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Weakly Nonlinear Integral Equations of the Hammerstein Type

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Abstract: By using the theory of Moore-Penrose pseudoinverse operators, the necessary and sufficient conditions for the solvability of a weakly nonlinear integral equation with a nondegenerate kernel are obtained. Equations for generating constants are constructed. A connection between the necessary and sufficient conditions has been established. The iterative procedure for finding a solution is proposed.

Keywords: *weakly nonlinear integral equations; Moore-Penrose pseudoinverse matrix; generating solution; equation for generating constants; iterative process.*

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1 Introduction

A lot of works are devoted to the investigation of different aspects of the theory of linear and nonlinear integral, differential and integro-differential equations [1, 2, 8–10, 12, 18, 19, 21]. A large part of such equations, in particular integral equations, belong to the equations with not everywhere invertible operator and arise in different areas of the natural science such as electrodynamics, mathematical physics, biology, economics and others [11, 22, 24]. The application of the theory of pseudoinverse operators enabled us to establish the conditions for the existence and the structure of solutions of such equations in the case where the kernel of integral equation is degenerate [5–7, 23, 25]. In the present paper, continuing the research mentioned above, we use one of possible approaches to finding the necessary and sufficient conditions for the solvability of weakly nonlinear integral equations with non-degenerate kernels and propose an algorithm for finding a solution. The obtained theoretical results can be used to study mathematical models and to create effective computational algorithms frequently encountered in applied research.

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2 Statement of the Problem.

We consider the weakly nonlinear integral equation

$$x(t)-\int_a^b K(t,s)x(s)ds=f(t)+\varepsilon\int_a^b K_1(t,s)Z(x(s,\varepsilon),s,\varepsilon)ds. \tag{1}$$

Our aim is to establish conditions for the existence of solution $x=x(t,\varepsilon)$: $x(\cdot,\varepsilon)\in L_2[a,b]$, $x(t,\cdot)\in C[0,\varepsilon_0]$, of equation (1), which turns into one of solutions $x_0(t,c_r)$ of the generating equation

$$x(t)-\int_a^b K(t,s)x(s)ds=f(t) \tag{2}$$

for $\varepsilon=0$. In what follows, the solution $x_0(t,c_r)$ is called a generating solution of the nonlinear equation (1).

Here, $K(t,s)$, $K_1(t,s)$ are square-summable kernels in $[a,b]\times[a,b]$, $f\in L_2[a,b]$, $x\in L_2[a,b]$, $Z(x(t,\varepsilon),t,\varepsilon)$ is the function nonlinear with respect to the first component and such that

$$Z(\cdot,t,\varepsilon)\in C^1[\|x-x_0\|\le q],\quad Z(x(\cdot,\varepsilon),\cdot,\varepsilon)\in L_2[a,b],\quad Z(x(t,\cdot),t,\cdot)\in C[0,\varepsilon_0], \tag{3}$$

where q,ε_0 are sufficiently small constants, $\varepsilon<<1$ is a small parameter.

As in [17], equation (1) can be reduced to a countably dimensional system of weakly nonlinear algebraic equations. Let $\{\varphi_i(t)\}_{i=1}^{\infty}$ be a complete orthonormal system of functions in space $L_2[a,b]$. Let us introduce into consideration the following notations:

$$x_i(\varepsilon)=\int_a^b x(t,\varepsilon)\varphi_i(t)dt,\quad f_i=\int_a^b f(t)\varphi_i(t)dt, \tag{4}$$

$$a_{ij}=\int_a^b\int_a^b K(t,s)\varphi_i(t)\varphi_j(s)dtds,\quad \tilde{a}_{ij}=\int_a^b\int_a^b K_1(t,s)\varphi_i(t)\varphi_j(s)dtds, \tag{5}$$

$$m_i(\varepsilon)=m_i(x_1(\varepsilon),x_2(\varepsilon),\dots,x_i(\varepsilon),\dots,\varepsilon)=\int_a^b Z(x(t,\varepsilon),t,\varepsilon)\varphi_i(t)dt. \tag{6}$$

Then we pass from equation (1) to the countably dimensional system of weakly nonlinear algebraic equations

$$x_i(\varepsilon)-\sum_{j=1}^{\infty}a_{ij}x_j(\varepsilon)=f_i+\varepsilon\sum_{j=1}^{\infty}\tilde{a}_{ij}m_j(\varepsilon),\quad i=\overline{1,\infty}, \tag{7}$$

$$\sum_{j=1}^{\infty}|x_j(\varepsilon)|^2<+\infty,\quad \sum_{j=1}^{\infty}|m_j(\varepsilon)|^2<+\infty,\quad \forall\varepsilon\in[0,\varepsilon_0].$$

We rewrite system (7) in the following vector form:

$$\Lambda z=g+\varepsilon\Lambda_1 V(z(\varepsilon),\varepsilon), \tag{8}$$

where

$$z(\varepsilon) = col\left(x_1(\varepsilon), \quad x_2(\varepsilon), \quad \dots, \quad x_i(\varepsilon), \quad \dots\right) \in \ell_2,$$

$$g = col\left(f_1, \quad f_2, \quad \dots, \quad f_i, \quad \dots\right) \in \ell_2,$$

$$\Lambda = \begin{pmatrix} 1-a_{11} & -a_{12} & \dots & -a_{1i} & \dots \\ -a_{21} & 1-a_{22} & \dots & -a_{2i} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -a_{i1} & -a_{i2} & \dots & 1-a_{ii} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \ \Lambda_1 = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1i} & \dots \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2i} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{a}_{i1} & \tilde{a}_{i2} & \dots & \tilde{a}_{ii} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$V(z(\varepsilon), \varepsilon) = col\left(m_1(\varepsilon), \quad m_2(\varepsilon), \quad \dots, \quad m_i(\varepsilon), \quad \dots\right) \in \ell_2,$$

$$V(\cdot, \varepsilon) \in C^1[\|z - z_0\| \le q], \quad V(z(\cdot), \cdot) \in C[0, \varepsilon_0].$$

The generating operator system for system (8) has the form

$$\Lambda z = g. \tag{9}$$

The following solvability condition is valid for system (9) [7, p. 57].

Theorem 2.1 *The homogeneous system (9) ($g = 0$) possesses an r-parameter family of solutions $z \in \ell_2$*

$$z(c_r) = P_{\Lambda_r} c_r, \quad \forall c_r \in \mathbb{R}^r.$$

The inhomogeneous system (9) is solvable if and only if r linearly independent conditions

$$P_{\Lambda_r^*} g = 0 \tag{10}$$

are satisfied. In this case, the inhomogeneous system (9) possesses an r-parameter family of solutions $z \in \ell_2$

$$z(c_r) = P_{\Lambda_r} c_r + \Lambda^+ g, \quad \forall c_r \in \mathbb{R}^r. \tag{11}$$

Here, P_{Λ_r} is a matrix composed of a complete system of r linearly independent columns of the matrix orthoprojector P_Λ, $P_{\Lambda_r^*}$ is the matrix composed of a complete system of r linearly independent rows of the matrix orthoprojector P_{Λ^*} and Λ^+ is the Moore–Penrose pseudoinverse matrix for the matrix Λ.

3 Necessary Condition for the Existence of Solution.

We now establish a necessary condition for the existence of solution $z(\varepsilon)$ of system (8), which turns into one of the generating solutions $z(c_r)$ of system (9) for $\varepsilon = 0$. The solvability condition of system (8) has the form

$$P_{\Lambda_r^*}(g + \varepsilon \Lambda_1 V(z(\varepsilon), \varepsilon)) = 0.$$

Taking into account (10), we obtain

$$P_{\Lambda_r^*} \Lambda_1 V(z(\varepsilon), \varepsilon) = 0. \tag{12}$$

Since $z(\varepsilon) \to z(c_r)$ as $\varepsilon \to 0$, by using the conditions imposed on the nonlinear function $V(z(\varepsilon), \varepsilon)$, we pass to the limit as $\varepsilon \to 0$ in (12) and obtain a necessary condition for the existence of solution of system (8)

$$P_{\Lambda_r^*} \Lambda_1 V(z(c_r), 0) = 0. \tag{13}$$

Thus, if system (13) has the root $c_r = c_r^0 \in R^r$, then c_r^0 specifies the generating solution $z(c_r^0)$, which may correspond to the solution $z(\varepsilon)$ of system (8). If system (13) has no solutions, then system (8) also does not have the required solution. Here, we speak about real solutions of system (13). We say that equation (13) is the equation for generating constants c_r^0 of the nonlinear system (8) [7]. The conditions of type (13) first emerged in the theory of periodic boundary-value problems for the systems of ordinary differential equations. In this case, the constants c_r have physical meaning: they are amplitudes of periodic solutions. Therefore, these equations are called the equations for generating amplitudes [3, 13, 20].

Theorem 3.1 *Assume that the weakly nonlinear system (8) possesses a solution* $z(\varepsilon)$*:*

$$z(\varepsilon) \in \ell_2, \quad z(\cdot) \in C[0, \varepsilon_0],$$

which turns into the generating solution (11) with constant $c_r = c_r^0 \in R^r$ *for* $\varepsilon = 0$*. Then the vector of constants* c_r^0 *is necessarily a real root of the equation for generating constants (13).*

4 Sufficient Condition for the Existence of Solution

To establish sufficient conditions for the existence of solution, we perform the following change of variables in system (8):

$$z(\varepsilon) = z(c_r^0) + y(\varepsilon),$$

where $z(c_r^0)$ is the generating solution, $c_r^0 \in R^r$ is a real root of equation (13).

We seek the conditions for the existence of a solution $y(\varepsilon)$,

$$y(\varepsilon) \in \ell_2, \quad y(\cdot) \in C[0, \varepsilon_0], \quad y(0) = 0,$$

of the following system

$$\Lambda y(\varepsilon) = \varepsilon \Lambda_1 V(z(c_r^0) + y(\varepsilon), \varepsilon). \tag{14}$$

By using the continuous differentiability of function $V(z, \varepsilon)$ with respect to z in the neighborhood of the generating solution, we separate the linear part in y and the zero-order terms with respect to ε of function $V(z(c_r^0) + y(\varepsilon), \varepsilon)$:

$$V(z(c_r^0) + y(\varepsilon), \varepsilon) = V(z_0(c_r^0), 0) + A_1 y(\varepsilon) + R(y(\varepsilon), \varepsilon), \tag{15}$$

where

$$V(z_0(c_r^0), 0) \in \ell_2, \quad A_1 = A_1(c_r^0) = \left.\frac{\partial V(z, 0)}{\partial z}\right|_{z = z(c_r^0)}, \quad R(y(\varepsilon), \varepsilon) \in \ell_2.$$

Here, we have

$$R(\cdot, \varepsilon) \in C^1(\|y\| \le q), \quad R(y(\cdot), \cdot) \in C[0, \varepsilon_0], \quad R(0, 0) = 0, \quad \frac{\partial R(0, 0)}{\partial y} = 0.$$

Thus, we consider the right-hand side of system (14) as an inhomogeneity. According to Theorem 2.1, system (14) has a solution

$$y(\varepsilon) = P_{\Lambda_r} c_r + \bar{y}(\varepsilon), \quad \forall c_r \in R^r, \tag{16}$$

$$\bar{y}(\varepsilon) = \varepsilon \Lambda^{+} \Lambda_1 V(z(c_r^0) + y(\varepsilon), \varepsilon).$$

The solvability condition of system (14) takes the form

$$P_{\Lambda_r^*} \Lambda_1 V(z(c_r^0) + y(\varepsilon), \varepsilon) = 0. \tag{17}$$

We substitute expansion (15) in equality (17)

$$P_{\Lambda_r^*} \Lambda_1 (V(z_0(c_r^0), 0) + A_1 y(\varepsilon) + R(y(\varepsilon), \varepsilon)) = 0.$$

In view of equation (13) and representation (16), we obtain

$$B_0 c_r = -P_{\Lambda_r^*} \Lambda_1 (A_1 \bar{y}(\varepsilon) + R(y(\varepsilon), \varepsilon)), \tag{18}$$

where B_0 is the $(r \times r)$-dimensional matrix of the form

$$B_0 = P_{\Lambda_r^*} \Lambda_1 A_1 P_{\Lambda_r}. \tag{19}$$

The algebraic system (18) is solvable if and only if the following condition

$$P_{B_0^*} P_{\Lambda_r^*} \Lambda_1 (A_1 \bar{y}(\varepsilon) + R(y(\varepsilon), \varepsilon)) = 0 \tag{20}$$

is satisfied. If

$$P_{B_0^*} P_{\Lambda_r^*} \Lambda_1 = 0, \tag{21}$$

then equality (20) is always satisfied and system (18) possesses a solution.

Thus, we arrive at the following system of operator equations for finding the solution of system (14)

$$\begin{aligned} y(\varepsilon) &= P_{\Lambda_r} c_r(\varepsilon) + \bar{y}(\varepsilon), \\ c_r(\varepsilon) &= -B_0^{+} P_{\Lambda_r^*} \Lambda_1 (A_1 \bar{y}(\varepsilon) + R(y(\varepsilon), \varepsilon)), \\ \bar{y}(\varepsilon) &= \varepsilon \Lambda^{+} \Lambda_1 (V(z_0(c_r^0), 0) + A_1 (P_{\Lambda_r} c_r(\varepsilon) + \bar{y}(\varepsilon)) + R(y(\varepsilon), \varepsilon)). \end{aligned} \tag{22}$$

Introducing a new variable $u = col(y(\varepsilon), c_r(\varepsilon), \bar{y}(\varepsilon))$, we obtain the equation

$$u = Lu + Fu, \tag{23}$$

where

$$L = \begin{pmatrix} 0 & P_{\Lambda_r} & I \\ 0 & 0 & L_1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_1 := -B_0^{+} P_{\Lambda_r^*} \Lambda_1 A_1,$$

$$Fu := \begin{pmatrix} 0 \\ -B_0^{+} P_{\Lambda_r^*} \Lambda_1 R(y(\varepsilon), \varepsilon) \\ \varepsilon \Lambda^{+} \Lambda_1 (V(z_0(c_r^0), 0) + A_1 (P_{\Lambda_r} c_r(\varepsilon) + \bar{y}(\varepsilon)) + R(y(\varepsilon), \varepsilon)) \end{pmatrix}.$$

Since the quasitriangular block matrix operator $I - L$ always possesses the inverse operator, equation (23) can be rewritten in the form

$$u = Su, \quad S := (I - L)^{-1} F. \tag{24}$$

The operator equation (24) belongs to the class of equations, which are solved with the use of the method of simple iterations [4, 7, 13]. We obtain the following iterative process for system (22).

The first approximation $\bar{y}_1(\varepsilon)$ to the element $\bar{y}(\varepsilon)$ is obtained as a particular solution of the equation

$$\Lambda\bar{y}_1(\varepsilon) = \varepsilon\Lambda_1 V(z(c_r^0), 0).$$

This solution exists due to the choice of constant $c_r^0 \in R^r$ from the equation for generating constants (13) and has the form

$$\bar{y}_1(\varepsilon) = \varepsilon\Lambda^+\Lambda_1 V(z(c_r^0), 0).$$

We set the first approximation $y_1(\varepsilon)$ to the solution $y(\varepsilon)$ of system (14) equal to $\bar{y}_1(\varepsilon)$:

$$y_1(\varepsilon) = \bar{y}_1(\varepsilon).$$

The second approximation $y_2(\varepsilon)$ to $y(\varepsilon)$ is obtained from the equation

$$\Lambda y_2(\varepsilon) = \varepsilon\Lambda_1(V(z_0(c_r^0), 0) + A_1(P_{\Lambda_r}c_r^1(\varepsilon) + \bar{y}_1(\varepsilon)) + R(\bar{y}_1(\varepsilon), \varepsilon)). \tag{25}$$

Equation (25) is solvable if and only if the following condition

$$B_0 c_r^1(\varepsilon) = -P_{\Lambda_r^*}\Lambda_1(A_1\bar{y}_1(\varepsilon) + R(\bar{y}_1(\varepsilon), \varepsilon)) \tag{26}$$

is satisfied.

The solvability condition of equation (26) has the form

$$P_{B_0^*}P_{\Lambda_r^*}\Lambda_1(A_1\bar{y}_1(\varepsilon) + R(\bar{y}_1(\varepsilon), \varepsilon)) = 0. \tag{27}$$

Under condition (21), equality (27) is satisfied and the first approximation $c_r^1(\varepsilon)$ to the parameter $c_r(\varepsilon)$ is obtained from equation (26)

$$c_r^1(\varepsilon) = -B_0^+ P_{\Lambda_r^*}\Lambda_1(A_1\bar{y}_1(\varepsilon) + R(\bar{y}_1(\varepsilon), \varepsilon)).$$

The second approximation $y_2(\varepsilon)$ to $y(\varepsilon)$ has the form

$$y_2(\varepsilon) = P_{\Lambda_r}c_r^1(\varepsilon) + \bar{y}_2(\varepsilon),$$

where

$$\bar{y}_2(\varepsilon) = \varepsilon\Lambda^+\Lambda_1(V(z_0(c_r^0), 0) + A_1(P_{\Lambda_r}c_r^1(\varepsilon) + \bar{y}_1(\varepsilon)) + R(\bar{y}_1(\varepsilon), \varepsilon)).$$

The third approximation $y_3(\varepsilon)$ to $y(\varepsilon)$ is obtained from the equation

$$\Lambda y_3(\varepsilon) = \varepsilon\Lambda_1(V(z_0(c_r^0), 0) + A_1(P_{\Lambda_r}c_r^2(\varepsilon) + \bar{y}_2(\varepsilon)) + R(y_2(\varepsilon), \varepsilon)). \tag{28}$$

Equation (28) is solvable if and only if the following condition

$$B_0 c_r^2(\varepsilon) = -P_{\Lambda_r^*}\Lambda_1(A_1\bar{y}_2(\varepsilon) + R(y_2(\varepsilon), \varepsilon)) \tag{29}$$

is satisfied.

The solvability condition of equation (29) has the form

$$P_{B_0^*}P_{\Lambda_r^*}\Lambda_1(A_1\bar{y}_2(\varepsilon) + R(y_2(\varepsilon), \varepsilon)) = 0. \tag{30}$$

Under condition (21), equality (30) is satisfied and the second approximation $c_r^2(\varepsilon)$ to the parameter $c_r(\varepsilon)$ is obtained from equation (29)

$$c_r^2(\varepsilon) = -B_0^+ P_{\Lambda_r^*} \Lambda_1 (A_1 \bar{y}_2(\varepsilon) + R(y_2(\varepsilon), \varepsilon)).$$

The third approximation $y_3(\varepsilon)$ to $y(\varepsilon)$ has the form

$$y_3(\varepsilon) = P_{\Lambda_r} c_r^2(\varepsilon) + \bar{y}_3(\varepsilon),$$

where

$$\bar{y}_3(\varepsilon) = \varepsilon \Lambda^+ \Lambda_1 (V(z_0(c_r^0), 0) + A_1(P_{\Lambda_r} c_r^2(\varepsilon) + \bar{y}_2(\varepsilon)) + R(y_2(\varepsilon), \varepsilon)).$$

Continuing the iterative process, we obtain the following procedure for finding $y(\varepsilon)$:

$$\begin{gathered} c_r^k(\varepsilon) = -B_0^+ P_{\Lambda_r^*} \Lambda_1 (A_1 \bar{y}_k(\varepsilon) + R(y_k(\varepsilon), \varepsilon)), \\ \bar{y}_{k+1}(\varepsilon) = \varepsilon \Lambda^+ \Lambda_1 (V(z_0(c_r^0), 0) + A_1(P_{\Lambda_r} c_r^k(\varepsilon) + \bar{y}_k(\varepsilon)) + R(y_k(\varepsilon), \varepsilon)), \\ y_{k+1}(\varepsilon) = P_{\Lambda_r} c_r^k(\varepsilon) + \bar{y}_{k+1}(\varepsilon), \quad k = \overline{0, \infty}, \\ y_0(\varepsilon) = \bar{y}_0(\varepsilon) = 0. \end{gathered} \tag{31}$$

Hence, the following theorem is true.

Theorem 4.1 *Assume that, under r linearly independent conditions (10), the generating system (9) for system (8) possesses an r-parameter family of solutions $z(c_r) \in \ell_2$ (11). Then, for each real value of vector $c_r = c_r^0 \in R^r$ satisfying the equation for generating constants (13) and under the condition*

$$P_{B_0^*} P_{\Lambda_r^*} \Lambda_1 = 0,$$

system (8) possesses a solution $z(\varepsilon) \in \ell_2$ continuous in ε, which turns into the generating solution $z(c_r^0)$ for $\varepsilon = 0$. This solution can be found from the following iterative process:

$$\begin{gathered} c_r^k(\varepsilon) = -B_0^+ P_{\Lambda_r^*} \Lambda_1 (A_1 \bar{y}_k(\varepsilon) + R(y_k(\varepsilon), \varepsilon)), \\ \bar{y}_{k+1}(\varepsilon) = \varepsilon \Lambda^+ \Lambda_1 (V(z_0(c_r^0), 0) + A_1(P_{\Lambda_r} c_r^k(\varepsilon) + \bar{y}_k(\varepsilon)) + R(y_k(\varepsilon), \varepsilon)), \\ y_{k+1}(\varepsilon) = P_{\Lambda_r} c_r^k(\varepsilon) + \bar{y}_{k+1}(\varepsilon), \\ z_k(\varepsilon) = z(c_r^0) + y_k(\varepsilon), \quad k = \overline{0, \infty}, \\ y_0(\varepsilon) = \bar{y}_0(\varepsilon) = 0. \end{gathered}$$

By using the obtained results for a countably dimensional system of weakly nonlinear algebraic equations (8), we can make conclusions about the existence of solution of weakly nonlinear integral equation (1). We achieve this using the approach applied in [17].

Assume that system (8) possesses at least one solution $z(\varepsilon) = col\left(x_1(\varepsilon), \quad x_2(\varepsilon), \quad \ldots, \quad x_i(\varepsilon), \quad \ldots\right)$. According to the Riesz–Fischer theorem, $x_i(\varepsilon)$ are the Fourier coefficients for the element $x = x(t, \varepsilon)$: $x(\cdot, \varepsilon) \in L_2[a, b]$, $x(t, \cdot) \in C[0, \varepsilon_0]$. Thus, the following representation is true:

$$x(t, \varepsilon) = \sum_{i=1}^{\infty} x_i(\varepsilon) \varphi_i(t) = \Phi(t) z(\varepsilon), \tag{32}$$

where
$$\Phi(t) = \big(\varphi_1(t), \quad \varphi_2(t), \quad \dots, \quad \varphi_i(t), \quad \dots\big),$$
$\{\varphi_i(t)\}_{i=1}^{\infty}$ is a complete orthonormal system of functions in $L_2[a,b]$.

By analogy with [14,15], we can conclude that the set of elements $x(t,\varepsilon)$, defined by the relation (32), is the required family of solutions of the original equation (1).

Hence, we can apply the results of Theorem 4.1 for system (8) to integral equation (1).

Theorem 4.2 *Assume that, under r linearly independent conditions (10), the generating equation (2) for equation (1) possesses an r-parameter family of solutions $x(t,c_r)$. Then, for each real value of vector $c_r = c_r^0 \in R^r$ satisfying the equation for generating constants (13) and under the condition*
$$P_{B_0^*} P_{\Lambda_r^*} \Lambda_1 = 0,$$
equation (1) possesses a solution $x = x(t,\varepsilon)$: $x(\cdot,\varepsilon) \in L_2[a,b]$, $x(t,\cdot) \in C[0,\varepsilon_0]$, which turns into the generating solution $x_0(t,c_r)$ for $\varepsilon = 0$. This solution can be found by using the convergent iterative process
$$\begin{gathered} c_r^k(\varepsilon) = -B_0^+ P_{\Lambda_r^*}\Lambda_1(A_1\bar{y}_k(\varepsilon) + R(y_k(\varepsilon),\varepsilon)),\\ \bar{y}_{k+1}(\varepsilon) = \varepsilon\Lambda^+\Lambda_1(V(z_0(c_r^0),0) + A_1(P_{\Lambda_r}c_r^k(\varepsilon) + \bar{y}_k(\varepsilon)) + R(y_k(\varepsilon),\varepsilon)),\\ y_{k+1}(\varepsilon) = P_{\Lambda_r}c_r^k(\varepsilon) + \bar{y}_{k+1}(\varepsilon),\\ z_k(\varepsilon) = z(c_r^0) + y_k(\varepsilon),\\ x_k(t,\varepsilon) = \Phi(t)z_k(\varepsilon), \quad k = \overline{0,\infty},\\ y_0(\varepsilon) = \bar{y}_0(\varepsilon) = 0. \end{gathered} \tag{33}$$

Remark 4.1 If the condition $P_{B_0} = 0$ is satisfied, then, according to the Fredholm property of index zero of matrix B_0, we obtain $P_{B_0^*} = 0$ and condition (21) is automatically satisfied. In this case $det B_0 \neq 0$ and in the iterative process (33) instead of B_0^+ it will be B_0^{-1}.

Remark 4.2 In the case, where $K(t,s) = 0$, $f(t) = 0$, $\varepsilon = 1$, $K_1(t,s)$ is a piecewise continuous, symmetric, positive-definite kernel, the results introduced in this paper coincide with the results established in [14].

Example 4.1 To illustrate the proposed procedure for the analysis of integral equation of the form (1), we consider the integral equation
$$x(t) - \frac{2}{\pi}\int_0^{\pi} \sin(t+s)x(s)ds = \sin t - \cos t +$$
$$+\,\varepsilon\int_0^{\pi} \cos t \sin s\,\big(\pi(2-\varepsilon^2) - 4(2+3\varepsilon^2)x(s) + 3\pi\varepsilon x^2(s)\big)\,ds \tag{34}$$
and the generating equation
$$x(t) - \frac{2}{\pi}\int_0^{\pi} \sin(t+s)x(s)ds = \sin t - \cos t. \tag{35}$$

Let us consider the orthonormal functions $\varphi_1(t) = \frac{1}{\sqrt{\pi}}(\sin t + \cos t)$ and $\varphi_2(t) = \frac{1}{\sqrt{\pi}}(\sin t - \cos t)$, which are eigenfunctions of the operator

$$(Kw)(t) = \frac{2}{\pi}\int_0^{\pi} \sin(t+s)w(s)ds,$$

and correspond to the characteristic numbers $\lambda_1 = 1$ and $\lambda_2 = -1$, respectively.

We reduce equations (34) and (35) to equations (8) and (9). By using the introduced notation (4)-(6), we obtain

$$\Lambda z = g + \varepsilon\Lambda_1 V(z(\varepsilon), \varepsilon), \tag{36}$$

$$\Lambda z = g, \tag{37}$$

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ \sqrt{\pi} \end{pmatrix}, \quad \Lambda_1 = \frac{\pi}{4}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \tag{38}$$

$$x_1 = \frac{1}{\sqrt{\pi}}\int_0^{\pi} x(t)(\sin t + \cos t)dt, \quad x_2 = \frac{1}{\sqrt{\pi}}\int_0^{\pi} x(t)(\sin t - \cos t)dt,$$

$$V(z(\varepsilon), \varepsilon) = 2\sqrt{\pi}(2-\varepsilon^2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 4(2+3\varepsilon^2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{2\varepsilon}{\sqrt{\pi}}\begin{pmatrix} 5x_1^2 + 2x_1x_2 + x_2^2 \\ x_1^2 + 2x_1x_2 + 5x_2^2 \end{pmatrix}.$$

By using the well-known formulas [7, p. 48], [16, p. 501], we get

$$\Lambda^+ = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad P_\Lambda = P_{\Lambda^*} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{39}$$

Taking into account (38), (39), it is easy to see that the condition for the solvability (10) is satisfied in this case. According to Theorem 2.1, system (37) possesses a solution

$$z(c_r) = \begin{pmatrix} c_r \\ \frac{\sqrt{\pi}}{2} \end{pmatrix}, \quad \forall c_r \in R,$$

and equation (35) has a solution

$$x(t, c_r) = \left(\frac{c_r}{\sqrt{\pi}} + \frac{1}{2}\right)\sin t + \left(\frac{c_r}{\sqrt{\pi}} - \frac{1}{2}\right)\cos t. \tag{40}$$

In the case, the necessary condition for the existence of a solution $z(\varepsilon)$ of system (36), which turns into one of the generating solutions $z(c_r)$ of system (37) for $\varepsilon = 0$, takes the form

$$P_{\Lambda_r^*}\Lambda_1 V(z(c_r), 0) = \frac{\pi}{4}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} m_1(0) \\ m_2(0) \end{pmatrix} =$$

$$= \frac{\pi}{4}(m_1(0) + m_2(0)) = 2\pi\left(\frac{\sqrt{\pi}}{2} - c_r\right) = 0. \tag{41}$$

Equation (41) possesses the unique solution $c_r^0 = \frac{\sqrt{\pi}}{2}$ that specifies the generating solution $z(c_r^0)$, which may correspond to the solution $z(\varepsilon)$ of system (36).

We now establish a sufficient condition for the existence of a solution of system (36). For this purpose, we perform the following change of variables:

$$z(\varepsilon) := z(c_r^0) + y(\varepsilon), \tag{42}$$

where

$$z(c_r^0) = \frac{\sqrt{\pi}}{2}\begin{pmatrix}1\\1\end{pmatrix} \tag{43}$$

is the generating solution of system (36).

System (14) for definition $y(\varepsilon)$ takes the form

$$\Lambda y(\varepsilon) = \varepsilon\Lambda_1 V(z(c_r^0) + y(\varepsilon), \varepsilon), \tag{44}$$

where the matrices Λ, Λ_1 have the form (38) and

$$V(z(c_r^0) + y(\varepsilon), \varepsilon) = 4\sqrt{\pi}(\varepsilon - 2\varepsilon^2)\begin{pmatrix}1\\1\end{pmatrix} -$$

$$-4\begin{pmatrix}3\varepsilon^2 - 3\varepsilon + 2 & -\varepsilon\\ -\varepsilon & 3\varepsilon^2 - 3\varepsilon + 2\end{pmatrix}\begin{pmatrix}y_1(\varepsilon)\\ y_2(\varepsilon)\end{pmatrix} + \frac{2\varepsilon}{\sqrt{\pi}}\begin{pmatrix}(y_1(\varepsilon) + y_2(\varepsilon))^2 + 4y_1^2(\varepsilon)\\ (y_1(\varepsilon) + y_2(\varepsilon))^2 + 4y_2^2(\varepsilon)\end{pmatrix}.$$

That is, in this case

$$V(z_0(c_r^0), 0) = 0, \quad A_1 = -8\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{45}$$

$$R(y(\varepsilon), \varepsilon) = 4\sqrt{\pi}(\varepsilon - 2\varepsilon^2)\begin{pmatrix}1\\1\end{pmatrix} +$$

$$+4\varepsilon\begin{pmatrix}3 - 3\varepsilon & 1\\ 1 & 3 - 3\varepsilon\end{pmatrix}\begin{pmatrix}y_1(\varepsilon)\\ y_2(\varepsilon)\end{pmatrix} + \frac{2\varepsilon}{\sqrt{\pi}}\begin{pmatrix}(y_1(\varepsilon) + y_2(\varepsilon))^2 + 4y_1^2(\varepsilon)\\ (y_1(\varepsilon) + y_2(\varepsilon))^2 + 4y_2^2(\varepsilon)\end{pmatrix}.$$

According to (19), (38), (39), (45), we obtain

$$B_0 = P_{\Lambda_r^*}\Lambda_1 A_1 P_{\Lambda_r} = -2\pi\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = -2\pi.$$

Thus,

$$B_0^+ = B_0^{-1} = -\frac{1}{2\pi}, \quad P_{B_0} = P_{B_0^*} = 0$$

and sufficient condition (21) for the existence of solution of system (36) is satisfied.

After appropriate calculations, we obtain that under condition (21), system (44) is equivalent to the system

$$\begin{gathered} c_r(\varepsilon) = \sqrt{\pi}(\varepsilon - 2\varepsilon^2) + \frac{1}{2}(4\varepsilon - 3\varepsilon^2)y_1(\varepsilon) + \frac{3\varepsilon}{2\sqrt{\pi}}(y_1(\varepsilon))^2,\\ \bar{y}_1(\varepsilon) = \bar{y}_2(\varepsilon) = y_2(\varepsilon) = 0, \quad y_1(\varepsilon) = c_r(\varepsilon). \end{gathered} \tag{46}$$

In this case, algorithm (31) takes the form

$$\begin{gathered} c_r^k(\varepsilon) = \sqrt{\pi}(\varepsilon - 2\varepsilon^2) + \frac{1}{2}(4\varepsilon - 3\varepsilon^2)y_1^k(\varepsilon) + \frac{3\varepsilon}{2\sqrt{\pi}}(y_1^k(\varepsilon))^2,\\ \bar{y}_1^{k+1}(\varepsilon) = \bar{y}_2^{k+1}(\varepsilon) = y_2^{k+1}(\varepsilon) = 0, \quad y_1^{k+1}(\varepsilon) = c_r^k(\varepsilon), \quad k = \overline{0, \infty},\\ y_1^0(\varepsilon) = y_2^0(\varepsilon) = \bar{y}_1^0(\varepsilon) = \bar{y}_2^0(\varepsilon) = 0. \end{gathered} \tag{47}$$

Convergence of the method of simple iterations (47) can be estimated by the method of Lyapunov majorants [3, 13]. The majorizing system for system (46) takes the form

$$v = U(v,\varepsilon) = \sqrt{\pi}(\varepsilon - 2\varepsilon^2) + \frac{1}{2}(4\varepsilon - 3\varepsilon^2)v + \frac{3\varepsilon}{2\sqrt{\pi}}v^2,$$
$$\bar{u}_1 = \bar{u}_2 = u_2 = 0, \quad u_1 = v.$$

To estimate the range of convergence for ε of the iterative process (47), we construct the system

$$v = U(v,\varepsilon), \quad 1 - \frac{\partial U}{\partial v} = 0.$$

This system possesses a real positive solution

$$\varepsilon^* = -\frac{1}{3}(2-\sqrt{10}) \approx 0,3874, \quad v^* = -\frac{\sqrt{\pi}}{3}(2-\sqrt{10}) \approx 0,6867.$$

Hence, system (36) has a solution $z(\varepsilon)$ in a neighborhood of $\varepsilon = 0$, which turns into the generating solution $z(c_r^0)$ for $\varepsilon = 0$. This solution can be found by the use of iterative process (47) convergent for $\varepsilon \in [0, \varepsilon^*]$ and equality (42).

We construct the first few approximations of the iterative process by scheme (47)

$$\begin{gathered} y_1^1(\varepsilon) = 0, \\ y_1^2(\varepsilon) = \sqrt{\pi}(\varepsilon - 2\varepsilon^2), \\ y_1^3(\varepsilon) = \sqrt{\pi}(\varepsilon - 4\varepsilon^3 - 3\varepsilon^4 + 6\varepsilon^5), \\ y_1^4(\varepsilon) = \frac{\sqrt{\pi}}{2}(2\varepsilon - 16\varepsilon^4 - 24\varepsilon^5 + 15\varepsilon^6 + 66\varepsilon^7 + 72\varepsilon^8 - 117\varepsilon^9 - 108\varepsilon^{10} + 108\varepsilon^{11}). \end{gathered} \tag{48}$$

As we see, the constructed approximate solutions in the neighborhood of $\varepsilon = 0$ lead to the vector

$$y^*(\varepsilon) = \begin{pmatrix} \sqrt{\pi}\varepsilon \\ 0 \end{pmatrix}. \tag{49}$$

One can easily verify by substitution that this vector is a solution of equation (44). Deviation of approximations (48) from the exact solution (49) is represented in the table.

Table 1: Approximation accuracy constructed by the method of simple iteration (47)

ε	$\lvert y_1^*(\varepsilon) - y_1^1(\varepsilon)\rvert$	$\lvert y_1^*(\varepsilon) - y_1^2(\varepsilon)\rvert$	$\lvert y_1^*(\varepsilon) - y_1^3(\varepsilon)\rvert$	$\lvert y_1^*(\varepsilon) - y_1^4(\varepsilon)\rvert$
0,3874	0,686648	0,532015	0,439177	0,375906
0,3000	0,531736	0,319042	0,208653	0,142307
0,2000	0,354491	0,141796	0,061823	0,027792
0,1000	0,177245	0,035449	0,007515	0,001611
0,0100	0,017725	0,000354	0,000007	0,000000

Thus, according to (42) and (49), the solution $z^*(\varepsilon)$ of system (36), which turns, for $\varepsilon = 0$, into the generating solution (43) of system (37), has the form

$$z^*(\varepsilon) = \frac{\sqrt{\pi}}{2}\begin{pmatrix} 1 + 2\varepsilon \\ 1 \end{pmatrix}.$$

And, according to Theorem 4.2, the solution of equation (34) takes the form

$$x(t) = (\varepsilon + 1)\sin t + \varepsilon \cos t. \tag{50}$$

It is easy to see that solution (50) is transformed, for $\varepsilon = 0$, into the generating solution (40) with a constant $c_r^0 = \frac{\sqrt{\pi}}{2}$. The constant $c_r^0 = \frac{\sqrt{\pi}}{2}$ is the root of the equation for generating constants (41).

5 Conclusion

We considered the weakly nonlinear integral equation of the Hammerstein type in space $L_2[a, b]$ with a parameter. The problem of existence and construction of solutions, which turn into one of solutions of the generating equation for zero value of the parameter, is investigated. The equation for generating constants is obtained and it is shown that for the existence of the required solution it is necessary that this equation possesses at least one real root. Sufficient conditions for the existence of such a solution are obtained and a constructive algorithm for its finding is proposed. An illustrative example is given. The obtained results are also valid for the case of weakly nonlinear Fredholm boundary-value problems for integral equations.

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