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A New Representation of Exact Solutions for Nonlinear Time-Fractional Wave-Like Equations with Variable Coefficients

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Abstract: In this paper, we give a new representation of exact solutions for nonlinear time-fractional wave-like equations with variable coefficients using a recent and reliable method, namely the fractional reduced differential transform method (FRDTM). Using the FRDTM, it is possible to find solution for this type of equations in the form of infinite series, this series in closed form gives the exact solution. It has been proven that the FRDTM is a convenient and effective method in its application. The accuracy and efficiency of the method is tested by means of three numerical examples.

Keywords: *nonlinear time-fractional wave-like equations; Caputo fractional derivative; fractional reduced differential transform method.*

Mathematics Subject Classification (2010): Primary 35R11, 26A33, Secondary 35C05, 74G10.

1 Introduction

The nonlinear fractional partial differential equations (NFPDEs) are increasingly used to model many problems in mathematical physics, including electromagnetics, fluid flow, diffusion, quantum mechanics, damping laws, viscoelasticity and other applications. Exact solutions of NFPDEs are sometimes too complicated to be attained by conventional techniques due to the computational complexities of nonlinear parts involving them. Therefore, for the study of solution of NFPDEs there are variety of analytical and approximate methods found in literature. Among them most useful and common methods are: the Adomian decomposition method (ADM) [8], variational iteration method

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(VIM) [10], fractional difference method (FDM) [4], generalized differential transform method (GDTM) [1], homotopy analysis method (HAM) [11], homotopy perturbation method (HPM) [9].

Recently, an efficient analytical technique for handling different types of NFPDEs has been developed called the fractional reduced differential transform method (FRDTM). The FRDTM was effectively used for finding the solution of various kinds of NFPDEs [5–7]. Further, this method does not require any discretization, linearization and therefore it reduces significantly the numerical computations compare with the existing methods such as the perturbation technique, differential transform method (DTM) and the Adomian decomposition method (ADM).

In [3], we solved the nonlinear time-fractional wave-like equations with variable coefficients by two different methods and compared between these two methods.

The main objective of this paper is to give a new representation of exact solutions for this type of equations using the FRDTM.

Consider the following nonlinear time-fractional wave-like equations:

$$\begin{aligned} D_t^{2\alpha} u &= \sum_{i,j=1}^{n} F_{1ij}(X,t,u) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}) \\ &\quad + \sum_{i=1}^{n} G_{1i}(X,t,u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X,t,u) + S(X,t), \end{aligned} \tag{1}$$

with the initial conditions

$$u(X,0) = a_0(X), \ u_t(X,0) = a_1(X), \tag{2}$$

where $D_t^{2\alpha}$ is the Caputo fractional derivative operator of order 2α, $\frac{1}{2} < \alpha \leq 1$.

Here $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, n \in \mathbb{N}^*$, F_{1ij}, G_{1i} $i,j \in \{1,2,...,n\}$, are nonlinear functions of X, t and u, F_{2ij}, G_{2i} $i,j \in \{1,2,...,n\}$, are nonlinear functions of derivatives of u with respect to x_i and x_j $i,j \in \{1,2,...,n\}$, respectively. Also H, S are nonlinear functions and k, m, p are integers.

These types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows.

2 Basic Definitions

In this section, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see [4].

Definition 2.1 A real function $u(x,t), x \in I \subset \mathbb{R}, t > 0$, is considered to be in the space $C_\mu(I \times \mathbb{R}^+)$, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $u(x,t) = t^p f(x,t)$, where $f(x,t) \in C\left(I \times \mathbb{R}^+\right)$, and it is said to be in the space C_μ^n if $u^{(n)}(x,t) \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2 Let $u(x,t) \in C_\mu(I \times \mathbb{R}^+), \mu \geq -1$. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of $u(x,t)$ is defined as follows:

$$I_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} u(x,\xi) d\xi, \alpha > 0, x \in I, t > \xi \geq 0, \\ u(x,t), \qquad \alpha = 0, \end{cases} \tag{3}$$

where $\Gamma(.)$ is the well-known gamma function.

Definition 2.3 The Caputo fractional derivative operator of order α of $u(x,t)$ is defined as follows:

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\xi)^{n-\alpha-1} u^{(n)}(x,\xi) d\xi, n-1 < \alpha < n, \\ u^{(n)}(x,t), \quad \alpha = n. \end{cases} \quad (4)$$

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation:

$$I_t^{\alpha} D_t^{\alpha} u(x,t) = u(x,t) - \sum_{k=0}^{n-1} u^{(k)}(x, 0^+) \frac{t^k}{k!}, x \in I, t > 0. \quad (5)$$

3 Fractional Reduced Differential Transform Method (FRDTM)

In this section, we apply the fractional reduced differential transform method for $(n+1)-$variable function $u(x_1, x_2, ..., x_n, t)$ which has been developed in [2].

On the basis of the properties of the one-dimensional differential transform, the function $u(x_1, x_2, ..., x_n, t)$ can be represented as

$$\begin{aligned} u(x_1, x_2, ..., x_n, t) &= \left(\sum_{k_1=0}^{\infty} F_1(k_1) x_1^{k_1} \right) \left(\sum_{k_2=0}^{\infty} F_2(k_2) x_2^{k_2} \right) \times ... \\ &\times \left(\sum_{k_n=0}^{\infty} F_n(k_n) x_n^{k_n} \right) \times \left(\sum_{k_m=0}^{\infty} F_m(k_m) t^{k_m} \right) \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} ... \sum_{k_n=0}^{\infty} \sum_{k_m=0}^{\infty} U(k_1, k_2, ..., k_n, k_m) x_1^{k_1} x_2^{k_2} ... x_n^{k_n} t^{k_m}, \end{aligned}$$

where $U(k_1, k_2, ..., k_n, k_m) = F_1(k_1) \times F_2(k_2) \times ... \times F_n(k_n) \times F_m(k_m)$ is called the spectrum of $u(x_1, x_2, ..., x_n, t)$. Next, we assume that $u(X,t), X = (x_1, x_2, ..., x_n)$ is a continuously differentiable function with respect to the space variable and time in the domain of interest.

Definition 3.1 Let $u(X,t)$ be an analytic function, then the FRDT of u is given by

$$U_k(X) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(X,t) \right]_{t=t_0}, \quad (6)$$

where α is a parameter describing the order of time fractional derivative in the Caputo sense. Here the lowercase $u(X,t)$ represents the original function while the uppercase $U_k(X)$ stands for the fractional reduced transformed function.

Definition 3.2 The inverse FRDT of $U_k(X)$ is defined by

$$u(X,t) = \sum_{k=0}^{\infty} U_k(X)(t-t_0)^{k\alpha}. \quad (7)$$

Combining equations (6) and (7), we have

$$u(X,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(X,t)\right]_{t=t_0}(t-t_0)^{k\alpha}. \tag{8}$$

In particular, for $t_0=0$, equation (8) becomes

$$u(X,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(X,t)\right]_{t=0}t^{k\alpha}. \tag{9}$$

Moreover, if $\alpha=1$, then the FRDT of equation (8) reduces to the classical RDT method. From the above definitions, the fundamental operations of the FRDTM are given by the following theorems.

Theorem 3.1 *Let $U_k(X), V_k(X)$ and $W_k(X)$ be the fractional reduced differential transform of the functions $u(X,t), v(X,t)$ and $w(X,t)$, respectively, then*

(1) if $w(X,t)=\lambda u(X,t)+\mu v(X,t)$, then $W_k(X)=\lambda U_k(X)+\mu V_k(X)$, λ, $\mu\in\mathbb{R}$.

(2) if $w(X,t)=u(X,t)v(X,t)$, then $W_k(X)=\sum_{r=0}^{k}U_r(X)V_{k-r}(X)$.

(3) if $w(X,t)=u^1(X,t)u^2(X,t)...u^n(X,t)$, then

$$\begin{aligned}W_k(X) &= \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}...\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U^1_{k_1}(X)U^2_{k_2-k_1}(X)\\ &\quad\times...\times U^{n-1}_{k_{n-1}-k_{n-2}}(X)U^n_{k-k_{n-1}}(X).\end{aligned}$$

(4) if $w(X,t)=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}u(X,t)$, then

$$W_k(X)=\frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)}U_{k+n}(X), n=1,2,....$$

4 FRDTM for Nonlinear Time-Fractional Wave-Like Equations

Theorem 4.1 *Consider the nonlinear time-fractional wave-like equations (1) with the initial conditions (2).*

Then, by FRDTM the solution of equations (1)-(2) is given in the form of infinite series as follows:

$$u(X,t)=\sum_{k=0}^{\infty}U_k(X)t^{k\alpha},$$

where $U_k(X)$ is the fractional reduced differential transformed function of $u(X,t)$.

Proof. In order to achieve our goal, we consider the following nonlinear time-fractional wave-like equations (1) with the initial conditions (2).

Applying the FRDTM to equation (1), we obtain the following recurrence relation formula:

$$U_{k+2}(X)=\frac{\Gamma\left(k\alpha+1\right)}{\Gamma\left(k\alpha+2\alpha+1\right)}\left[A_k(X)+B_k(X)+C_k(X)+D_k(X)\right], \tag{10}$$

where $A_k(X), B_k(X), C_k(X)$ and $D_k(X)$ are the transformed form of the nonlinear terms, $\sum\limits_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(u_{x_i},u_{x_j})$, $\sum\limits_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p}{\partial x_i^p}G_{2i}(u_{x_i})$, $H(X,t,u)$ and $S(X,t)$, rerspectively.

Now, using the FRDTM under the initial conditions (2), we obtain

$$U_0(X)=a_0(X), U_1(X)=a_1(X). \tag{11}$$

We substitute equation (11) into equation (10), we get

$$\begin{aligned} U_0(X) &= a_0(X),\ U_1(X)=a_1(X), \\ U_2(X) &= \frac{1}{\Gamma\left(2\alpha+1\right)}\left[A_0(X)+B_0(X)+C_0(X)+D_0(X)\right], \\ U_3(X) &= \frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(3\alpha+1\right)}\left[A_1(X)+B_1(X)+C_1(X)+D_1(X)\right], \\ U_4(X) &= \frac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(4\alpha+1\right)}\left[A_2(X)+B_2(X)+C_2(X)+D_2(X)\right]. \\ &\cdots \end{aligned} \tag{12}$$

Then, the solution of equations (1)-(2) in the form of infinite series is given by

$$u(X,t)=\sum_{k=0}^{\infty} U_k(X)t^{k\alpha}. \tag{13}$$

The proof is complete.

5 Numerical Examples

In this section, we describe the method explained in Section 4. Three numerical examples of nonlinear time-fractional wave-like equations with variable coefficients are considered to validate the capability, reliability and efficiency of the FRDTM.

Example 5.1 Consider the 2-dimensional nonlinear time-fractional wave-like equation with variable coefficients:

$$D_t^{2\alpha}u=\frac{\partial^2}{\partial x\partial y}(u_{xx}u_{yy})-\frac{\partial^2}{\partial x\partial y}(xyu_xu_y)-u, t>0, \frac{1}{2}<\alpha\leq 1, \tag{14}$$

with the initial conditions

$$u(x,y,0)=e^{xy}, u_t(x,y,0)=e^{xy}, (x,y)\in\mathbb{R}^2. \tag{15}$$

Applying the FRDTM to equations (14)-(15), we obtain the following recurrence relation formula:

$$\begin{aligned} U_0(x,y) &= e^{xy},\ U_1(x,y)=e^{xy}, \\ U_{k+2}(x,y) &= \frac{\Gamma\left(k\alpha+1\right)}{\Gamma\left(k\alpha+2\alpha+1\right)}\left[\frac{\partial^2}{\partial x\partial y}A_k(x,y)-\frac{\partial^2}{\partial x\partial y}B_k(x,y)-U_k(x,y)\right], \end{aligned} \tag{16}$$

where $A_k(x,y)$ and $B_k(x,y)$ are the transformed form of the nonlinear terms, $u_{xx}u_{yy}$ and xyu_xu_y. For the convenience of the reader, the first few nonlinear terms are as follows:

$$\begin{aligned} A_0 &= U_{0xx}U_{0yy}, \\ A_1 &= U_{0xx}U_{1yy}+U_{1xx}U_{0yy}, \\ A_2 &= U_{0xx}U_{2yy}+U_{1xx}U_{1yy}+U_{2xx}U_{0yy}, \end{aligned}$$

$$\begin{aligned} B_0 &= xyU_{0x}U_{0y}, \\ B_1 &= xyU_{0x}U_{1y}+xyU_{1x}U_{0y}, \\ B_2 &= xyU_{0x}U_{2y}+xyU_{1x}U_{1y}+xyU_{2x}U_{0y}. \end{aligned}$$

From the relationship in (16), we obtain

$$\begin{aligned} U_0(x,y) &= e^{xy},\ U_1(x,y)=e^{xy}, U_2(x,y)=-\frac{1}{\Gamma(2\alpha+1)}e^{xy}, \\ U_3(x,y) &= -\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}e^{xy}, U_4(x,y)=\frac{1}{\Gamma(4\alpha+1)}e^{xy}... \end{aligned}$$

So, the solution of equations (14)-(15) is given in the form of infinite series as follows:

$$u(x,y,t)=\left(1+t^{\alpha}-\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{1}{\Gamma(4\alpha+1)}t^{4\alpha}+...\right)e^{xy}.$$

In particular, for $\alpha=1$, the solution of equations (14)-(15) has the general pattern form which coincides with the following exact solution in terms of infinite series:

$$u(x,y,t)=\left(1+t-\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+...\right)e^{xy}.$$

Therefore, the exact solution of equations (14)-(15) in a closed form of elementary function will be given by

$$u(x,y,t)=(\cos t+\sin t)\,e^{xy},$$

which is the same result as those obtained by the NIM and NHPM [3].

Example 5.2 Consider the following nonlinear time-fractional wave-like equation with variable coefficients:

$$D_t^{2\alpha}u=u^2\frac{\partial^2}{\partial x^2}(u_xu_{xx}u_{xxx})+u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3)-18u^5+u,\quad t>0,\frac{1}{2}<\alpha\leq 1, \tag{17}$$

with the initial conditions

$$u(x,0)=e^x, u_t(x,0)=e^x,\quad x\in\,]0,1[\,. \tag{18}$$

Applying the FRDTM to equations (17)-(18), we obtain the following recurrence relation formula:

$$\begin{aligned} U_0(x) &= e^x, U_1(x)=e^x, \qquad (19) \\ U_{k+2}(x) &= \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left[A_k(x)+B_k(x)-18C_k(x)+U_k(x)\right], \end{aligned}$$

where $A_k(x)$, $B_k(x)$ and $C_k(x)$ are the transformed form of the nonlinear terms, $u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}), u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3)$ and u^5. For the convenience of the reader, the first few nonlinear terms are as follows:

$$\begin{aligned}
A_0 &= U_0^2\frac{\partial^2}{\partial x^2}\left[U_{0x}U_{0xx}U_{0xxx}\right],\\
A_1 &= 2U_0U_1\frac{\partial^2}{\partial x^2}\left[U_{0x}U_{0xx}U_{0xxx}\right]+U_0^2\frac{\partial^2}{\partial x^2}\left[U_{1x}U_{0xx}U_{0xxx}\right.\\
&\quad \left.+U_{0x}U_{1xx}U_{0xxx}+U_{0x}U_{0xx}U_{1xxx}\right],
\end{aligned}$$

$$\begin{aligned}
B_0 &= U_{0x}^2\frac{\partial^2}{\partial x^2}U_{0xx}^3,\\
B_1 &= 2U_{0x}U_{1x}\frac{\partial^2}{\partial x^2}U_{0xx}^3+3U_{0x}^2\frac{\partial^2}{\partial x^2}\left[U_{0xx}^2U_{1xx}\right],
\end{aligned}$$

$$C_0=U_0^5,\ C_1=5U_0^4U_1.$$

From the relationship in (19), we obtain

$$\begin{aligned}
U_0(x) &= e^x,\ U_1(x)=e^x,\\
U_2(x) &= \frac{1}{\Gamma(2\alpha+1)}e^x, U_3(x)=\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(3\alpha+1\right)}e^x...
\end{aligned}$$

So, the solution of equations (17)-(18) is given in the form of infinite series as follows:

$$u(x,t)=\left(1+t^\alpha+\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}+...\right)e^x.$$

In particular, for $\alpha=1$, the solution of equations (17)-(18) has the general pattern form which coincides with the following exact solution in terms of infinite series:

$$u(x,t)=\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\right)e^x.$$

Therefore, the exact solution of equations (17)-(18) in a closed form of elementary function will be given by

$$u(x,t)=e^{x+t},$$

which is the same result as those obtained by the NIM and NHPM [3].

Example 5.3 Consider the following one-dimensional nonlinear time-fractional wave-like equation with variable coefficients:

$$D_t^{2\alpha}u=x^2\frac{\partial}{\partial x}(u_xu_{xx})-x^2(u_{xx})^2-u,\quad t>0,\frac{1}{2}<\alpha\leq 1, \tag{20}$$

with the initial conditions

$$u(x,0)=0,u_t(x,0)=x^2,\quad x\in\left]0,1\right[. \tag{21}$$

Applying the FRDTM to equations (20)-(21), we obtain the following recurrence relation formula:

$$\begin{aligned} U_0(x) &= 0, U_1(x) = x^2, \\ U_{k+2}(x) &= \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left[x^2\frac{\partial}{\partial x}A_k(x) - x^2 B_k(x) - U_k(x)\right], \end{aligned} \tag{22}$$

where $A_k(x)$ and $B_k(x)$ are the transformed form of the nonlinear terms, $u_x u_{xx}$ and u_{xx}^2. For the convenience of the reader, the first few nonlinear terms are as follows:

$$\begin{aligned} A_0 &= U_{0x}U_{0xx}, \\ A_1 &= U_{0x}U_{1xx} + U_{1x}U_{0xx}, \\ A_2 &= U_{0x}U_{2xx} + U_{1x}U_{1xx} + U_{2x}U_{0xx}, \end{aligned}$$

$$\begin{aligned} B_0 &= U_{0xx}^2, \\ B_1 &= 2U_{0xx}U_{1xx}, \\ B_2 &= 2U_{0xx}U_{2xx} + U_{1xx}^2. \end{aligned}$$

From the relationship in (22), we obtain

$$\begin{aligned} U_0(x) &= 0, U_1(x) = x^2, U_2(x) = 0, \\ U_3(x) &= -\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}x^2, U_4(x) = 0... \end{aligned}$$

So, the solution of equations (20)-(21) is given in the form of infinite series as follows:

$$u(x,t) = \left(t^\alpha - \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}t^{5\alpha} + ...\right)x^2.$$

In particular, for $\alpha = 1$, the solution of equations (20)-(21) has the general pattern form which coincides with the following exact solution in terms of infinite series:

$$u(x,t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + ...\right)x^2.$$

Therefore, the exact solution of equations (20)-(21) in a closed form of elementary function will be given by

$$u(x,t) = x^2 \sin t,$$

which is the same result as those obtained by the NIM and NHPM [3].

6 Numerical Results and Discussion

In this section the numerical results for all Examples 5.1, 5.2 and 5.3 are presented. Figures 1, 3 and 5 represent the surface graph of the exact solution and the 6−term approximate solution at $\alpha = 0.6, 0.8, 1$. Figures 2, 4 and 6 represent the behavior of the exact solution and the 6−term approximate solution at $\alpha = 0.7, 0.8, 0.95, 1$ in the case when $x = y = 0.5$ for Example 5.1 and $x = 0.5$ for Examples 5.2 and 5.3. Tables 1, 2 and 3 show the absolute errors between the exact solution and the 6−term approximate solution at $\alpha = 1$ and different values of x, y and t. The numerical results afirm that when α approaches 1, our results obtained by the FRDTM approach the exact solutions.

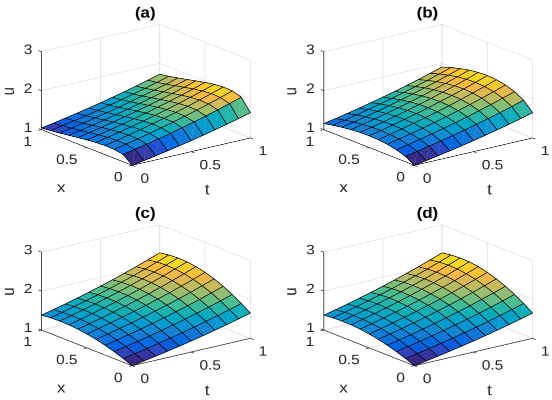

Figure 1: The surface graph of the exact solution and the 6−term approximate solution by the FRDTM for Example 5.1 when $y = 0.5$: (a) u when $\alpha = 0.6$, (b) u when $\alpha = 0.8$, (c) u when $\alpha = 1$, and (d) u is exact.

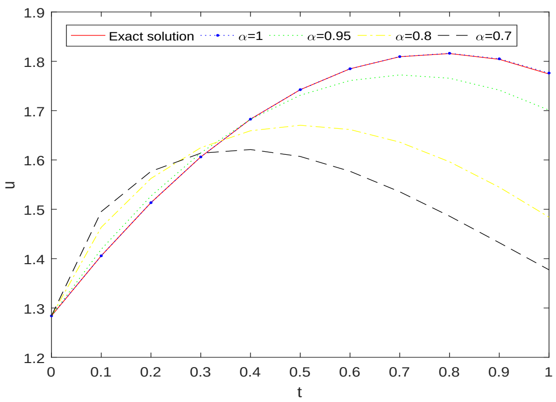

Figure 2: The behavior of the exact solution and the 6−term approximate solution by the FRDTM for different values of α for Example 5.1 when $x = y = 0.5$.

$t/x, y$	0.1	0.3	0.5	0.7
0.1	1.4226×10^{-9}	1.5411×10^{-9}	1.8085×10^{-9}	2.2991×10^{-9}
0.3	1.0648×10^{-6}	1.1535×10^{-6}	1.3536×10^{-6}	1.7208×10^{-6}
0.5	2.3382×10^{-5}	2.5330×10^{-5}	2.9725×10^{-5}	3.7787×10^{-5}
0.7	1.8000×10^{-4}	1.9499×10^{-4}	2.2882×10^{-4}	2.9089×10^{-4}
0.9	8.2963×10^{-4}	8.9872×10^{-4}	1.0547×10^{-3}	1.3407×10^{-3}

Table 1: Comparison of the absolute errors for the obtained results and the exact solution for Example 5.1, when $n = 6$ and $\alpha = 1$.

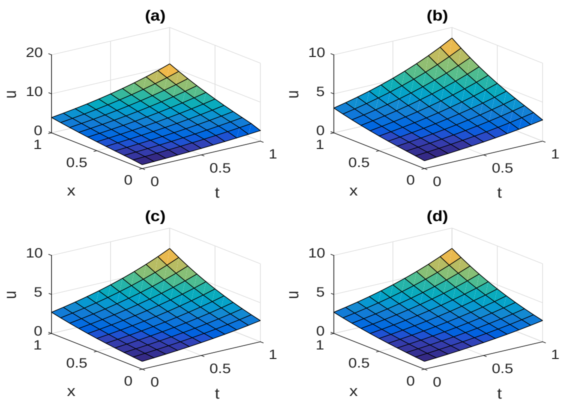

Figure 3: The surface graph of the exact solution and the 6−term approximate solution by the FRDTM for Example 5.2 : (a) u when $\alpha = 0.6$, (b) u when $\alpha = 0.8$, (c) u when $\alpha = 1$, and (d) u is exact.

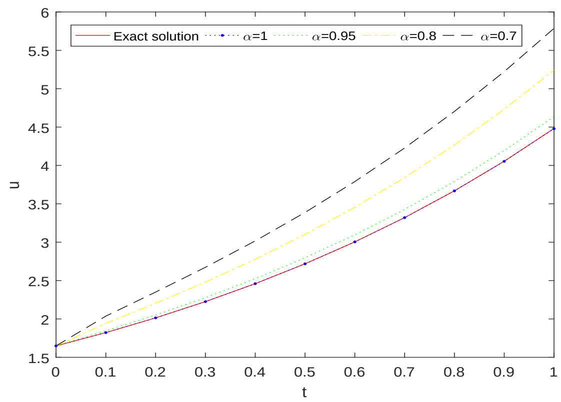

Figure 4: The behavior of the exact solution and the 6−term approximate solution by the FRDTM for different values of α for Example 5.2 when $x = 0.5$.

t/x	0.1	0.3	0.5	0.7
0.1	1.5572×10^{-9}	1.9019×10^{-9}	2.3230×10^{-9}	2.8373×10^{-9}
0.3	1.1688×10^{-6}	1.4276×10^{-6}	1.7436×10^{-6}	2.1297×10^{-6}
0.5	2.5810×10^{-5}	3.1525×10^{-5}	3.8504×10^{-5}	4.7029×10^{-5}
0.7	2.0036×10^{-4}	2.4472×10^{-4}	2.9890×10^{-4}	3.6507×10^{-4}
0.9	9.3372×10^{-4}	1.1404×10^{-3}	1.3929×10^{-3}	1.7013×10^{-3}

Table 2: Comparison of the absolute errors for the obtained results and the exact solution for Example 5.2, when $n = 6$ and $\alpha = 1$.

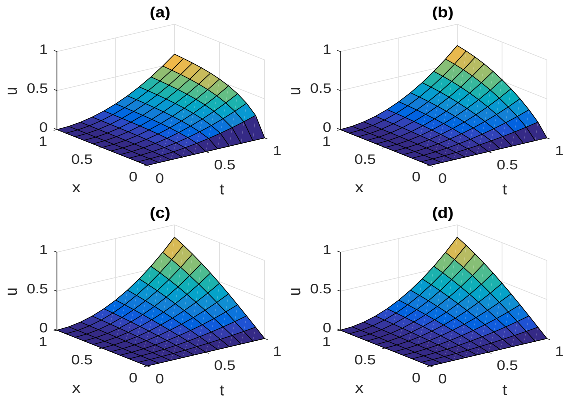

Figure 5: The surface graph of the exact solution and the 6−term approximate solution by the FRDTM for Example 5.3 : (a) u when $\alpha = 0.6$, (b) u when $\alpha = 0.8$, (c) u when $\alpha = 1$, and (d) u is exact.

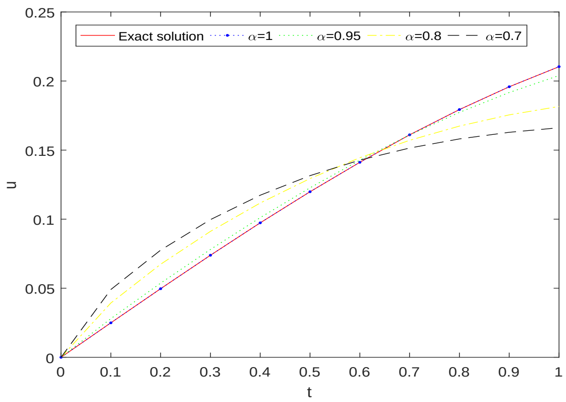

Figure 6: The behavior of the exact solution and the 6−term approximate solution by the FRDTM for different values of α for Example 5.3 when $x = 0.5$.

t/x	0.1	0.3	0.5	0.7
0.1	1.9839×10^{-13}	1.7855×10^{-12}	4.9596×10^{-12}	9.7209×10^{-12}
0.3	4.3339×10^{-10}	3.9005×10^{-9}	1.0835×10^{-8}	2.1236×10^{-8}
0.5	1.5447×10^{-8}	1.3903×10^{-7}	3.8618×10^{-7}	7.5692×10^{-7}
0.7	1.6229×10^{-7}	1.4606×10^{-6}	4.0574×10^{-6}	7.9524×10^{-6}
0.9	9.3840×10^{-7}	8.4456×10^{-6}	2.3460×10^{-5}	4.5982×10^{-5}

Table 3: Comparison of the absolute errors for the obtained results and the exact solution for Example 5.3, when $n = 6$ and $\alpha = 1$.

7 Conclusion

In this paper, a new representation of exact solutions for nonlinear time-fractional wave-like equations with variable coefficients was presented by using the fractional reduced differential transform method (FRDTM). The method was applied to three numerical examples. In the numerical examples, our method gave us the solutions in the form of infinite series, this series in closed form gives the corresponding exact solutions for these equations without any transformation, discretization and any other restrictions, therefore it reduces significantly the numerical computations compare with the existing methods such as the perturbation technique, differential transform method (DTM) and the Adomian decomposition method (ADM). Also, our results obtained in this paper are in a good agreement with the exact solutions; hence, this technique is powerful and efficient as an alternative method for finding approximate and exact solutions for many other nonlinear fractional partial differential equations.

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