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Analysis and Adaptive Control Synchronization of a Novel 3-D Chaotic System

F. Hannachi *

Department of Management Sciences, University of Tebessa, (12002), Algeria

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Abstract: In this paper, a new 3D chaotic system is introduced. Basic dynamical characteristics and properties of this new chaotic system are studied, namely the equilibrium points and their stability, the Lyapunov exponent, Lyapunov exponent spectrum and the Kaplan-Yorke dimension. Also, we derive new control results via the adaptive control method based on Lyapunov stability theory and the adaptive control theory of this new chaotic system with unknown parameters. The results are validated by numerical simulation using Matlab.

Keywords: *chaotic system; strange attractor; Lyapunov exponent; Lyapunov stability theory; adaptive control; synchronization.*

Mathematics Subject Classification (2010): 37B55, 34C28, 34D08, 37B25, 37D45, 93C40, 93D05.

1 Introduction

In mathematics and physics, chaos theory deals with the behavior of certain nonlinear dynamical systems that under certain conditions exhibit a phenomenon known as chaos, which is characterised by a sensitivity to initial conditions [1]. Chaos as an important nonlinear phenomenon has been studied in mathematics, engineering and many other disciplines. Since Lorenz discovered a three-dimensional autonomous chaotic system [2], many other systems have been introduced and analysed, we mention the Chen, Rössler and Lü systems [3,4,5]. After that hyperchaotic systems were constructed using many different methods. The synchronization of two chaotic systems was introduced in the work of Pecora and Carroll [6], then many different methodologies have been developed for synchronization of chaotic systems such as the OGY method [7], active contol method [8], sliding mode control [9], backstepping control [10], function projective method [11], adaptive control [12-14], etc.

In this work, a new chaotic system is introduced and we derive new control results via the adaptive control method based on Lyapunov stability theory and the adaptive control theory for this new chaotic system with unknown parameters. The results are validated by numerical simulation using Matlab.

* Corresponding author: mailto:farehhannachi@yahoo.com

1.1 Description of the novel chaotic system

In this research work, we propose a new 3D chaotic system with two quadratic nonlinearities, which is given in the system form as

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1), \\ \frac{dx_2}{dt} = cx_1 + x_1x_3, \\ \frac{dx_3}{dt} = -x_1x_2 + b(x_1 - x_3), \end{cases} \tag{1}$$

where a, b, c are positive reals parameters. In the first part of this paper, we shall show that the system (1) is chaotic when the system parameters a, b and c take the values:

$$a = 13, b = 2.5, c = 50. \tag{2}$$

1.2 Basic properties

In this section, some basic properties of system (1) are given. We start with the equilibrum points of the system and check their stability at the initial values of the parameters a, b, c.

1.3 Equilibrum points

Putting equations of system (1) equal to zero, i.e.

$$a(x_2 - x_1) = 0, \quad cx_1 + x_1x_3 = 0, \quad -x_1x_2 + b(x_1 - x_3) = 0, \tag{3}$$

gives the three equilibrium points

$$p_0 = (0,0,0), \quad p_{1,2} = \left(\frac{1}{2}b \mp \frac{1}{2}\sqrt{4bc + b^2}, \frac{1}{2}b \mp \frac{1}{2}\sqrt{4bc + b^2}, -c\right). \tag{4}$$

1.4 Stability

In order to check the stability of the equilibrum points we derive the Jacobian matrix at a point $p(x, y, z)$ of the system (1)

$$J(p) = \begin{pmatrix} -a & a & 0 \\ c + z & 0 & x \\ b - y & -x & -b \end{pmatrix}. \tag{5}$$

For p_0, we obtain $J(p_0) = \begin{pmatrix} -a & a & 0 \\ c & 0 & 0 \\ b & 0 & -b \end{pmatrix}$, with the characteristic polynomial equation $\lambda^3 + (a + b)\lambda^2 + (ab - ac)\lambda - abc = 0$, which has three eigenvalues

$$\lambda_1 = 19.811, \lambda_2 = -2.5, \lambda_3 = -32.811. \tag{6}$$

Since all the eigenvalues are real, the Hartma-Grobman theorem implies that p_0 is a saddle point which is unstable according to the Lyapunov theorem of stability.

By the same method, the eigenvalues of the Jacobian at p_1 are:

$$\lambda_1 = 0.993\,85 - 12.895i, \lambda_2 = 0.993\,85 + 12.895i, \lambda_3 = -17.488. \tag{7}$$

The eigenvalues of the Jacobian at p_2 are:

$$\lambda_1 = 0.763\,22 - 14.\,634i, \lambda_2 = 0.763\,22 + 14.\,634i, \lambda_3 = -17.\,026. \tag{8}$$

Then p_1 and p_2 are two unstable saddle-foci because none of the eigenvalues have zero real part and λ_1, λ_2 are complex.

1.5 Dissipativity

A dissipative dynamical system satisfies the condition

$$\nabla.V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} < 0. \tag{9}$$

In the case of the system (1), we have

$$\nabla.V = -(a+b). \tag{10}$$

For $a = 13, b = 2.5, c = 50$ we obtain $\nabla.V = -15.\,5 < 0$, and threfore dissipativity condition holds for this system. Also,

$$\frac{dV}{dt} = e^{-(a+b)} = 1.\,855\,4 \times 10^{-7}. \tag{11}$$

Then the volume of the attractor decreases by a factor of 0.00000018554.

2 Lyapunov Exponents and Kaplan-Yorke Dimension

Lyapunov exponents are used to measure the exponential rates of divergence and convergence of nearby trajectoiries, which is an important characterstic to judge whether the system is chaotic or not. The existence of at least one positive Lyapunov exponent implies that the system is chaotic.

For the chosen parameter values (2), the Lyapunov exponents of the novel chaotic system (1) are obtained using Matlab as:

$$L_1 = 1.4375, L_2 = -0.000166417, L_3 = -16.9373. \tag{12}$$

The Lyapunov exponents spectrum is shown in Fig. 1.

Since the spectrum of Lyapunov exponents (13) has a positive term L_1, it follows that the novel 3-D chaotic system (1) is chaotic. The Kaplan-Yorke dimension of system (1) is calculated as

$$D_{KL} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0849. \tag{13}$$

3 Adaptive Control of the Novel 3-D Chaotic System

This section describes an adaptive design of a globally stabilizing feedback controller for the chaotic system (1) with unknown parameters. The design is carried out using the adaptive control theory and Lyapunov stability theory.

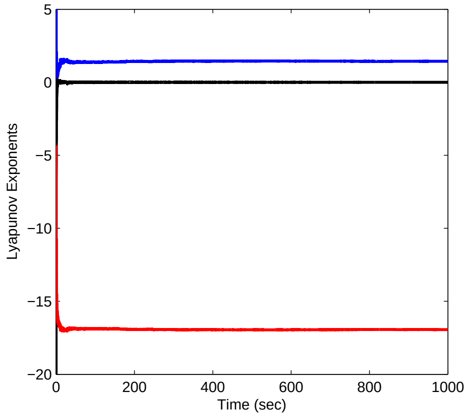

Figure 1: Lyapunov exponents spectrum.

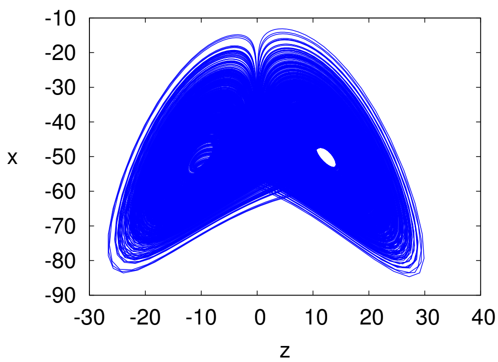

Figure 2: Projection of the strange attractor of the system (1) into the (z; x)-plane.

A controlled chaotic system of (1) is given by

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1) + u_1, \\ \frac{dx_2}{dt} = cx_1 + x_1x_3 + u_2, \\ \frac{dx_3}{dt} = -x_1x_2 + b(x_1 - x_3) + u_3, \end{cases} \tag{14}$$

where a, b, c are unknown constant parameters, and u_1, u_2, u_3 are adaptive controllers to be found using the states x_1, x_2, x_3 and estimates $a_1(t), b_1(t), c_1(t)$ of the unknown parameters a, b, c, respectively.

We take the adaptive control law defined by

$$\begin{cases} u_1 = -a_1(t)(x_2 - x_1) - k_1x_1, \\ u_2 = -c_1(t)x_1 - x_1x_3 - k_1x_2, \\ u_3 = x_1x_2 - b_1(t)(x_1 - x_3) - k_3x_3, \end{cases} \tag{15}$$

where k_1, k_2, k_3 are positive gain constants.

Substituting (15) into (14), we obtain the closed-loop control system as

$$\begin{cases} \frac{dx_1}{dt} = (a - a_1(t))(x_2 - x_1) - k_1 x_1, \\ \frac{dx_2}{dt} = (c - c_1(t))x_1 - k_2 x_2, \\ \frac{dx_3}{dt} = (b - b_1(t))(x_1 - x_3) - k_3 x_3. \end{cases} \tag{16}$$

We define the parameter estimation errors as

$$e_a(t) = a - a_1(t), \quad e_c(t) = c - c_1(t), \quad e_b(t) = b - b_1(t). \tag{17}$$

By using (17), we rewrite the closed-loop system (16) as

$$\begin{cases} \frac{dx_1}{dt} = e_a(t)(x_2 - x_1) - k_1 x_1, \\ \frac{dx_2}{dt} = e_c(t)x_1 - k_2 x_2, \\ \frac{dx_3}{dt} = e_b(t)(x_1 - x_3) - k_3 x_3. \end{cases} \tag{18}$$

Differentiating (17) with respect to t, we obtain

$$\begin{cases} \frac{de_a(t)}{dt} = -\frac{da_1(t)}{dt}, \\ \frac{de_c(t)}{dt} = -\frac{dc_1(t)}{dt}, \\ \frac{de_b(t)}{dt} = -\frac{db_1(t)}{dt}. \end{cases} \tag{19}$$

To find an update law for the parameter estimates, we shall use the Lyapunov stability theory. We consider the quadratic Lyapunov function given by

$$V(x_1, x_2, x_3, e_a, e_b, e_c) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2\right). \tag{20}$$

which is a positive definite function on $\mathbb{R}^6$.

Differentiating V along the trajectories of the systems (18) and (19), we obtain the following:

$$\dot{V} = -\sum_{i=1}^{3} k_i x_i^2 + e_a\left(x_1x_2 - x_1^2 - \frac{da_1(t)}{dt}\right) + e_b\left(x_1x_3 - x_3^2 - \frac{db_1(t)}{dt}\right) + e_c\left(x_1x_2 - \frac{dc_1(t)}{dt}\right). \tag{21}$$

In view of (21), we take the parameter update law as follows

$$\begin{cases} \frac{da_1(t)}{dt} = x_1x_2 - x_1^2, \\ \frac{db_1(t)}{dt} = x_1x_3 - x_3^2, \\ \frac{dc_1(t)}{dt} = x_1x_2. \end{cases} \tag{22}$$

Theorem 3.1 *The 3-D novel chaotic system (14) with unknown parameters is globally and exponentially stabilized by the adaptive feedback control law (15) and the parameter update law (22), where k_1, k_2, k_3 are positive constants* 3.1.

Proof. Substituting the parameter update law (21) into (20), we obtain the time derivative of V as:

$$\dot{V} = -k_1x_1^2 - k_2x_2^2 - k_3x_3^2, \tag{23}$$

which is a negative definite function on $\mathbb{R}^6$. By the direct method of Lyapunov [15], it follows that $x_1, x_2, x_3, e_a, e_b, e_c$ are globally exponentially stable. □

3.1 Numerical simulations

We used the classical fourth-order Runge-Kutta method with the step size $h = 10^{-8}$ to solve the system of differential equations (14) and (22), when the adaptive control law (15) is applied.

The parameter values of the novel 3-D chaotic system (14) are chosen as in the chaotic case (2). The positive gain constants are taken as $k_i = 3$, for $i = 1, 2, 3$.

The initial conditions of the novel chaotic system (14) are chosen as $x_1(0) = 6.4, x_2(0) = -4.7, x_3(0) = 2.5$. Furthermore, as initial conditions of the parameter estimates of the unknown parameters, we have chosen: $a_1(0) = 2.5, b_1(0) = 5.3, c_1(0) = 4.8$.

In Figs. 3-4, the exponential convergence of the controlled states $x_1(t), x_2(t), x_3(t)$ and the time-history of the parameter estimates $a_1(t); b_1(t); c_1(t)$ are depicted, when the adaptive control law (15) and parameter update law (22) are implemented.

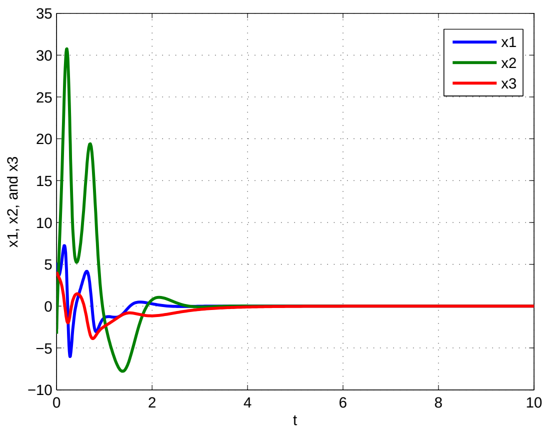

Figure 3: Exponential convergence of the controlled states $x_1(t); x_2(t); x_3(t)$.

4 Adaptive Synchronization of the Identical Novel 3-D Chaotic Systems

In this section, we derive an adaptive control law for globally and exponentially synchronizing the identical novel 3-D chaotic systems with unknown system parameters. Thus, the master system is given by the novel chaotic system dynamics

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1), \\ \frac{dx_2}{dt} = cx_1 + x_1 x_3, \\ \frac{dx_3}{dt} = -x_1 x_2 + b(x_1 - x_3). \end{cases} \tag{24}$$

Also, the slave system is given by the novel chaotic system dynamics

$$\begin{cases} \frac{dy_1}{dt} = a(y_2 - y_1) + u_1, \\ \frac{dy_2}{dt} = cy_1 + y_1 y_3 + u_2, \\ \frac{dy_3}{dt} = -y_1 y_2 + b(y_1 - y_3) + u_3. \end{cases} \tag{25}$$

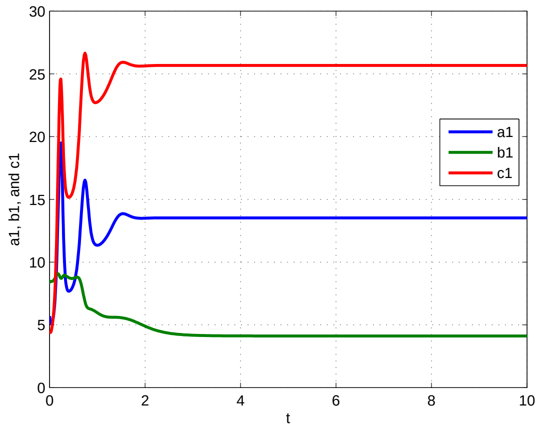

Figure 4: Time-history of the parameter estimates $a_1(t); b_1(t); c_1(t)$.

In (24) and (25), the system parameters a, b, c are unknown and the design goal is to find an adaptive feedback controls u_1, u_2, u_3 using the states x_1, x_2, x_3 and estimates $a_1(t), b_1(t), c_1(t)$ of the unknown parameters a, b, c, respectively. The synchronization error between the novel chaotic systems (24) and (25) is defined as

$$e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3. \tag{26}$$

Then (26) implies

$$\begin{cases} \dot{e}_1 = \dot{y}_1 - \dot{x}_1, \\ \dot{e}_2 = \dot{y}_2 - \dot{x}_2, \\ \dot{e}_3 = \dot{y}_3 - \dot{x}_3. \end{cases} \tag{27}$$

Thus, the synchronization error dynamics is obtained as

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + u_1, \\ \dot{e}_2 = ce_1 + y_1 y_3 - x_1 x_3 + u_2, \\ \dot{e}_3 = b(e_1 - e_3) - y_1 y_2 + x_1 x_2 + u_3. \end{cases} \tag{28}$$

We take the adaptive control law defined by

$$\begin{cases} u_1 = -a_1(e_2 - e_1) - k_1 e_1, \\ u_2 = -c_1 e_1 - y_1 y_3 + x_1 x_3 - k_2 e_2, \\ u_3 = -b_1(e_1 - e_3) + y_1 y_2 - x_1 x_2 - k_3 e_3. \end{cases} \tag{29}$$

where k_1, k_2, k_3 are positive gain constants.

Substituting (29) into (28), we obtain the closed-loop error dynamics as

$$\begin{cases} \dot{e}_1 = (a - a_1)(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 = (c - c_1)e_1 - k_2 e_2, \\ \dot{e}_3 = (b - b_1)(e_1 - e_3) - k_3 e_3. \end{cases} \tag{30}$$

The parameter estimation errors are defined as

$$e_a(t) = a - a_1(t), \quad e_c(t) = c - c_1(t), \quad e_b(t) = b - b_1(t). \tag{31}$$

Differentiating (31) with respect to t, we obtain

$$\begin{cases} \frac{de_a(t)}{dt} = -\frac{da_1(t)}{dt}, \\ \frac{de_c(t)}{dt} = -\frac{dc_1(t)}{dt}, \\ \frac{de_b(t)}{dt} = -\frac{db_1(t)}{dt}. \end{cases} \tag{32}$$

By using (31), we rewrite the closed-loop system (30) as

$$\begin{cases} \dot{e}_1 = e_a(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 = e_c e_1 - k_2 e_2, \\ \dot{e}_3 = e_b(e_1 - e_3) - k_3 e_3. \end{cases} \tag{33}$$

We consider the quadratic Lyapunov function given by

$$V(x_1, x_2, x_3, e_a, e_b, e_c) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2\right). \tag{34}$$

which is a positive definite function on $\mathbb{R}^6$.

Differentiating V along the trajectories of the systems (33) and (32), we obtain the following:

$$\dot{V} = -\sum_{i=1}^{3} k_i e_i^2 + e_a\left(e_1e_2 - e_1^2 - \frac{da_1(t)}{dt}\right) + e_b\left(e_1e_3 - e_3^2 - \frac{db_1(t)}{dt}\right) + e_c\left(e_1e_2 - \frac{dc_1(t)}{dt}\right). \tag{35}$$

In view of (35), we take the parameter update law as follows:

$$\begin{cases} \frac{da_1(t)}{dt} = e_1e_2 - e_1^2, \\ \frac{db_1(t)}{dt} = e_1e_3 - e_3^2, \\ \frac{dc_1(t)}{dt} = e_1e_2. \end{cases} \tag{36}$$

Substituting (36) into (35), we get

$$\dot{V} = -\sum_{i=1}^{3} k_i e_i^2, \tag{37}$$

which is a negative definite function on $\mathbb{R}^3$. Hence, by the Lyapunov stability theory [15], it follows that $e_i(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for $i = 1, 2, 3$. Hence, we have proved the following theorem.

Theorem 4.1 *The novel 3-D chaotic systems (24) and (25) with unknown parameters are globally and exponentially synchronized for all initial conditions by the adaptive feedback control law (29) and the parameter update law (36), where k_1, k_2, k_3 are positive constants* 4.1.

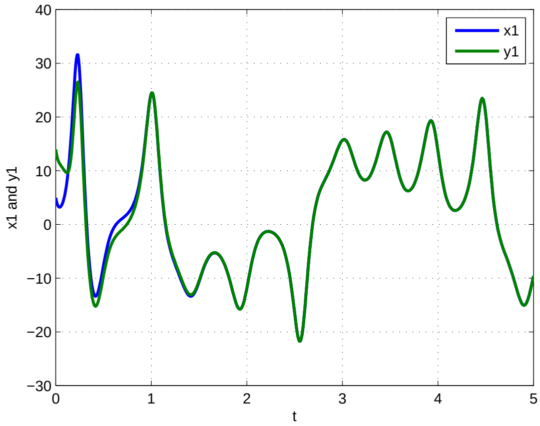

Figure 5: Synchronization of the states $x_1(t)$ and $y_1(t)$.

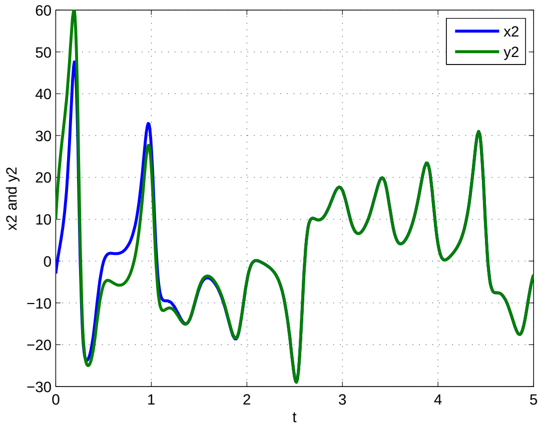

Figure 6: Synchronization of the states $x_2(t)$ and $y_2(t)$.

4.1 Numerical simulations

We used the classical fourth-order Runge-Kutta method with the step size $h = 10^{-8}$ to solve the system of differential equations (24), (25) and (36), when the adaptive control law (29) is applied.

The parameter values of the novel 3-D chaotic system (24) are chosen as in the chaotic case (2). The positive gain constants are taken as $k_i = 4$, for $i = 1, 2, 3$.

The initial conditions for the master system (24) are chosen as $x_1(0) = 5, x_2(0) =$

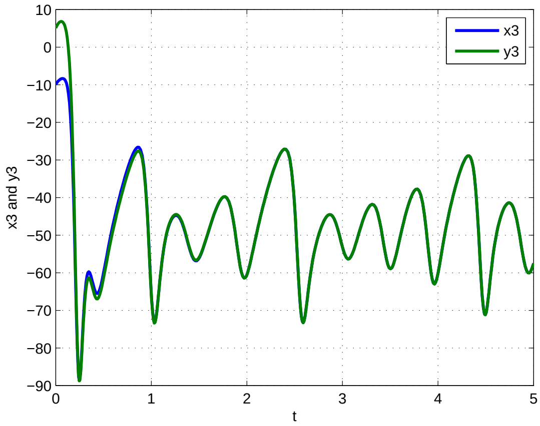

Figure 7: Synchronization of the states $x_3(t)$ and $y_3(t)$.

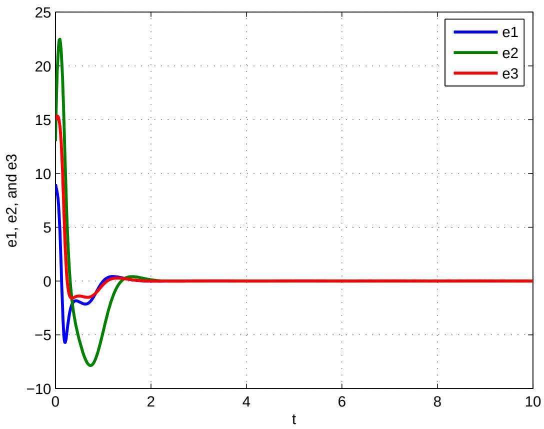

Figure 8: Time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$.

$-3, x_3(0) = -10$ and those for the slave system (25) are chosen as $y_1(0) = 14, y_2(0) = 10, y_3(0) = 5$. Furthermore, as initial conditions of the parameter estimates of the unknown parameters, we have chosen $a_1(0) = 10, b_1(0) = 15, c_1(0) = 20$. In Figs. 5-7, the synchronization of the states of the master system (24) and slave system (25) is depicted, when the adaptive control law (29) and parameter update law (36) are implemented. In Fig. 8, the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$ is depicted.

5 Conclusion

In this paper, a new chaotic system is introduced. Basic properties of this system are studied, namely, the equilibrum points and their stability, the Lyapunov exponent and the Kaplan-Yorke dimension. Moreover, adaptive control schemes have been proposed to stabilize and synchronize such two new chaotic systems. Numerical simulations using MATLAB have been made to illustrate our results for the new chaotic system with unknown parameters.

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