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A Piecewise Orthogonal Functions-Based Approach for Minimum Time Control of Dynamical Systems

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Abstract: This paper introduces a numerical technique for solving minimum time control problems. These problems are addressed to linear time invariant systems in feedforward and feedback control. The mathematical formulation of the control problem is expanded in several piecewise orthogonal bases, namely, the Walsh, block-pulse and Haar wavelets. Operational matrices are used to transform the integration procedure into a product. A numerical optimization problem is formulated to determine the final time and the control sequence (switching times) necessary to steer the system from an initial to a target position. The used numerical method shows that the employed piecewise orthogonal function generates better results than other functions.

Keywords: *orthogonal functions; operational matrices; minimum time control; linear systems; closed loop scheme.*

Mathematics Subject Classification (2010): 93C35, 93D15.

1 Introduction

After the introduction of human operated machines, there was a need to enhance further the productivity and reduce costs. Therefore, automatic machines (i.e. robots) were designed and introduced. Today, many engineering systems, from manufacturing machines to vehicles and airplanes, require optimal control algorithms in order to operate efficiently. Pontryagin [1] developed the theoretical background needed to formulate and then solve these problems. Nevertheless, due to the nature of these engineering systems, finding a solution to these control problems remains a challenging task and requires multidisciplinary knowledge, from ordinary differential equation (ODE) discretization to optimisation so that to obtain a numerical solution. The control problems can be derived

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in two categories: linear and nonlinear. The nonlinear problems feature nonlinear ODEs and are not the scope of this paper. This paper focuses on solving an optimal control problem for linear systems(i.e. the ODE is linear in states and control, even though the problem formulation is non-linear), and particularly, on the determination of a minimum time optimal control.

Finding a solution to the minimum time control problem is a difficult task. The intent of these problems is to steer a system from a given initial state to a target state in minimum time. Often, and due to the complexity of the mathematical formulation, it is difficult to find an analytical solution even for linear systems. In fact, very few examples have an analytical solution obtained through the Pontryagin maximum principle, it is well known that when constrains over system inputs are considered, the obtained minimum time control is necessarily of a bang-bang form [2].

Nevertheless, the minimum control problem could be undertaken with numerical approaches based on nonlinear optimization techniques like the shooting method [3]. Other approaches in literature are typically based on geometric or graphical resolution [4], however, despite of accuracy, these techniques are of limited usage to low order LTI systems.

The orthogonal functions constitue a considerable tool to solve various optimal control problems [5]. Generally, when orthogonal polynomials are used, it is called a pseudo-parametrization technique. In fact, that issue could be an interesting alternative to the securitization technique and could save considerably computational effort since it reduces unknown parameters in the nonlinear optimization problem.

There are different types of orthogonal functions:

- Piecewise functions (block-pulse, Walsh and Haar wavelets) [6, 7];
- Polynomials (Legendre, Chebyshev,...) [8];
- Trigonometric functions (sine, cosine,...) [9].

Researchers have tried to solve the minimum time control problem using the Chebyshev orthogonal functions for open loop linear systems [8], multivariable systems [10] and PID control [11].

Since the type of control is known a priori (i.e. the bang-bang control), it is suitable to use piecewise orthogonal functions thus allowing the capture of discontinuities in the inputs. This method is simpler compared to the methods proposed in [8] and [10] where the Chebyshev orthogonal polynomials had been used. In fact, in those works a set of equalities are derived where each one contains two unknown variables. Then, the authors [11] formulated a parameter optimisation problem to find the final time t_f and using the latter variables they determine the control sequence also.

In this effort, we use a simpler method that exploits the operational matrix of integration [6], and thus, there is no need to find the coefficient in [8, 10] making the problem formulation easier. Furthermore, in addition to the open loop optimization, a closed loop algorithm is formulated.

This paper is organized as follows. The second section is reserved to the formulation of the minimum time control problem. In the third section, a description of the orthogonal functions used and their algebraic properties are provided. The formulation of the proposed method and simulation results are presented in the fourth section. Finally, conclusions and future works are given in the last section.

2 Time-Optimal Constrained Feedforward Control Problem

We consider an LTI system described by the following state space model:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{cases} \tag{1}$$

where $y \in \mathbb{R}^p$ is the output, $u \in \mathbb{R}^m$ is the input control signal and $x \in \mathbb{R}^n$ is the state vector. In general, if the final state is not zero, we define a new system state X given in equation (2) such that the system becomes normalized and the target remains the origin of state space. Then the system (1) can be written as

$$\begin{aligned} X &= x - x_f, \\ \dot{X} &= AX + Bu + Ax_f, \\ X_0 &= x_0 - x_f, \end{aligned} \tag{2}$$

where x_0 is the initial position of the system and x_f is the target position to reach.

To minimize the final time, the cost function is taken as [12]

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt. \tag{3}$$

Applying the Pontryagin maximum principle (PMP) [1], we define the Hamiltonian [13] for (1)

$$H(.) = -1 + \lambda^T (AX + Ax_f + Bu). \tag{4}$$

The canonical equation of Hamilton is given by

$$\dot{X} = H_\lambda = AX + Ax_f + Bu, \tag{5a}$$

$$\dot{\lambda} = -H_x = -A^T \lambda. \tag{5b}$$

The target state being the origin is

$$X(t_f) = 0. \tag{6}$$

Minimizing the Hamiltonian we obtain the following control signal:

$$u(t) = sign(\lambda^T B). \tag{7}$$

This can be written as follows:

$$u(t) = \begin{cases} u_{min}, & if \quad \lambda^T B < 0, \\ u_{max}, & if \quad \lambda^T B > 0. \end{cases} \tag{8}$$

Thus, the obtained control is bang-bang.

3 Orthogonal Functions and Algebraic Properties

Using orthogonal functions to construct operational matrices was firstly proposed in the study of dynamic systems for modeling [14], identification [15] and control purposes [16].

3.1 Principle

Let $\phi_i(t)$ be a set of orthogonal polynomials, piecewise functions. Any analytical function absolutely integrable on the time interval $[0, T]$ can be approximated as follows:

$$f(t) = \sum_{i=0}^{\infty} f_i \phi_i(t), \tag{9}$$

where the coefficients f_i are evaluated by the following scalar product:

$$f_i = \int_0^T f(t)\phi_i(t)dt. \tag{10}$$

For numerical purposes, a truncation of equation (9) until a convenient number of elementary functions is considered in practice.

$$f(t) \cong \sum_{i=0}^{N-1} f_i \phi_i(t) = F_N^T \Phi_N(t), \tag{11}$$

where $\Phi_N^T = [\varphi_0(t)\varphi_1 \cdots \varphi_{N-1}(t)]$ is the orthogonal basis and $F_N^T = [f_0 f_1 \cdots \ f_{N-1}]$ is the coefficient vector.

Integrating equation (11), we obtain:

$$\int f(t) \cong F_N^T P_N \Phi_N(t), \tag{12}$$

where $P_N \in \mathbb{R}^{n \times n}$ is the operational matrix of integration depending on the considered orthogonal basis. As a result, the differential equations describing dynamic processes can be reduced into algebraic relations allowing important simplifications in the synthesis problems.

In this paper, we focus on three types of piecewise orthogonal functions, which are block-pulse, Walsh and Haar wavelets. They present different characteristics. The main difference and properties of each one will be detailed in the next section.

3.2 Block-pulse functions

Block-pulse functions constitute a complete set of orthogonal functions and are defined as follows [7,17]:

$$b_i(t) = \begin{cases} 1, & if\ t \in [iT\ ,\ (i+1)T], \\ & i = 0, ..., N-1, \\ 0, & otherwise. \end{cases} \tag{13}$$

A function $f(t)$ can be approximated by

$$f(t) \simeq \sum_{i=0}^{N-1} f_i b_i(t) = F_N^T B(t), \tag{14}$$

with: $F_N = [f_0, f_1, ..., f_{N-1}]^T$ is the coefficient vector, $B(t) = [b_0(t), b_1(t), ..., b_{N-1}(t)]^T$ is the block-pulse basis vector and f_i are given by

$$f_i = N \int_{(i-1)T}^{iT} f(t) b_i(t) dt, \tag{15}$$

where N is the order of block-pulse functions.

The operational matrix for the block-pulse functions denoted $P_{N,bp}$ is given by [5]

$$P_{N,bp} = \frac{T}{N} \begin{bmatrix} \frac{1}{2} & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{2} & 1 & \dots & 1 \\ \vdots & \ddots & \frac{1}{2} & \dots & 1 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \frac{1}{2} \end{bmatrix}. \tag{16}$$

The representation of this basis for $N = 8$ can be described in Figure 1.

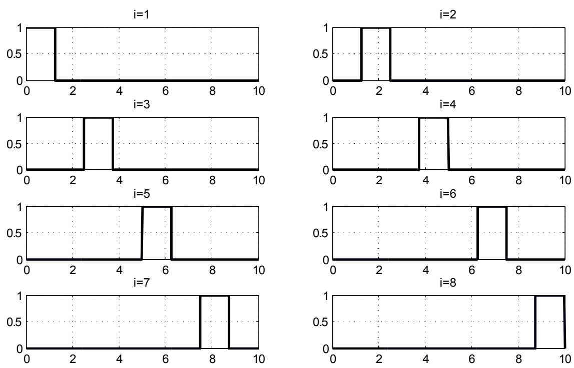


Figure 1: A set of block-pulse functions.

3.3 Walsh functions

Walsh functions belong to the family of piecewise orthogonal functions [18]. They can have only two values +1 or -1 over the interval of interest.

A function $f(t)$, absolutely integrable in $[0, 1]$, may be expanded into the Walsh series as

$$f(t) \simeq \sum_{i=0}^{N-1} f_i w_i(t) = F_N^T W(t). \tag{17}$$

The Walsh functions $w_0(t), w_1(t), \cdots, w_{N-1}(t)$ are orthonormal square waves.

To determine the operational matrix, i.e. $P_{N,w}$, of integration, the equation (18) is used:

$$P_{N,w} = \begin{bmatrix} P_{\frac{N}{2}\times\frac{N}{2}} & \frac{-1}{2N} I_{\frac{N}{2}\times\frac{N}{2}} \\ \frac{1}{2N} I_{\frac{N}{2}\times\frac{N}{2}} & 0 \end{bmatrix}, \tag{18}$$

Where I is the identity matrix. Then the same state space transformation for the block-pulse function is used.

In fact there is a matrical relation between block-pulse and Walsh operational matrix of integration [17]:

$$P_{N,w} = W_{N\times N} \times P_{N,bp} \times W_{N\times N}^{-1}, \tag{19}$$

where W denotes the transition matrix from block-pulse to Walsh basis. For $N = 4$ the Walsh transformation is

$$W_{4\times4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \quad (20)$$

This basis can be described in Figure 2 with N=8.

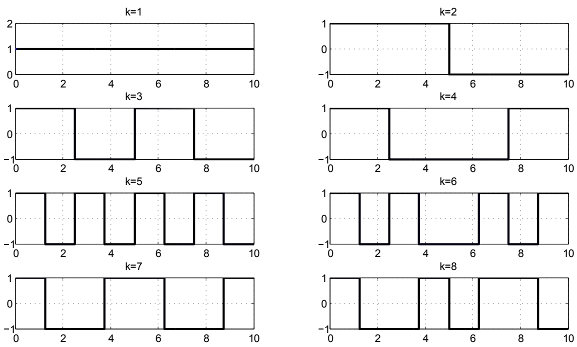


Figure 2: A set of Walsh functions.

3.4 Haar wavelets

The orthogonal set of Haar functions defined in [19] is a group of square waves with magnitude of ±1 in some intervals and zeros elsewhere. The first function is $h_0 = 1$ $\forall x \in [0,1]$. It is commonly referred to as the scaling function. The second is the fundamental square or the mother wavelet which spans the hole interval $[0,1]$, for N=4, for example,

$$h_1(t) = [1\ \ 1\ -1\ -1]\phi_4(t). \quad (21)$$

All the other subsequent curves are generated from $h_1(t)$ with two operations: translation and dilation. $h_2(t)$ is obtained from $h_1(t)$ with dilation, namely, $h_1(t)$ is compressed from the whole interval $[0,1]$ to the half interval $[0,1/2]$ to generate $h_2(t)$. $h_3(t)$ is the same as $h_2(t)$ but shifted to the right by 1/2. Similarly, $h_2(t)$ is compressed from the half interval to the quarter interval to generate $h_4(t)$. $h_4(t)$ is translated to the right by 1/4, 1/2 and 3/4 to generate $h_5(t)$, $h_6(t)$ and $h_7(t)$, respectively.

The general description of the square waves is given as follows:

$$h_0(t) = 1,$$
$$h_i(t) = \begin{cases} 2^{j/2}, \frac{k-1}{2^j} \le t < \frac{k-1/2}{2^j}, \\ -2^{j/2}, \frac{k-1}{2^j} \le t < \frac{k-1/2}{2^j}, \\ 0, otherwise\ in[0,1). \end{cases} \quad (22)$$

The index $i = 1, 2, \ldots, N-1$, j and k represent the integer decomposition of i as follows: $i = 2^j + k - 1$.

The description of the Haar wavelets can be seen in Figure 3 for N=8.

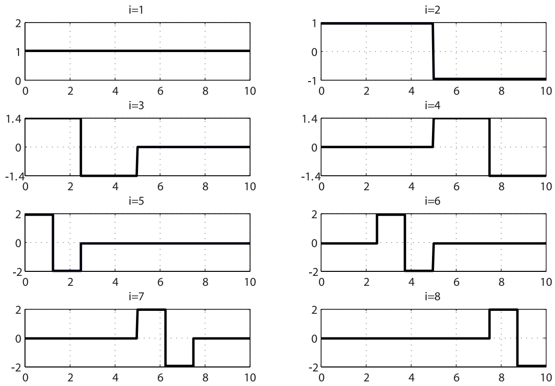


Figure 3: A set of Haar functions.

The operational matrix of integration for the Haar wavelets denoted $P_{N,h}$ is given as follows:

$$P_{N,h} = \frac{1}{2N} \begin{bmatrix} 2NP_{\frac{N}{2}\times\frac{N}{2}} & -H_{\frac{N}{2}\times\frac{N}{2}} \\ H^{-1}_{\frac{N}{2}\times\frac{N}{2}} & 0 \end{bmatrix}, \tag{23}$$

where

$$H_{N\times N} \triangleq \begin{bmatrix} h_N(t_0) & h_N(t_1) & \cdots & h_N(t_{N-1}) \end{bmatrix}.$$

As the Walsh function, the operational matrix of the Haar wavelets can be expressed by the block-pulse operational matrix [17]

$$P_{N,h} = H_{N\times N} \times P_{N,bp} \times H^{-1}_{N\times N}. \tag{24}$$

For N=4, $H_{N\times N}$ is as follows:

$$H_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}. \tag{25}$$

4 Orthogonal Function Based Minimum Time Control Problem Formulation

4.1 The original open loop problem

Minimum time control is an open loop control problem. It is described by Figure 4, where $X(0)$ is the known initial system state and $u(t)$ is the control vector. This framework is dedicated to the class of systems described by equation (1). Here $x(t)$ is the system state trajectory that is needed to search for a prefixed target state in a minimum time t_f to be calculated.

In this work, we intent to develop a numerical method that is able to return the final time t_f and the control sequence (or precisely the control coefficient over an orthogonal function basis), while the initial $X(0)$ (i.e. its coefficients over the same basis) should be provided to the algorithm.

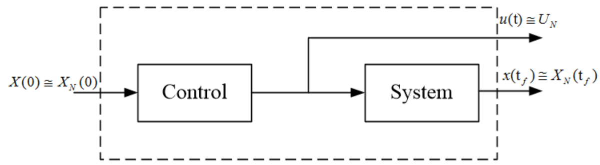


Figure 4: An open loop control structure.

4.2 Main developement

Finding the solution of (5b) means solving multiple differential equations, which is mathematically delicate. To overcome this difficulty we will make use of the set of orthogonal functions described in the last section.

In order to derive the final time, a variable change is introduced:

$$t = \tau t_f. \tag{26}$$

This change of variable allows a transformation of the time domain from $t \in [0, t_f]$ to $\tau \in [0, 1]$, then system states becomes

$$X(t) = \tilde{X}(\tau). \tag{27}$$

Notice that the latter variable change leads to a constant time interval [0,1] for the used series since the final time t_f is unknown.

Consequently, we deduce

$$\dot{X}(t) = \frac{d\tilde{X}(\tau)}{d\tau} . \frac{d\tau}{dt} = \frac{1}{t_f}\dot{\tilde{X}}(\tau). \tag{28}$$

The original state equation of system (1) is now equivalent to

$$\frac{1}{t_f}\dot{\tilde{X}}(\tau) = A\tilde{X}(\tau) + B\tilde{u}(\tau). \tag{29}$$

Using orthogonal functions consists in developing both, the system states and the input over that basis:

$$\tilde{X}(\tau) = \tilde{X}_N^T \cdot \phi_N(\tau), \quad \tilde{u}(\tau) = \tilde{u}_N^T \cdot \phi_N(\tau), \tag{30}$$

where $\phi_N(\tau) \in B(\tau), W(\tau), H(\tau)$. Furthermore, integrating equation (29) leads to

$$\frac{1}{t_f}(\tilde{X}(\tau) - \tilde{X}(0)) = A\int_0^1 (\tilde{X}(\tau))d\tau + B\int_0^1 \tilde{u}(\tau)d\tau. \tag{31}$$

Introducing coefficients of $\tilde{X}(\tau)$, $\tilde{u}(\tau)$ and the operational matrix of integration we obtain

$$\int_0^1 \tilde{X}(\tau)d\tau = \tilde{X}_N \int_0^1 \phi_N(\tau)d\tau = \tilde{X}_N^T P_N \phi_N(\tau), x \tag{32}$$

then we can write

$$(\tilde{X}_N^T - \tilde{X}_{N_0}^T)\phi_N = t_f(A\tilde{X}_N^T P_N + B\tilde{U}_N^T P_N)\phi_N, \tag{33}$$

thus

$$\tilde{X}_N^T - \tilde{X}_{N0}^T = t_f(A\tilde{X}_N^T P_N + B\tilde{U}_N^T P_N), \tag{34}$$

where $\tilde{X}_{N0}$ is a projection of the initial state over orthogonal functions and depends on the chosen set of functions.

4.3 OFs Optimization problem formulation

To find the transition time from the initial to the target position, we need to solve the following nonlinear problem:

Original optimization problem

$$\textbf{min}\ (t_f) \tag{35}$$

subject to:

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t), \quad 0 \le t \le t_f, \\ u \in [u_{min}, u_{max}], \\ x(0) = x_0, x(t_f) = x_f. \end{array} \tag{36}$$

This problem is reported to the domain $[0, \tau]$. The optimization algorithm in the orthogonal basis has the following form:

Orthogonal function optimization problem

$$\textbf{min}\ (t_f) \tag{37}$$

subject to linear constraints: initial state expansion:

- for the block-pulse function

$$\tilde{X}_{N0,bp} = \left[\begin{array}{cccc} \tilde{X}(0) & \tilde{X}(0) & \cdots & \tilde{X}(0) \end{array} \right],$$

- for the Walsh functions and Haar wavelets

$$\tilde{X}_{N0,w} = \tilde{X}_{N0,h} = \left[\begin{array}{cccc} \tilde{X}(0) & 0 & \cdots & 0 \end{array} \right]$$

final state expansion:

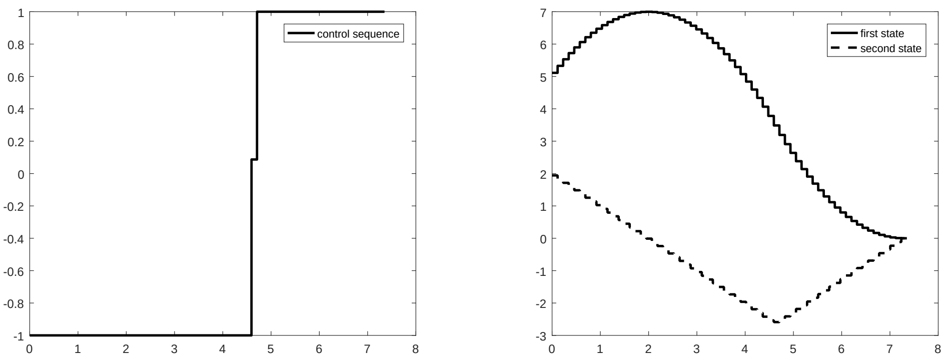
- For the block-pulse functions:
  $\tilde{U}_{Nmin} \le \tilde{U}_N \le \tilde{U}_{Nmax}$,
  $\tilde{X}_{Nf,bp} = \left[\begin{array}{cccc} 0 & 0 & \cdots & \tilde{X}_f \end{array} \right]$.
- For the Walsh functions:
  $\tilde{U}_{Nmin} \le \tilde{U}_N\ \phi_{N,w} \le \tilde{U}_{Nmax}$,
  $\tilde{X}_{Nf,w} = \left[\begin{array}{cccc} 0 & 0 & \cdots & \tilde{X}_f \end{array} \right] W_{N\times N}$.
- For the Haar functions:
  $\tilde{U}_{Nmin} \le \tilde{U}_N\ \phi_{N,h} \le \tilde{U}_{Nmax}$,
  $\tilde{X}_{Nf,h} = \left[\begin{array}{cccc} 0 & 0 & \cdots & \tilde{X}_f \end{array} \right] H_{N\times N}$,

where $\tilde{X}_{N,f}$ denotes the projection of the final sate over orthogonal functions. $W_{N\times N}$ and $H_{N\times N}$ are, respectively, the Walsh and Haar transition matrices,
nonlinear constraints:

$$\tilde{X}_N - \tilde{X}_{N0} = t_f(A\tilde{X}_N P_N + B\tilde{U}_N P_N). \tag{38}$$

To solve this optimization problem, an interior point method the same as the one implemented in the function "fmincon" of Matlab is used.

Figure 5: Example 1.



(a) Control sequence for Example 1.

(b) System trajectory for Example 1.

4.4 Simulation and validation

In this subsection, a comparison between our results and some other results available in the literature is presented.

4.4.1 Example 1

We consider a simple double integrator system in which its analytic solution using the PMP is well known. Its state space representation is given in equation (39):

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{39}$$

The system needs to be shifted from an initial state $x(0) = [5, 2]^T$ to the origin of state space with the input constraint $u \in [-1, 1]$. The analytical solution for this system is as follows: the system has one switching point at $t_c = 4.64$ and the final time is $t_f = 7.29$. Determining the solution of the double integrator system using optimization algorithm for a base of dimension $N = 64$, we obtain comparable results with the analytical solution. From Figure 5(a) and Figure5(b), it is clear that the system has only one switching point at $t_c = 4.594$ which is almost the same one found by the analytical solution.

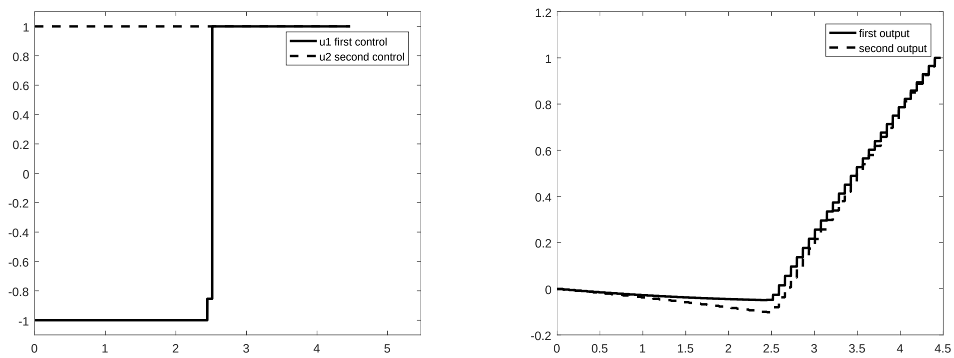
We can also see that the control sequence is bang-bang and that the final time $t_f = 7.3508$ is also the same as the analytical solution.

4.4.2 Example 2

Take the example given in [10] which is a fourth order MIMO system with two real double poles $\lambda_1 = 5, 2833$ and $\lambda_2 = -0.0833$. The state space representation of the system is given as follows:

$$A = \begin{bmatrix} -\frac{1}{10} & 0 & 0 & 0 \\ 0 & -\frac{1}{15} & 0 & 0 \\ 0 & 0 & -\frac{1}{15} & 0 \\ 0 & 0 & 0 & -\frac{1}{10} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{3}{5} & 0 & \frac{8}{15} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{3}{5} \end{bmatrix}.$$

Figure 6: Example 2.



(a) Control sequence for Example 2.

(b) System trajectory for Example 2.

This system needs to be shifted from $y = [0, 0]^T$ to $y_f = [1, 1]^T$, the input constraints are u_1, $u_2 \in [-1, 1]^T$.

Computing the MIMO system including its constraints we obtain the control sequence described in Figure 6(a).

We can see from Figure 6(a) that the control sequence is bang-bang, that u_1 does not contain any switching time and that u_2 contains only one at $t_c = 2.5$.

This proves that our method is also effective for MIMO constrained systems. By comparing this result to the result obtained in [10] where $t_f = 54.1$ we can see from Figure 6(b) that the target is reached before at $t_f = 4.47$.

It is clear that the obtained results through the orthogonal piecewise functions using operational matrices are far better than the one obtained using the Chebyshev technique [10].

5 Closed Loop Online Suboptimal Minimum Time Control Algorithm

In the past section we have elaborated an algorithm to compute the minimum time control for open loop systems. Such solution can not recover from perturbations, so we determined an offline suboptimal control structure. In this part, we introduce an online suboptimal minimum time control.

5.1 Principle

The control problem is now described by Figure 7. Z is a perturbation that may affect the system states, $X_N^T(kh) = [x_0(kh) \;\; x_1(kh) \;\; \cdots \;\; x_{N-1}(kh)]$ is the output state vector at $t = kh$ which represents the discrete time. Besides, h is chosen as small as possible in order to take into account correctly an eventual disturbance over the system states.

The optimization problem formulation in the closed loop is similar to that in the open loop case. However, it is computed k times. The initial state is continuously actualized to the final state of the previous optimization step.

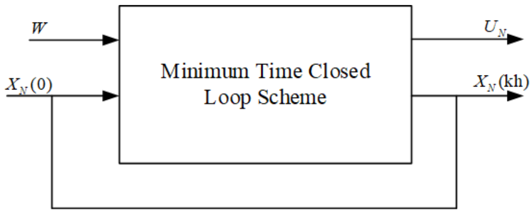


Figure 7: Feedback control scheme.

5.2 Algorithm description

Algorithm 5.1 Minimum time closed-loop algorithm

begin
initialization
$k \longleftarrow 0$
$h \longleftarrow 10^{-2}$
$\tilde{X}_{N0,bp} = [\tilde{X}(0)\ \tilde{X}(0) \quad \cdots \quad \tilde{X}(0)]$
While $\tilde{x} \neq x_f$ **do**
Find $\mathbf{min}(t_f)$
Subject to:
Linear constraints:
$\tilde{U}_{Nmin} \leq \tilde{U}_N \times \phi_N \leq \tilde{U}_{Nmax}$
$\tilde{X}_{N0} = [\tilde{x}_0(kh)\ \tilde{x}_1(kh) \quad \tilde{x}_{N-1}(kh)]$
$\tilde{X}_{N0,f} = [0 \quad 0 \quad \cdots \quad \tilde{x}_f]$
Nonlinear constraints:
$\tilde{X}_N - \tilde{X}_{N0} = t_f(A\tilde{X}_N P_N + B\tilde{U}_N P_N)$
$k \longleftarrow k+1$
end
end

5.3 Simulation and comparison results

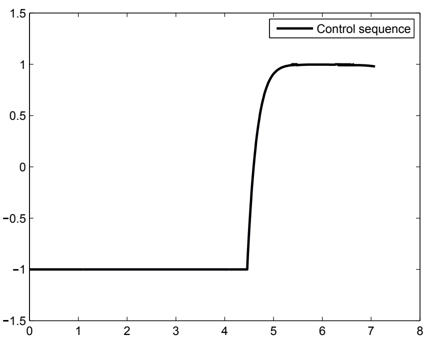
We consider the same system: a simple double integrator described previously.

In this section we intent to apply the closed loop optimization procedure to the system using the orthogonal block-pulse, Walsh or Haar wavelets for $N = 64$. This will be considered for various cases, namely, the system without disturbance (here the closed loop performance should meet the open loop one to prove the correctness of the algorithm), and after that the presence of perturbation case is examined. That disturbance is seen as an exterior event that discards the system state from its trajectory at time instant denoted t_p.

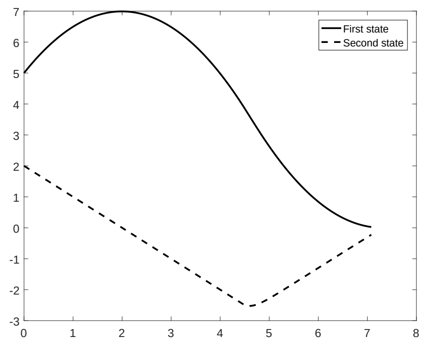
From Figure 8(a), it is clear that the system has only one switching point at $t_c = 4.594$ which is almost the same one found by the analytical solution.

We can also see that the control sequence is bang-bang in the closed loop and that the final time $t_f = 7.323$ is also the same as the analytical solution. It is clear from Figure 8(b) that the system without perturbations in the closed loop reaches the target at the same time of the open loop.

Figure 8: Example 1 without perturbation.

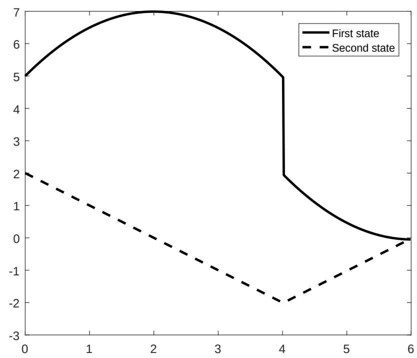


(a) Control sequence for the example 1 without perturbation.

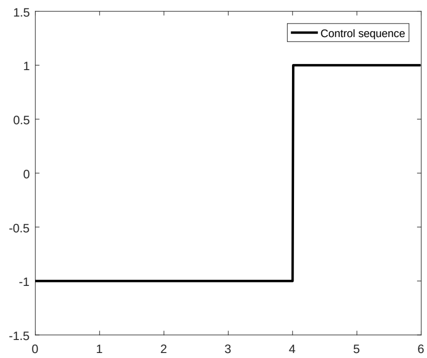


(b) System trajectory for the example 1 without perturbation.

Figure 9: Example 1 with perturbation.



(a) System trajectory for the example 1 with perturbation on the first state.



(b) Control sequence for the example 1 with perturbation on the first state.

5.4 Example for perturbed system

In this part, the system is perturbed at $t_p = 4s$. We can see from Figure 9(b) that the control sequence is still bang-bang but there is a change of the switching time.

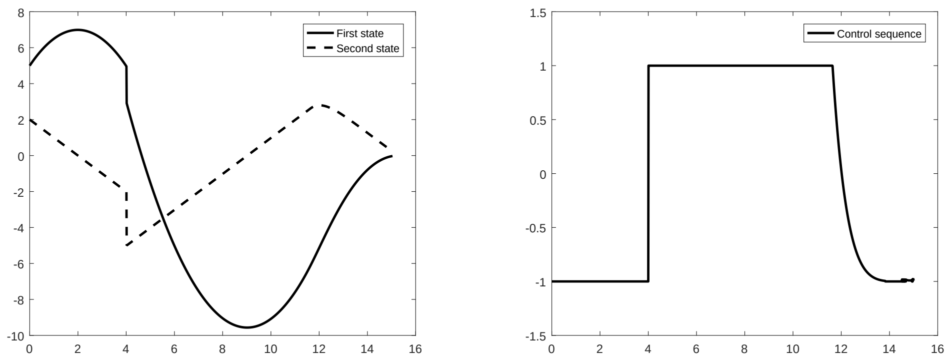
It is clear from Figure 9(a) that the system is able to recover from the perturbation and reaches the target faster for the case of this perturbation. In fact, the perturbation signal on the first state has brought it closer to the target. This explains why $t_f < t_{f_{ol}}$.

Another simulation context could be verified. In fact, the system described by the state space form in (39) is perturbed at $t_p = 4s$, where $x = [x_1 - 2; x_2 - 3]$.

We can see from Figure 10(b) that the control sequence is still bang-bang but the system needs two switches to steer the system to the origin.

It is clear from Figure 10(a) that the system is able to recover from the perturbation and reaches the target. In fact, the perturbation signal on the two states has made the

Figure 10: Example 1 with perturbation on two states.



(a) System trajectory for the example 1 with perturbation on the two states.

(b) Control sequence for the example 1 with perturbation on the two states.

final time t_f bigger than the one without perturbation. Then the perturbation signal has a direct effect on the final time t_f.

6 Conclusion

In this paper, we focused on the problem of a minimum time control determination for linear systems in both cases of control structures: an open loop and a closed loop control. The key of the developed method is the approximation of the dynamic equation of the system under consideration using a complete basis of orthogonal functions and its operational properties. We have opted for the use of the piecewise orthogonal functions: block-pulse, Walsh and Haar wavelets. The results suggest that the proposed development yields a new formulation of the optimization problem which is simpler than those developed in the literature.

Other advantages of the proposed method include a better final time estimate and fewer switches for high order systems. The developed algorithm was also tested for a number of examples (i.e. SISO and MIMO systems), the results showed perfect agreement with the exact analytic results, which ensures the availability of the proposed technique. In the closed loop case, two algorithms were introduced to take into account the effects of perturbations on the system. They are: an offline algorithm which, compared to open loop results, has a great deterioration of results, and an online one which has a slight deterioration of performances even with perturbations.

In future work, we expect to generalize the proposed approach to the synthesis of minimum time control laws for nonlinear and fuzzy systems using state variable observation [20].

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