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A New Integral Transform for Solving Higher Order Ordinary Differential Equations

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Abstract: In this work a new integral transform is introduced and applied to solve higher order linear ordinary differential equations with constants coefficients and variable coefficients as well as. We compare the present transform with other existing transforms such as the Laplace, Elzaki, Sumudu and other ones.

Keywords: *integral transform; ordinary differential equations; Laplace transform; Sumudu transform; Elzaki transform.*

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1 Introduction

The differential equations have played a fundamental role in every aspect of applied mathematics for a very long time [1, 5–8, 10, 15, 19, 20]. Integral transform methods have been modified to solve several dynamic equations with initial or boundary conditions in many ways the Laplace, Sumudu and Elzaki transforms are such typical tools [4, 5, 9, 11–15]. In this paper we introduce a new integral transform and then some relationship between this transform and the Laplace, Sumudu, Elzaki and natural transforms; further, for the comparison purpose, we apply all transforms to solve differential equations to see the differences and similarities. Finally, we provide some examples relating to the second order differential equations with non-constant coefficients as a special case. For the function $f(t)$ that is piecewise continuously differentiable in every finite interval and is absolutely integrable on the whole real line the following integral equations hold true in the domain $-\infty < t < +\infty$:

$$\mathcal{F}\{f(t)\} = F(k) = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} e^{-ikt} f(t)\, dt, \quad t \neq 0. \tag{1}$$

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This transform is called the Fourier transform of f(t). The natural transform of f(t) for t$\in (0,+\infty)$, obtained from the Fourier transform, is defined as.

$$\mathcal{N}\left\{f\left(t\right)\right\} = \int_0^{+\infty} e^{-st} f\left(ut\right)dt; \quad s > 0, u > 0. \tag{2}$$

If we assign $u = 1$ in (2), then it is called the Laplace transform and written as

$$\mathcal{L}\left\{f\left(t\right)\right\} = \int_0^{+\infty} e^{-st} f\left(t\right)dt; \quad s > 0. \tag{3}$$

If we assign $s = 1$ in (2), then it is called Sumudu transform and written as

$$\mathcal{G}\left\{f\left(t\right)\right\} = \int_0^{+\infty} e^{-t} f\left(ut\right)dt; \quad u > 0. \tag{4}$$

Finally, we assign $u = 1$ *and* $s = \frac{1}{\lambda}$ in (2) and multiply it by λ, then it is called the Elzaki transform and written as

$$\mathcal{T}\left\{f\left(t\right)\right\} = \lambda \int_0^{+\infty} e^{-\frac{1}{\lambda}t} f\left(t\right)dt. \tag{5}$$

The above mentioned integral transforms has been applied to solve higher order linear ordinary differential equation (ODEs), partial differential equations (PDEs), a system of ordinary and partial differential equations and integral equations.

In this paper, we introduced a new integral transform. We have compared the new transform with other exiting transforms. We solve ODEs with variable and constant coefficients using the new transform. Moreover, we obtain the solution of integral equations using this transform. The new transform is very effective for the solution of the response of linear and nonlinear differential equations.

2 A New Integral Transform and Its Properties

Definition 2.1 The *HY* integral transform is defined by

$$\mathbf{P}\left(\boldsymbol{\nu}\right) = HY\{f\left(t\right)\} = \boldsymbol{\nu} \int_0^{+\infty} e^{-\boldsymbol{\nu}^2 t} f\left(t\right)dt. \tag{6}$$

The *HY* integral transform states that, if $f\left(t\right)$ is piecewise continuous on every finite interval in $t \in [0,+\infty)$ satisfying

$$\left|f\left(t\right)\right| \le Me^{at},\ \exists M > 0 \tag{7}$$

for all $t \in [0,\ \infty)$, then $HY\{f\left(t\right)\}\left(\boldsymbol{\nu}\right)$ exists for all $\nu > a$.

If we assign $u = 1$ *and* $s = v^2$ in (2) and multiply it by v, then we obtain our defined new integral transform which is (6).

Theorem 2.1 ***(Criteria for convergence)***
The HY integral transform of $f\left(t\right)$ exists, if it has exponential order and the integral

$\int_0^b \left|f\left(t\right)\right| dt$

exists for any b>0.

Proof: Since we only need to show convergence for sufficiently large ν, assume $\boldsymbol{\nu}>\sqrt{c}$ and $\boldsymbol{\nu}>0$.

$$\begin{aligned}
\boldsymbol{\nu}\int_0^{\infty}\left|f(t)\,e^{-\boldsymbol{\nu}^2 t}\right|dt &= \boldsymbol{\nu}\int_0^{n}\left|f(t)\,e^{-\boldsymbol{\nu}^2 t}\right|dt+\boldsymbol{\nu}\int_n^{\infty}\left|f(t)\,e^{-\boldsymbol{\nu}^2 t}\right|dt\\
&\le \boldsymbol{\nu}\int_0^{n}|f(t)|\,dt+\boldsymbol{\nu}\int_n^{\infty}e^{-\boldsymbol{\nu}^2 t}|f(t)|\,dt, \qquad 0\le e^{-\boldsymbol{\nu}^2 t}\le 1\\
&\le \boldsymbol{\nu}\int_0^{n}|f(t)|\,dt+\boldsymbol{\nu}\int_n^{\infty}e^{-\boldsymbol{\nu}^2 t}Me^{ct}\,dt\\
&\le \boldsymbol{\nu}\int_0^{n}|f(t)|\,dt+\boldsymbol{\nu}M\left[\frac{e^{\left(c-\boldsymbol{\nu}^2\right)t}}{c-\boldsymbol{\nu}^2}\right]_n^{\infty}, \qquad \boldsymbol{\nu}>\sqrt{c}\\
&\le \boldsymbol{\nu}\int_0^{n}|f(t)|\,dt+\boldsymbol{\nu}M\left[\frac{e^{\left(c-\boldsymbol{\nu}^2\right)n}}{c-\boldsymbol{\nu}^2}\right], \qquad \boldsymbol{\nu}>\sqrt{c}.
\end{aligned}$$

The first integral exists by assumption, and the second term is finite for $\boldsymbol{\nu}^2>c$, so the integral $\nu\int\limits_0^{+\infty}e^{-\nu^2 t}f(t)\,dt$ is convergent absolutely and the *HY* transform of $f(t)$ exist. □

Theorem 2.2 ***(Linear property of HY transform)***
Let $P(v)$ and $Q(v)$ be HY transforms of $f(t)$ and $g(t)$, respectivly, for each constants of c_1 and c_2, Then

$$HY\{c_1 f(t)+c_2 g(t)\}=c_1 HY\{f(t)\}+c_2 HY\{g(t)\}=c_1P(v)+c_2Q(v).$$

Proof: Beacuase of the linear property of the Integrals, the proof is obvious. □

Theorem 2.3 *Let $P(v)$ be the HY transform of $f(t)$. Then*

$$HY\left\{f'(t)\right\} = \boldsymbol{\nu}^2P(\boldsymbol{\nu})-\boldsymbol{\nu}f(0), \tag{8}$$

$$HY\left\{f''(t)\right\} = \boldsymbol{\nu}^4P(\boldsymbol{\nu})-\boldsymbol{\nu}^3f(0)-\boldsymbol{\nu}f'(0), \tag{9}$$

$$HY\left\{f^{(n)}(t)\right\} = \boldsymbol{\nu}^{2n}P(\boldsymbol{\nu})-\sum_{k=0}^{n-1}\boldsymbol{\nu}^{2(n-k)-1}f^{(k)}(0), \qquad n\ge 1. \tag{10}$$

Proof: Replacing $f(t)$ with $f'(t)$ in (6) gives $HY\left\{f'(t)\right\}=\boldsymbol{\nu}\int\limits_0^{\infty}e^{-\boldsymbol{\nu}^2 t}f'(t)dt$. Integrate by parts to find that $HY\left\{f'(t)\right\}=\boldsymbol{\nu}^2\mathrm{P}(\boldsymbol{\nu})-\boldsymbol{\nu}f(0)$.

Let g(t)= $f'(t)$, then $g'(t)=f''(t)$, thus by using (9), we get $HY\left\{f''(t)\right\}=\boldsymbol{\nu}^4\mathrm{P}(\boldsymbol{\nu})-\boldsymbol{\nu}^3f(0)-\boldsymbol{\nu}f'(0)$. (10) can be provided by mathematical induction. □

In the following table, we showed the *HY* transform of some important functions, where $\delta(t)$ is the unit impulse function.

Function (f(t))	$\boldsymbol{HY}$ **Transform**	$\boldsymbol{HY}^{-1}$ **Transform**
$\delta(t)$	$\boldsymbol{\nu}$	$\delta(t)$
1	$\frac{1}{\boldsymbol{\nu}}$	1
at^n	$\frac{a.n!}{\boldsymbol{\nu}^{2n+1}}$, a constant	at^n
t^a	$\frac{\Gamma(a+1)}{\boldsymbol{\nu}^{2n+1}}$, a>-1	t^a
$e^{\pm\alpha t}$	$\frac{\boldsymbol{\nu}}{\boldsymbol{\nu}^2 \pm\alpha}$	$e^{\pm\alpha t}$
sinat	$\frac{\alpha\boldsymbol{\nu}}{\boldsymbol{\nu}^4 + a^2}$	sinat
cosat	$\frac{\boldsymbol{\nu}^3}{\boldsymbol{\nu}^4 + a^2}$	cosat
sinhat	$\frac{\alpha\boldsymbol{\nu}}{\boldsymbol{\nu}^4 - a^2}$	sinhat
coshat	$\frac{\boldsymbol{\nu}^3}{\boldsymbol{\nu}^4 - a^2}$	coshat

Table 1: HY transforms for some basic functions.

3 Application

3.1 Solving higher order ODEs with constant coefficient using new integral transform

In this section we use new the integral transform for solving higher order ODEs with constant and variable coefficients and integral equations. At first, we solve the linear equation of order n with constant coefficients as

$$L(D)[y(t)] = D^n y(t) + a_1 D^{n-1} y(t) + a_2 D^{n-2} y(t) + \cdots + a_n y(t) = \phi(t) \quad (11)$$

with the initial conditions

$$y(t_0) = y_0, Dy(t_1) = y_1, D^2 y(t_2) = y_2, \cdots, D^{n-1} y(t_{n-1}) = y_{n-1},$$

where $D = \frac{d}{dt}$ is a differential operator. $y_0,\ y_1,\ \cdots,\ y_{n-1}$ and $a_1,\ a_2, \cdots,\ a_n$ are constants. We apply the new integral transform on both side of (11)

$$HY(D^n y(t) + a_1 D^{n-1} y(t) + a_2 D^{n-2} y(t) + \cdots + a_n y(t)) = HY(\phi(t)).$$

By using the linear property of this transform we have

$$HY(D^n y(t)) + a_1 HY\left(D^{n-1} y(t)\right) + a_2 HY\left(D^{n-2} y(t)\right) + \cdots + a_n HY(y(t))$$

$$= HY(\phi(t)) = \phi(\boldsymbol{\nu})$$

$$\left\{\boldsymbol{\nu}^{2n}\mathrm{P}(\boldsymbol{\nu}) - \sum_{k=0}^{n-1} \boldsymbol{\nu}^{2(n-k)-1} f^{(k)}(0)\right\} + a_1\left\{\boldsymbol{\nu}^{2(n-1)}\mathrm{P}(\boldsymbol{\nu}) - \sum_{k=0}^{n-2} \boldsymbol{\nu}^{2(n-k)-2} f^{(k)}(0)\right\}$$

$$+ \cdots + a_n \mathrm{P}(\boldsymbol{\nu}) = \phi(\boldsymbol{\nu}), \quad (12)$$

where $\phi(\boldsymbol{\nu})$ is the HY transform of $\phi(t)$. (10) can be writen in the following form:

$$\underbrace{\left[\boldsymbol{\nu}^{2n} + a_1\boldsymbol{\nu}^{2(n-1)} + \cdots + a_n\right]}_{f(\boldsymbol{\nu})} \mathrm{P}(\boldsymbol{\nu}) = \phi(\boldsymbol{\nu}) + \psi(\boldsymbol{\nu}),$$

$$\mathrm{P}(\boldsymbol{\nu}) = \frac{\phi(\boldsymbol{\nu}) + \psi(\boldsymbol{\nu})}{f(v)}, \qquad \mathrm{P}(\boldsymbol{\nu}) = HY(y(t)) = \frac{\phi(\boldsymbol{\nu})}{f(v)} + \frac{\psi(\boldsymbol{\nu})}{f(\boldsymbol{\nu})}.$$

Inversion yields:

$$y\,(t) = HY^{-1}\left(\frac{\phi\,(\boldsymbol{\nu})}{f\,(\boldsymbol{\nu})}\right) + HY^{-1}\left(\frac{\psi\,(\boldsymbol{\nu})}{f\,(\boldsymbol{\nu})}\right). \tag{13}$$

The inverse operation on the right hand can be carried out by a partial fraction or any method.

Example 3.1 Solve the following initial value problem:

$$\begin{aligned}&y^{'''}\,(t) + 2y^{''}\,(t) + 2\;y^{'}\,(t) + 3y\,(t) = \sin t + \cos t, \qquad (14)\\ &\mathrm{y}\,(0) = \mathrm{y}^{'}\,(0) = \mathrm{y}^{''}\,(0) = 0.\end{aligned}$$

Taking the HY transform on both sides (14) , we get

$$HY\left\{y^{'''}\,(t)\right\}+2HY\left\{y^{''}\,(t)\right\}+2HY\left\{y^{'}\,(t)\right\}+3\;HY\{y\,(t)\} = HY\{sin\,(t)\}+HY\{cos\,(t)\}\,,$$

$\boldsymbol{\nu}^6\mathrm{P}\,(\boldsymbol{\nu}) + 2\boldsymbol{\nu}^4\mathrm{P}\,(\boldsymbol{\nu}) + 2\boldsymbol{\nu}^2\mathrm{P}\,(\boldsymbol{\nu}) + 3P\,(v) = \frac{\boldsymbol{\nu}}{\boldsymbol{\nu}^4+1} + \frac{\boldsymbol{\nu}^3}{\boldsymbol{\nu}^4+1} + v^3 + 2v,$
$\mathrm{P}\,(\boldsymbol{\nu}) = \frac{\boldsymbol{\nu}}{\boldsymbol{\nu}^4+1}$. Take the inverse HY transform $y\,(t) = sin\,(t)$, which is an exact solution of (14).

3.2 Solving higher order ODEs with variable coefficient using new integral transform

Now we want to apply the new integral transform to solve ODE with variable coefficient. Before doing this, process we present few theorems which are useful in our work.

Theorem 3.1 *Let $P(v)$ be the HY of function $f\,(t)$, then*

$$HY\{tf\,(t)\} = \left(\frac{-1}{2}\right)\frac{d}{d\boldsymbol{\nu}}\left(\frac{p\,(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right), \tag{15}$$

$$HY\left\{t^2 f\,(t)\right\} = \left(\frac{-1}{2}\right)^2\frac{d}{d\boldsymbol{\nu}}\left(\frac{1}{\boldsymbol{\nu}}\frac{d}{d\boldsymbol{\nu}}\left(\frac{p\,(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right)\right), \tag{16}$$

$$HY\{t^n f\,(t)\} = \left(\frac{-1}{2}\right)^n\frac{d}{d\boldsymbol{\nu}}\left(\underbrace{\frac{1}{\boldsymbol{\nu}}\cdots\frac{d}{d\boldsymbol{\nu}}}_{n\ times}\left(\frac{p\,(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right)\cdots\right), \qquad n \geq 1. \tag{17}$$

Proof: Since $p\,(\boldsymbol{\nu}) = HY\{\mathrm{f(t)}\} = \boldsymbol{\nu}\int\limits_0^\infty e^{-\boldsymbol{\nu}^2 t}f\,(t)\,dt,$

$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\left(\frac{p(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right) = -2\underbrace{\boldsymbol{\nu}\int_0^\infty e^{-\boldsymbol{\nu}^2 t}tf\,(t)\,dt}_{HY\{tf(t)\}}$. First, divide both sides of the above equation by $\boldsymbol{\nu}$ and take the derivative with respect to $\boldsymbol{\nu}$, we get

$$HY\{tf\,(t)\} = \left(\frac{-1}{2}\right)^1\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\left(\frac{p\,(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right).$$

From (15). we have $HY\{tf(t)\} = \left(\frac{-1}{2}\right)^1 \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\left(\frac{p(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right)$,
$\boldsymbol{\nu}\int\limits_0^\infty e^{-\boldsymbol{\nu}^2 t} tf(t)\,dt = \left(\frac{-1}{2}\right)^1 \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\left(\frac{p(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right)$ divide both sides of this equation by $\boldsymbol{\nu}$ and taking derivative with respect to $\boldsymbol{\nu}$ again, result in

$$HY\{t^2 f(t)\} = \left(\frac{-1}{2}\right)^2 \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\left(\frac{1}{\boldsymbol{\nu}}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\left(\frac{p(\boldsymbol{\nu})}{\boldsymbol{\nu}}\right)\right).$$

(17) can be provided by mathematical induction. □

Theorem 3.2 *Let $P(v)$ be the HY of function $f(t)$, then*

$$HY\left\{tf'(t)\right\} = \left(\frac{-1}{2}\right)^1 \left[\frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f'(t)\right\}\right]\right]\right], \tag{18}$$

$$HY\left\{tf''(t)\right\} = \left(\frac{-1}{2}\right)^1 \left[\frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f''(t)\right\}\right]\right]\right], \tag{19}$$

$$HY\left\{t^2 f'(t)\right\} = \left(\frac{-1}{2}\right)^2 \frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f'(t)\right\}\right]\right]\right], \tag{20}$$

$$HY\left\{t^2 f''(t)\right\} = \left(\frac{-1}{2}\right)^2 \frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f''(t)\right\}\right]\right]\right], \tag{21}$$

$$HY\left\{t^n f^{(m)}(t)\right\} = \left(\frac{-1}{2}\right)^n \frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\frac{d}{d\boldsymbol{\nu}}\left[\cdots\left[\frac{1}{\boldsymbol{\nu}}\frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f^{(m)}(t)\right\}\right]\right]\cdots\right]\right], \tag{22}$$

where $n, m \geq 1$.

Proof: As we know $HY\left\{\mathrm{f}'(\mathrm{t})\right\} = \boldsymbol{\nu}\int_0^\infty e^{-\boldsymbol{\nu}^2 t} f'(t)\,dt$, $\frac{1}{\boldsymbol{\nu}}HY\left\{\mathrm{f}'(\mathrm{t})\right\} = \int_0^\infty e^{-\boldsymbol{\nu}^2 t} f'(t)\,dt$. First, divide both sides of this equation by $\boldsymbol{\nu}$ and take the derivative with respect to $\boldsymbol{\nu}$, we get
$HY\left\{tf'(t)\right\} = \left(\frac{-1}{2}\right)^1 \left[\frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f'(t)\right\}\right]\right]\right]$ is (18),
replacing $f''(t)$ with $f'(t)$ in (18) results in

$HY\left\{tf''(t)\right\} = \left(\frac{-1}{2}\right)^1 \left[\frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f''(t)\right\}\right]\right]\right]$ is (19).

Rewrite formula (18) and divide both sides of this equation by $\boldsymbol{\nu}$ and take the derivative with respect to $\boldsymbol{\nu}$ again, we get

$HY\left\{t^2 f'(t)\right\} = \left(\frac{-1}{2}\right)^2 \frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f'(t)\right\}\right]\right]\right]$ is (20),
replacing $f''(t)$ with $f'(t)$ in (20) results in

$HY\left\{t^2 f''(t)\right\} = \left(\frac{-1}{2}\right)^2 \frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[HY\left\{f''(t)\right\}\right]\right]\right]$ is (21).
(22) can be provided by mathematical induction. □

Example 3.2 Solve the following equation with initial values:

$$y''(t) + 3ty'(t) - 6y(t) = 2, \qquad \mathrm{y}(0) = \mathrm{y}'(0) = 0. \tag{23}$$

Taking the HY transform on both sides of (23), we get

$HY\left\{y^{''}(t)\right\} + 3tHY\left\{y^{'}(t)\right\} - 6\ HY\{y(t)\} = HY\{2\}$,
$\boldsymbol{\nu}^4\mathrm{P}(\boldsymbol{\nu}) - \boldsymbol{\nu}^3\mathrm{y}^{'}(0) - \boldsymbol{\nu}\mathrm{y}(0) + 3\left(\left(\frac{-1}{2}\right)\left[\frac{d}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[\boldsymbol{\nu}^2\mathrm{P}(\boldsymbol{\nu}) - \boldsymbol{\nu}y(0)\right]\right]\right]\right) - 6P(\boldsymbol{\nu}) = \frac{2}{\boldsymbol{\nu}}$,
$-\frac{3}{2}\boldsymbol{\nu}\ \mathrm{P}^{'}(\boldsymbol{\nu}) - \frac{15}{2}\mathrm{P}(\boldsymbol{\nu}) + \boldsymbol{\nu}^4\mathrm{P}(\boldsymbol{\nu}) = \frac{2}{v}$.
This equation is written as follows:

$$\mathrm{P}^{'}(\boldsymbol{\nu}) + \frac{\left(15\boldsymbol{\nu} - 2\boldsymbol{\nu}^5\right)}{3\boldsymbol{\nu}^2}\ \mathrm{P}(\boldsymbol{\nu}) = \frac{4}{-3\boldsymbol{\nu}^2}.$$

Thus, we get a linear first order differential equation that must be solved in order to get transform for the solution. In this equation $f(\boldsymbol{\nu}) = \frac{15\boldsymbol{\nu} - 2\boldsymbol{\nu}^5}{3\boldsymbol{\nu}^2}$ and $g(\boldsymbol{\nu}) = \frac{4}{-3\boldsymbol{\nu}^2.}$

Thus the general solution of this equation is

$$P(\boldsymbol{\nu}) = e^{-\int f(\boldsymbol{\nu})d\boldsymbol{\nu}}\left(\int g(\boldsymbol{\nu})\,e^{\int f(\boldsymbol{\nu})d\boldsymbol{\nu}}d\boldsymbol{\nu}\right)$$

$$= e^{-\int \frac{\left(15\boldsymbol{\nu} - 2\boldsymbol{\nu}^5\right)}{3\boldsymbol{\nu}^2}d\boldsymbol{\nu}}\left(\int \frac{4}{-3\boldsymbol{\nu}^2}e^{\int \frac{\left(15\boldsymbol{\nu} - 2\boldsymbol{\nu}^5\right)}{3\boldsymbol{\nu}^2}d\boldsymbol{\nu}}d\boldsymbol{\nu}\right) = 2\boldsymbol{\nu}^{-5}e^{\ \frac{1}{6}\boldsymbol{\nu}^4}e^{-\ \frac{1}{6}\boldsymbol{\nu}^4} = 2\boldsymbol{\nu}^{-5},$$

$$HY\{y(t)\} = 2\boldsymbol{\nu}^{-5}.$$

Take the inverse HY transform $y(t) = t^2$, which is an exact solution of (23).

Example 3.3 Solve the following initial value problem:

$$2t^2y^{'''}(t) + 9ty^{''}(t) + 9y^{'}(t) = 60t^2, \qquad y(0) = y^{'}(0) = y^{''}(0) = 0. \tag{24}$$

Apply the HY transform to equation (24), and make use of the initial conditions and the above mentioned theorems, then we get
$2HY\left\{t^2y^{'''}(t)\right\} + 9HY\left\{ty^{''}(t)\right\} + 9HY\left\{y^{'}(t)\right\} = HY\left\{60\mathrm{u}^2\right\}$,
$\frac{15}{2}\boldsymbol{\nu}^2\ \mathrm{P}(\boldsymbol{\nu}) + \frac{9}{2}\boldsymbol{\nu}^3P^{'}(\boldsymbol{\nu}) + \frac{1}{2}\boldsymbol{\nu}^4P^{''}(\boldsymbol{\nu}) - \frac{27}{2}\boldsymbol{\nu}^2\mathrm{P}(\boldsymbol{\nu}) - \frac{9}{2}\boldsymbol{\nu}^3\mathrm{P}^{'}(\boldsymbol{\nu}) + 6\boldsymbol{\nu}^2\mathrm{P}(\boldsymbol{\nu}) = \frac{60\times 2!}{\boldsymbol{\nu}^5}$,
$\mathrm{P}^{''}(\boldsymbol{\nu}) = \frac{240}{\boldsymbol{\nu}^7}$, $\mathrm{P}(\boldsymbol{\nu}) = \frac{30}{7}\frac{1}{\boldsymbol{\nu}^7}$, $y(t) = \frac{5}{7}\mathrm{t}^3$, which is an exact solution of (24).

Theorem 3.3 *Let $P(v)$ be the HY of function $f(t)$, then the solution of the Euler-Cauchy equation*

$$t^2y^{''}(t) + aty^{'}(t) + by(t) = 0 \tag{25}$$

can be represented by $y = HY^{-1}(\boldsymbol{\nu}^m)$, *where* $m = (a-2) \pm \sqrt{(a-1)^2 - 4b}$ *for* $y(t) = \boldsymbol{\nu}^m$.

Proof: Taking the HY transform on both sides, we have
$HY\left\{t^2y^{''}(t)\right\} + aHY\left\{ty^{'}(t)\right\} + bHY\{y(t)\} = 0$,
$\left(\frac{-1}{2}\right)^2\frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[\boldsymbol{\nu}^4\mathrm{P}(\boldsymbol{\nu}) - \boldsymbol{\nu}^3f(0) - \boldsymbol{\nu}f^{'}(0)\right]\right]\right] + a\left(\frac{-1}{2}\right)\left[\frac{\mathrm{d}}{d\boldsymbol{\nu}}\left[\frac{1}{\boldsymbol{\nu}}\left[\boldsymbol{\nu}^2\mathrm{P}(\boldsymbol{\nu}) - \boldsymbol{\nu}f(0)\right]\right]\right] + b\mathrm{P}(\boldsymbol{\nu}) = 0$,
$\boldsymbol{\nu}^2P^{''}(\boldsymbol{\nu}) + (5-2a)\boldsymbol{\nu}P^{'}(\boldsymbol{\nu}) + (3-2a+4b) = 0$.

For $Y = HY\{f(t)\} = P(\boldsymbol{\nu})$. Since Y is a function of $\boldsymbol{\nu}$, let us put $Y = P(\boldsymbol{\nu}) = \boldsymbol{\nu}^m$ as m is constant. Then we have $P^{'}(\boldsymbol{\nu}) = m\boldsymbol{\nu}^{m-1}$ and $P^{''}(\boldsymbol{\nu}) = m(m-1)\boldsymbol{\nu}^{m-2}$, and the given equation becomes

$m(m-1)\boldsymbol{\nu}^m + (5-2a)m\boldsymbol{\nu}^m + (3-2a+4b)\boldsymbol{\nu}^m = 0.$
As we know $\boldsymbol{\nu}^m \neq 0$, then $m(m-1) + (5-2a)m + (3-2a+4b) = 0.$

Organizing this equality, we have $m^2 + (4-2a)m + (3-2a+4b) = 0.$

Hence, $m = log_{\boldsymbol{\nu}} Y = (a-2) \pm \sqrt{(a-1)^2 - 4b}$ and the solution is $y(t) = HY^{-1}(Y) = HY^{-1}\left(\boldsymbol{\nu}^{(a-2)\pm\sqrt{(a-1)^2-4b}}\right).$

Example 3.4 Solve the following Euler-Cauchy equation:

$$t^2 y^{''}(t) - 3ty^{'}(t) + 3y(t) = 0. \tag{26}$$

According to (25), we have $a = -3,\quad b = 3$ and $m = (a-2) \pm \sqrt{(a-1)^2 - 4b},\quad$ so $m = (-3-2) \pm \sqrt{(-3-1)^2 - 4(3)},\quad$ then first, $m = -3$ and second, $m = -7$,
$y_1 = HY^{-1}\left(v^{-3}\right) = t,\quad y_2 = HY^{-1}\left(v^{-7}\right) = t^3$. Then
the solution is $y = c_1 t +\ c_2 t^3$, Which is an exact solution of (26).

Example 3.5 Solve the following Euler-Cauchy equation:

$$t^2 y^{''}(t) + \frac{3}{2}ty^{'}(t) - \frac{1}{2}y(t) = 0. \tag{27}$$

According to (25), we have $a = \frac{3}{2},\quad b = -\frac{1}{2},\quad m = (a-2) \pm \sqrt{(a-1)^2 - 4b},\quad$ so
$m = \left(\frac{3}{2} - 2\right) \pm \sqrt{\left(\frac{3}{2} - 1\right)^2 - 4(-\frac{1}{2})},\qquad$ then first, $m = 1$ and second, $m = -2$,
$y_1 = HY^{-1}\left(v^{+1}\right) = \frac{1}{t},\quad y_2 = HY^{-1}\left(v^{-2}\right) = \sqrt{\frac{t}{\pi}}$. Then
the solution is $\quad y = c_1 \frac{1}{t} +\ c_2 \sqrt{\frac{t}{\pi}}\ $ which is an exact solution of (27).

3.3 Application of new integral transform for integral equations

The integral equations can be solved by our new transform. Before the application of convolution of two functions f(x) and g(x), a theorem should be proved.

Theorem 3.4 *Let $P(\nu)$ and $Q(\nu)$ be the $HY\{f(x)\}$ and $HY\{g(x)\}$ transforms of function $f(x)$ and $g(x)$. Then the HY transform of the convolution of $f(x)$ and $g(x)$, $(f * g)(t) = \int_0^\infty f(t)\, g(t-\tau)\, d\tau$ is given by*

$$HY\{(f*g)(t)\} = \frac{1}{\nu} P(\nu)\, Q(\nu). \tag{28}$$

Proof: The *HY* transform of $(f*g)(t)$ is defined by

$$HY\{(f*g)(t)\}$$

$$= \nu \int_0^\infty e^{-\nu^2 t} \int_0^\infty f(t)\,.g(t-\tau)\, d\tau\ dt = \nu \int_0^\infty f(\tau)\, d\tau \int_0^\infty e^{-\nu^2 t}.g(t-\tau)\, dt.$$

Now setting $t - \tau = u\quad$ results in:
$\nu 0 \overset{\infty}{\int} e^{-\nu^2 \tau} f(\tau)\, d\tau . \underset{0}{\overset{\infty}{\int}} e^{-\nu^2 t}.g(t)\ dt = \nu[\frac{1}{\nu} \underbrace{\nu \int_0^\infty e^{-\nu^2 \tau} f(\tau)\, d\tau}_{P(\nu)} \frac{1}{\nu} \underbrace{\nu \int_0^\infty e^{-\nu^2 \tau} g(t)\, dt}_{Q(\nu)}].$

then $HY\{(f*g)(t)\} = \frac{1}{\nu} P(\nu)\, Q(\nu)$. $\square$

Transform	**Natural**	**Laplace**	**Sumudu**	**Elzaki**	***HY***
Natural	$R\left(s,u\right)$	$R\left(1,u\right)$	$R\left(s,1\right)$	$vR(\frac{1}{v},1)$	$vR(v^2,1)$
Laplace	$\frac{1}{u}F(\frac{s}{u})$	$F(s)$	$\frac{1}{u}F(\frac{1}{u})$	$vF(\frac{1}{v})$	$vF(v^2)$
Sumudu	$\frac{1}{s}G(\frac{u}{s})$	$\frac{1}{s}G(\frac{1}{s})$	$G(u)$	$v^2G(v)$	$\frac{1}{v}G(\frac{1}{v^2})$
Elzaki	$\frac{s}{v^2}T(\frac{v}{s})$	$sT(\frac{1}{s})$	$\frac{1}{v}T(v)$	$T(v)$	$\frac{1}{\sqrt{V}}T(v)$
HY	$\frac{1}{\sqrt{sv}}P(\sqrt{\frac{s}{v}})$	$\frac{1}{\sqrt{s}}P(\sqrt{s})$	$\sqrt{\frac{1}{v}}P(\sqrt{\frac{1}{v}})$	$\frac{1}{v^3}P(\frac{v}{s})$	$P(v)$

Table 2: Relation between mentioned transforms.

Example 3.6 Solve the following Volterra integral equation:

$$u\left(x\right)=1-sinhx+\int_0^x\left(x-t+2\right)u\left(t\right)\ dt. \tag{29}$$

Upon taking the HY transforms of (29) we get

$$HY\{u\left(x\right)\}=HY\{1\}-HY\{sinhx\}+HY\left\{\int_0^x\left(x-t+2\right)u\left(t\right)\ dt\right\}.$$

Let $HY\{u\left(x\right)\}=P\left(u\right)$, $P\left(u\right)=\frac{1}{\nu}-\frac{\nu}{\nu^4-1}+\frac{1}{\nu}\left(\frac{1}{\nu^3}+\frac{2}{\nu}\right)P(u)$, $HY\{u\left(x\right)\}=P\left(u\right)=\frac{\nu^3}{\nu^4-1}$. Then $u\left(x\right)=cosh(x)$, which is an exact solution of (29).

In Table 2, we adjust the relationship between the mentioned transform with the Laplace, Elzaki, Sumudu and natural transforms.

4 Conclusion

In this paper, we have introduced a new integral transform. Namely, an HY transform for solving some of differential and integral equations with constant and non-constant coefficients which were not solved by other transforms like the Sumudu and Laplace once. Some of differential equations like the Euler-Cauchy equations that were solved by the power series only, are solved by this new transform. In a large domain we will discuss the HY transform for solving some of well known differential equations like the Legendre and Bessel equations.

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