\Big]\end{aligned}$$

or

$$\begin{aligned}u_1(t,\alpha) &= \frac{8\alpha^4\Gamma(4\alpha+4)t^{5\alpha+3}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+4)} + \frac{48\alpha^3\Gamma(4\alpha+4)t^{5\alpha+3}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+4)} - \frac{64\alpha^3\Gamma(4\alpha+5)t^{5\alpha+4}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+5)} \\ &+ \frac{108\alpha^2\Gamma(4\alpha+4)t^{5\alpha+3}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+4)} - \frac{288\alpha^2\Gamma(4\alpha+5)t^{5\alpha+4}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+5)} + \frac{192\alpha^2\Gamma(4\alpha+6)t^{5\alpha+5}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+6)} \\ &- \frac{t^{\alpha+1}}{1260\Gamma(\alpha+2)} + \frac{108\alpha\Gamma(4\alpha+4)t^{5\alpha+3}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+4)} + \frac{81\Gamma(4\alpha+4)t^{5\alpha+3}}{2\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+4)} \\ &- \frac{216\Gamma(4\alpha+5)t^{5\alpha+4}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+5)} + \frac{576\alpha\Gamma(4\alpha+6)t^{5\alpha+5}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+6)} + \frac{432\Gamma(4\alpha+6)t^{5\alpha+5}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+6)} \\ &- \frac{256\alpha\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+7)} - \frac{384\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+7)} + \frac{128\Gamma(4\alpha+8)t^{5\alpha+7}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+8)} \\ &- \frac{432\alpha\Gamma(4\alpha+5)t^{5\alpha+4}}{\Gamma\left(\alpha+\frac{5}{2}\right)^4\Gamma(5\alpha+5)}.\end{aligned}$$

To obtain the inverse Sumudu transform of $\mathbb{S}[u_2(t)]$ from (15), we use *mathematica* to avoid lengthy calculations. The approximate solution is given by $u_a(t) = u_0(t) + u_1(t) + u_2(t)$. When $\alpha = 0.5$, then $u_a(t) = t^2 - t$ which is the exact solution. The value of $\alpha = 0.5$ is the only case for which we know the exact solution, and our approximate solution is in excellent agreement with the exact values as shown in Figure 2.

Example 4.2 Consider the following nonlinear fourth-order fractional integro-differential equation [8, 17]

$$D^{\alpha}u(x) = 1 + \int_0^x e^{-t}u^2(t))dt,\ 0 \le x \le 1,\ \ 3 < \alpha \le 4, \tag{17}$$

subject to the boundary conditions $u(0) = u'(0) = 1,\ u(1) = u'(1) = e$. Since $3 < \alpha \le 4$, in equation (7), we take $n = 4$. Applying the Sumudu transform to both sides of equation (17), we get

$$\mathbb{S}[D^{\alpha}u(x)] = \mathbb{S}[1] + \mathbb{S}\Big[\int_0^x e^{-t}u^2(t))dt\Big].$$

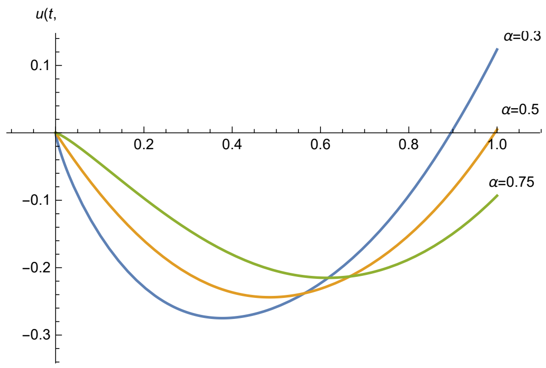

Figure 1: The approximate solution when $0 < t < 1$, for Example 4.1 for different values of α.

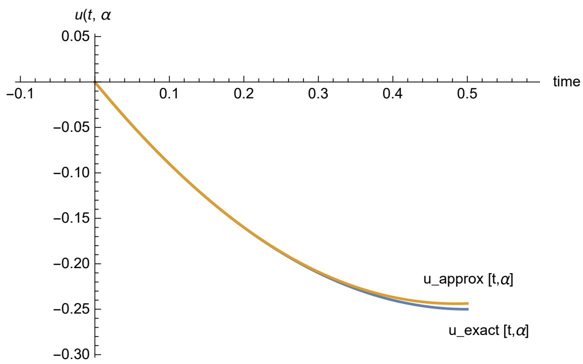

Figure 2: Comparison between the approximate solution when $\alpha = 0.5$ and the exact solution when $0 < t < 0.5$, for Example 4.1.

Use the initial conditions together with equation (7) to obtain

$$\frac{\mathbb{S}[u(x)]}{u^\alpha} - \frac{1}{u^\alpha} - \frac{1}{u^{\alpha-1}} - \frac{A}{u^{\alpha-2}} - \frac{B}{u^{\alpha-3}} = 1 + \mathbb{S}\Big[\int_0^x e^{-t}u^2(t)]dt\Big], \tag{18}$$

where the constants $A = u''(0)$ and $B = u'''(0)$ are to be determined by imposing the other two boundary conditions $u(1) = u'(1) = e$ on the obtained approximate solution. Simplify equation (18), we get

$$\mathbb{S}[u(x)] = 1 + u + Au^2 + Bu^3 + u^\alpha + u^\alpha \mathbb{S}\Big[\int_0^x e^{-t}u^2(t)]dt\Big]. \tag{19}$$

Substituting the decomposition series (8) for $u(x)$, and the series $\sum_{n=0}^{\infty} A_n(t)$ for the nonlinear term $u^2(t)$, we have

$$\mathbb{S}[\sum_{n=0}^{\infty} u_n(x)] = 1 + u + Au^2 + Bu^3 + u^\alpha + u^\alpha \mathbb{S}\Big[\int_0^x e^{-t}\sum_{n=0}^{\infty} A_n(t))dt\Big], \tag{20}$$

where the first two Adomian polynomials are given by $A_0(t) = u_0^2(t)$, $A_1(t) = 2u_0(t)u_1(t)$. The modified decomposition technique introduces the use of the recursive

Sumudu-Adomian decomposition algorithm as

$$\mathbb{S}[u_0(x)] = 1 + u + u^{\alpha}, \tag{21}$$

and

$$\mathbb{S}[u_1(x)] = Au^2 + Bu^3 + u^{\alpha}\mathbb{S}\Big[\int_0^x e^{-t}A_0(t)dt\Big]. \tag{22}$$

The 2−term approximation is given by

$$\phi_2(x, A; B) = u_0(x) + u_1(x), \tag{23}$$

where the constants A and B can be determined using the remaining boundary conditions. Table 1 shows some numerical values for different values of α. The exact solution of the problem in equation (17) is $u(x) = e^x$, and the values in Table 1 corresponding to $\alpha = 4$ are in an excellent agreement with the exact values. Table 2 shows some numerical values for different values of α. In the theory of fractional calculus, it is obvious that when the fractional derivative α $(m - 1 < \alpha \leq m)$ tends to positive integer m, then the approximate solution *continuously* tends to the exact solution of the problem with derivative m. A closer look at the values obtained by our method in Table 1 do have this characteristic. In Table 2, we compare the approximate solution for problem (17) obtained by the proposed method for different values of α with those obtained by the Adomian method [8], and the Legendre collocation method [17]. In Table 2, our results for $\alpha = 4$, which is the only case where we know the exact solution, are in better agreement than those obtained by the methods described in [8] and [17].

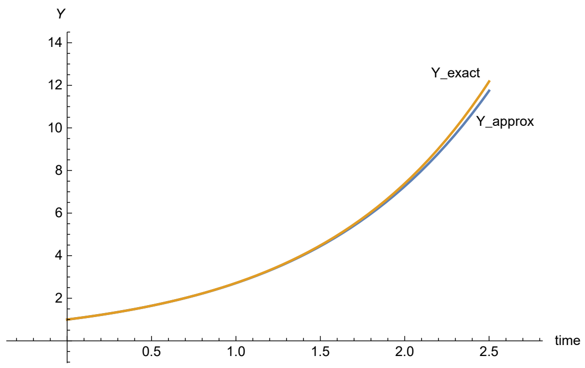

Figure 3: The approximate solution for $0 < t < 2.5$, when $\alpha = 4$, compared to the exact solution for Example 4.2.

5 Conclusion

The fundamental goal of this paper is to propose an efficient algorithm for the solution of fractional integro-differential equations. This goal has been achieved by using the Sumudu-Adomian decomposition method. The validity and accuracy of our approach is examined by solving two examples found in [15, 17]. In order to illustrate the technique, plots of the behavior of the approximate solutions are provided which ensure that the

α (A, B)	3.10 $(1.005, 0.163)$	3.90 $(1.0055, 0.9351)$	3.99 $(0.9974, 1.0056)$	4.00 $(0.996, 1.01)$	*Exact* e^x
$x = 0.1$	1.10517	1.10519	1.10516	1.10516	1.10517
$x = 0.2$	1.22137	1.22145	1.22136	1.22135	1.22140
$x = 0.3$	1.34974	1.34993	1.34978	1.34976	1.34986
$x = 0.4$	1.49161	1.49189	1.49170	1.49168	1.49182
$x = 0.5$	1.64842	1.64877	1.64857	1.64855	1.64872
$x = 0.6$	1.82177	1.82214	1.82197	1.82195	1.82212
$x = 0.7$	2.01344	2.01375	2.01362	2.01361	2.01375
$x = 0.8$	2.22533	2.22552	2.22546	2.22545	2.22554
$x = 0.9$	2.45957	2.45959	2.45957	2.45957	2.45960
$x = 1.0$	2.71828	2.71828	2.71828	2.71828	2.71828

Table 1: Numerical values for Example 4.2 with different values of the order α.

	$\alpha = 3.25$			$\alpha = 3.5$		
x_i	Results [15]	Results [17]	Our Method	Results [15]	Results [17]	Our Method
0.1	1.10101	1.10655	1.10517	1.10675	1.10679	1.10516
0.2	1.21402	1.22393	1.22137	1.22432	1.22441	1.22136
0.3	1.34119	1.35320	1.34974	1.35375	1.35388	1.34978
0.4	1.48170	1.49560	1.49161	1.49627	1.49642	1.49170
0.5	1.63876	1.65255	1.64842	1.65327	1.65343	1.64857
0.6	1.81365	1.82565	1.82177	1.82635	1.82651	1.82197
0.7	2.00662	2.01668	2.01344	2.01729	2.01744	2.01362
0.8	2.22023	2.22763	2.22533	2.22808	2.22819	2.22546
0.9	2.45691	2.46069	2.45953	2.46093	2.46099	2.45957
1.0	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828

Table 2: Comparison of the methods for solving equation (17) for $\alpha = 3.25, 3.5$.

Sumudu decomposition method is a very helpful and efficient method to produce the approximate solutions. Finally, we would like to claim that the method presented in this work for solving nonlinear fractional integro-differential equation is an excellent one in terms of its simplicity, implementation and high accuracy. Also, we conclude that it can be applied to several sophisticated linear and nonlinear equations.

As future work, we aim to apply alternative methods based on different versions of the fractional power series technique [26–31] to solve different types of fractional integro-differential problems and other fractional problems arising in engineering and science applications.

References

[1] K. Al-Khaled and F. Allan. Decomposition method for solving nonlinear integro-differential equations. *J. Appl. Math. and Computing* **19** (1-2) (2005) 415–425.

[2] K. Al-Khaled and E. Az-Zobi. A new convergence proof of the Adomian decomposition method for a mixed hyperbolic elliptic system of conservation laws. *Appl. Math. and Comput.* **217** (8) 4248–4256.

[3] A.M. Nagy and N.H. Sweilam. An efficient method for solving fractional Hodgkin-Huxley model. *Phy. Lett. A* **378** (2014) 1980–1984.

[4] K.B. Oldham and J. Spanier. *The Fractional Calculus.* Academic Press. New York, 1974.

[5] I. Podlubny. *Fractional Differential Equations.* Academic Press. New York, 1999.

[6] I. Podlubny. What Euler could further write, or the unnoticed "big bang" of the fractional calculus. *Frac. Calc. Appl. Anal.* **16** (2013) 501–506.

[7] K.S. Miller. and B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons Inc. New York, 1993.

[8] S. Momani and Z. OdibatZ. Numerical methods for nonlinear paartial differential equations of fractional order. *Appl. Math. Modeling* **32** (2008) 28–39.

[9] I. Podlubny. Gemmetric and physical interpretaion of fractional integration and fractional differentaition. *Frac. Calc. Appl. Anal.* **5** (2002) 367–386.

[10] G.K. Watuagala. Sumudu transform: a new integral transform to solve differential equations in control engineering problems. *Mathematical Engineering in Industry* **6**(4) (1998) 319–329.

[11] G.K. Watuagala. Sumudu transform: an integral transform to solve differential equation and control engineering problems. *Int. J. Math. Educ. Sci. and Technol.* **24**(1) (1993) 35-42.

[12] K. Abdolamir, M.M. Mohammad and H.M. Hamed. Exact Solution of Time-Fractional Partial Differential Equations Using Sumudu Transform. *WSEAS Trans. Math.* **13** (2014) 142–151.

[13] Q. Katatbeh and F. Belgacem. Applications of the Sumudu transform to fractional differential equations. *Nonlinear Studies* **18** (1) (2011) 99–112.

[14] F. Jarad, K. Bayram, T. Abdeljawad and D. Baleanu. On the disrete Sumudu transform. *Romanian Reports in Phys.* **64**(2) (2012) 356–356.

[15] K. Al-Khaled, M. Alquran., A. Al-Saidi, G.M. NGuerekata and J. Chattopadhyay. Numerical Methods for Solving Nonlinear Fractional Integro-Differential Equations. *Nonlinear Studies* **22** (4) (2015) 647–657.

[16] M. Alquran, I. Jaradat and S. Sivasundaram. Elegant scheme for solving Caputo-time-fractional integro-differential equations. *Nonlinear Studies* **25** (2) (2018) 385–393.

[17] A. Saadatmandi and M. Dehghan. A Legendre collocation method for fractional integro-differential equations. *J. of Vibration and Control* **17** (13) (2011) 2050–2058.

[18] A. Ahmed, K.P. Ghadle and S.M. Atshan. The approximate solutions of fractional integro-differential equations by using modified Adomian decomposition method. *Khayyam J. Math.* **5** (1) (2019) 21–39.

[19] C. Yang and J. Hou. Numerical solution of integro-differential equations of fractional order by Laplace decomposition method. *J. Wseas Transactions on Mathematics* **12** (12) (2013) 1173–2880.

[20] K. Al-Khaled. Numerical solution of time-fractional partial differential equations using Sumudu decomposition method. *Rom. J. of Phy.* **60** (1-2) (2015) 99–110.

[21] M. Caputo. Linear models of dissipation whose Q is almost frequency independent-II. *Geophys. J. R. Astron. Soc.* **13** (1967) 529–539.

[22] F. Belgacem and A.A. KaraballiA. Sumudu transform fundamental properties investigations and applications. *International J. Appl. Math. Stoch. (2005) 1–23*, DOI 10.1155/JAMSA/2006/91083.

[23] G. Adomian. A review of the decomposition method in applied mathematics. *J. Math. Anal. Appl.* **135** (1988) 501–544.

[24] G. Adomian. *Solving Frontier Problems of Physics: The Decompsition Method.* Kluwer Academic Publishers. Boston, 1994.

[25] A.M. Wazwaz. A new algorithm for calculating Adomian polynomials for nonlinear operators, *Applied Mathematics and Computation* **111** (2000) 53–69.

[26] M. Alquran, K. Al-Khaled, S. Sivasundaram and H.M. JaradatM. Mathematical and numerical study of existence of bifurcations of the generalized fractional Burgers-Huxley equation. *Nonlinear Studies* **24** (1) (2017) 235–244.

[27] I. Jaradat, M. Alquran and K. Al-Khaled. An analytical study of physical models with inherited temporal and spatial memory. *Eur. Phys. J. Plus.* (2018) 133: 162.

[28] I. Jaradat, M. Al-Dolat, K. Al-Zoubi and M. Alquran. Theory and applications of a more general form for fractional power series expansion. *Chaos Solitons Fract.* **108** (2018) 107–110.

[29] I. Jaradat, M. Alquran and M. Al-Dolat. Analytic solution of homogeneous time-invariant fractional IVP. *Adv. Differ. Equ.* (2018) 2018: 143.

[30] I. Jaradat, M. Alquran and R. Abdel-Muhsen. An analytical framework of 2D diffusion, wave-like, telegraph, and Burgers' models with twofold Caputo derivatives ordering. *Nonlinear Dynamics* **93** (4) (2018) 1911–1922.

[31] H.M. Jaradat, I. Jaradat, M. Alquran, M.M.M. Jaradat, Z. Mustafa, K. Abohassan and Abdelkarim R. Approximate solutions to the generalized time-fractional Ito system. *Italian journal of pure and applied mathematics* **37** (2017) 699–710.

[32] M. Alquran and I. Jaradat. A novel scheme for solving Caputo time-fractional nonlinear equations: theory and application. *Nonlinear Dynamics* **91** (4) (2018) 2389–2395.