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Singular Analysis of Reduced ODEs of Rotating Stratified Boussinesq Equations Through the Mirror Transformations

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Abstract: In this paper we have considered the system of six coupled non-linear ordinary differential equations (ODEs), which arose in the reduction of uniformly stratified fluid contained in a rotating rectangular box of dimension $L \times L \times H$ which is completely integrable if the Rayleigh number $Ra = 0$. In our investigations, we have shown that there exits a regular mirror system near movable singularities of these integrable ODEs. Moreover, we have used the mirror system to prove the convergence of Laurent series solutions obtained by the Painlevé method.

Keywords: *mirror transformation; mirror system; Painlevé test.*

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1 Introduction

In general, we believed that the differential system is integrable due to some sort of underlying linear structure(s). But, when it comes to this concept, it is never clear what does it mean. On the other hand the integrability of nonlinear system is quite ambiguous. In this connection many mathematicians started to work over the investigation of integrability of nonlinear system. In 1889, Sophie Kowalevski [12] proved the complete integrability of the system of ordinary differential equations (ODEs) governing the motion of a spinning top moving under the influence of gravity. In her study, she was seeking

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analytic solutions whose singularities are movable poles. This was done by substituting a Frobenius series into the system of ODEs. Then, few years later, that is in 1897, Paul Painlevé [6] classified first and second order algebraic differential equations whose solutions exist in the complex domain and are devoid of movable essential singularities or movable branch points. ODEs possessing this property are said to be of the Painlevé type. Painlevé test in view of partial differential equations is generally known as WTC (Weiss-Tabor-Carnevale, [7]) test, which is further modified by S. Kichenassamy and G. K. Srinivasan [3]. So far various properties are considered as indicator of integrability: solitons, the Lax pair, the Bäcklund transformations, the underlying Hamiltonian formulations, Hirota's bilinear representation, etc. The relation between these properties has yet to be understood.

In 1999–2000, Hu J. and Yan M. [8, 9] introduced the mirror transformation, which is a new tool used in the singularity analysis of ODEs. With the help of this method we constructed the mirror system of given PDEs or ODEs successfully; we could focus commonly at the singularity structure and symplectic structure of the Hamiltonian system for each principle balance in the Painlevé test. Further to this study, Hu et al [11] proved that the mirror transformation is canonical for finite-dimensional Hamiltonian systems. Furthermore, in 2001 Yee [13] showed that linearization of the mirror systems near movable poles provides the possibility to construct the associated Backlund transformations. In continuous development of mirror transformations in 2011, Tat-Leung Yee [14] extended the mirror method with perturbations which was utilized for finer analysis of certain nonlinear equations possessing negative Fuchsian indices.

In connection with the basin scale dynamics, Maas [5] has considered the flow of fluid contained in a rectangular basin of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moment of mass and heat. The container is assumed to be steady, uniform rotation of an f-plane. With this assumption Maas [5] reduces the rotating stratified Boussinesq equation to a beautiful six coupled system of ODEs. Srinivasan et al. [4] extended this work and gave a detail mathematical analysis of the reduced system of six coupled ODEs. Furthermore, Desale and Patil [2] tested the system of six coupled ODEs (5) for complete integrability using the Painlevé test. Also, they investigated the case of non-integrability for $Ra \neq 0$ and thereby they have obtained weak solutions (in the form of logarithmic psi-series) in the different branches of leading order.

In this paper we have successfully implemented the mirror transformations and constructed the mirror system of (5) for $Ra = 0$ which is regular near movable singularity. Further, with the help of mirror transformation, we have proved that the Laurent series obtained by using the Painlevé test are convergent. In the following section we imploy the mirror transformation to find the mirror system of ideal rotating stratified Boussinesq equations.

2 Mirror System of Six Coupled Non-Linear ODEs

Consider the rotating stratified Boussinesq equations (see Majda [1], p. 1)

$$\begin{aligned} \frac{D\vec{\mathbf{v}}}{Dt} + f(\hat{\mathbf{e_3}} \times \vec{\mathbf{v}}) &= -\nabla p + \nu(\Delta \vec{\mathbf{v}}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e_3}}, \\ \operatorname{div} \vec{\mathbf{v}} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= \kappa \Delta \tilde{\rho}, \end{aligned} \tag{1}$$

where $\vec{\mathbf{v}}$ denotes the velocity field, ρ is the density which is the sum of constant reference density ρ_b and perturbation density $\tilde{\rho}$, p is the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e_3}}$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ is the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{\mathbf{v}} \cdot \nabla)$ is a convective derivative. For more about the rotating stratified Boussinesq equations one may see Majda [1]. Maas [5] reduces the system of equations (1) to the following system of six coupled ODEs:

$$\begin{aligned} Pr^{-1}\frac{d\vec{\mathbf{w}}}{dt} + f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} &= \hat{\mathbf{e_3}} \times \vec{\mathbf{b}} - (w_1,\, w_2,\, rw_3) + \hat{T}\vec{\mathbf{T}}, \\ \frac{d\vec{\mathbf{b}}}{dt} + \vec{\mathbf{b}} \times \vec{\mathbf{w}} &= -(b_1,\, b_2,\, \mu b_3) + Ra\vec{\mathbf{F}}. \end{aligned} \tag{2}$$

In these equations, $\vec{\mathbf{b}} = (b_1, b_2, b_3)$ is the center of mass, $\vec{\mathbf{w}} = (w_1, w_2, w_3)$ is the basin averaged angular momentum vector, $\vec{\mathbf{T}}$ is the differential momentum, $\vec{\mathbf{F}}$ are buoyancy fluxes, $f' = f/2r_h$ is the earth rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ is the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [5] considers the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction. $\vec{\mathbf{F}} = (0, 1, 0)$, the wind effect is neglected, i.e. $\vec{\mathbf{T}} = 0$. For the Prandtl number Pr, equal to one, the system of equations (2) reduces to the following ideal rotating, uniformly stratified system of six coupled ODEs

$$\begin{aligned} \frac{d\vec{\mathbf{w}}}{dt} &= -f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} + \hat{\mathbf{e_3}} \times \vec{\mathbf{b}}, \\ \frac{d\vec{\mathbf{b}}}{dt} &= -\vec{\mathbf{b}} \times \vec{\mathbf{w}} + Ra\vec{\mathbf{F}}. \end{aligned} \tag{3}$$

The system of ODEs (3) can be written component wise as

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \quad \dot{w_3} = 0, \\ &\dot{b_1} = w_2b_3 - w_3b_2, \quad \dot{b_2} = w_3b_1 - w_1b_3 + Ra, \quad \dot{b_3} = w_1b_2 - w_2b_1. \end{aligned} \tag{4}$$

Since $\dot{w_3} = 0$, this gives $w_3 = \text{constant} = k_1$. Consequently, we have the following system of ODEs:

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \\ &\dot{b_1} = w_2b_3 - k_1b_2, \quad \dot{b_2} = k_1b_1 - w_1b_3 + Ra, \quad \dot{b_3} = w_1b_2 - w_2b_1. \end{aligned} \tag{5}$$

In our earlier study [2], we have shown that the system of ODEs (5) is completely integrable provided that $Ra = 0$ and we have determined the solutions in the form of Laurent series with the help of the Painlevé method. Now our aim is to determine the mirror system of (5) and its solutions in the following form:

$$\begin{aligned} &w_1(t) = \theta^{-m_1}, \quad \theta' = l_0 + l_1\theta + l_2\theta^2 + l_3\theta^3 + l_4\theta^4 + \cdots, \\ &w_2(t) = \theta^{-m_2}\Big(w_{20} + w_{21}\theta + w_{22}\theta^2 + w_{23}\theta^3 + w_{24}\theta^4 + \cdots\Big), \\ &b_1(t) = \theta^{-m_3}\Big(b_{10} + b_{11}\theta + b_{12}\theta^2 + b_{13}\theta^3 + b_{14}\theta^4 + \cdots\Big), \\ &b_2(t) = \theta^{-m_4}\Big(b_{20} + b_{21}\theta + b_{22}\theta^2 + b_{23}\theta^3 + b_{24}\theta^4 + \cdots\Big), \\ &b_3(t) = \theta^{-m_5}\Big(b_{30} + b_{31}\theta + b_{32}\theta^2 + b_{33}\theta^3 + b_{34}\theta^4 + \cdots\Big), \end{aligned} \tag{6}$$

where $\theta = t - t_0$ and t_0 is an arbitrary position of singularity. We found that there were several possible cases of dominant balance of the system (5) similar to those in the Painlevé test. Among the several possible cases of principle dominant balance we have obtained the singular solution only in the following case of principle dominant balance:

$$\dot{w_1} = -b_2, \quad \dot{w_2} = b_1, \quad \dot{b_1} = w_2 b_3, \quad \dot{b_2} = -w_1 b_3, \quad \dot{b_3} = w_1 b_2 - w_2 b_1, \tag{7}$$

and the exponent with this principle dominant balance are as follows:

$$m_1 = m_2 = -1, \quad m_3 = m_4 = m_5 = -2. \tag{8}$$

Since w_1, w_2 are of order 1 near the movable singularity, we can introduce the indicial normalization $w_1(t) = \theta^{-1}$ and try to calculate the formal $\theta-$ series of (6) with $m_2 = -1$, $m_3 = m_4 = m_5 = -2$. Since the system (5) is autonomous, the coefficients appearing in the series given by (5) are to be constant. Substituting the values of exponents from (8) into the equations (6) and then substituting these series into the system (5) and hence equating the like powers of θ on both sides, we obtain the following equations in leading order coefficients:

$$\begin{aligned} &l_0 = b_{20}, \quad -w_{20} l_0 = b_{10}, \quad -2b_{10} l_0 = w_{20} b_{30}, \\ &2b_{20} l_0 = b_{30}, \quad -2b_{30} l_0 = b_{20} - w_{20} b_{10}. \end{aligned} \tag{9}$$

Solving equations (9), we find two possible branches of leading order coefficients which are as follows:

$$l_0 = r_1', \; w_{20} = \pm\sqrt{-1-4r_1'^2}, \; b_{10} = \mp r_1' \sqrt{-1-4r_1'^2}, \; b_{20} = r_1', \; b_{30} = 2r_1'^2, \tag{10}$$

where r_1' is an arbitrary constant.

Definition 2.1 The leading exponents m_1, m_2, m_3, m_4, m_5 for system of ODEs (5) are Fuchsian, if the m_*-weighted degree of the right-hand side of (5) is $\leq m_i + 1$.

The m_*-weighted degree of polynomial in w_1, w_2, b_1, b_2, b_3 is found by taking the degree of $w_i's$, $i = 1, 2$, $b_i's$, $i = 1, 2, 3$ to be m_i, $i = 1, 2, 3, 4, 5$. And we verified that the exponents m_i's, $i = 1, 2, 3, 4, 5$ are Fuchsian for the system (5).

Remark 2.1 Since all leading order coefficients given by (10) are nonzero, the selection of leading exponents is natural and these exponents satisfy the Fuchsian condition.

So far in the employment of mirror transformations we have completed the two steps of algorithm, that is, we have determined leading order coefficients in principle dominant balance and exponents. Now, in the following section we will implement the third step of the algorithm and determine the resonances in the following way.

2.1 Resonances

Now we substitute the assumed θ-series (6) with the values of exponents given by (8) into the system of ODEs (5) and after doing some algebraic calculations we specify the following recursive relations to determine the coefficients w_{1j}, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 1, 2, 3, \ldots$ which are valid for $j \geq 2$:

$$M(j) \begin{pmatrix} l_j \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} = \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \\ E_j \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} A_j &= f' w_{2(j-1)}, \quad B_j = -\sum_{k=1}^{j-1} l_k w_{2(j-k)}, \\ C_j &= -k_1 b_{2(j-1)} + \sum_{k=1}^{j-1} w_{2k} b_{3(j-k)} - \sum_{k=1}^{j-1} l_k b_{1(j-k)}, \\ D_j &= k_1 b_{1(j-1)} - \sum_{k=1}^{j-1} l_k b_{2(j-k)}, \\ E_j &= -\sum_{k=1}^{j-1} w_{2k} b_{1(j-k)} - \sum_{k=1}^{j-1} l_k b_{3(j-k)} \ , \end{aligned} \tag{12}$$

and matrix $M(j)$ is

$$M(j) = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ -w_{20} & (j-1)l_0 & -1 & 0 & 0 \\ -2b_{10} & -b_{30} & (j-2)l_0 & 0 & -w_{20} \\ -2b_{20} & 0 & 0 & (j-2)l_0 & 1 \\ -b_{30} & b_{10} & w_{20} & -1 & (j-2)l_0 \end{pmatrix}. \tag{13}$$

The above recursive relations $(11, 12)$ determine the unknown expansion coefficients uniquely unless the determinant of matrix $M(j)$ is zero. Those values of j at which the determinant of matrix $M(j)$ vanishes are called the *resonances*. Here, we observe that for both possible branches of leading order coefficients given in equations (10), the resonances are $j = 0, 2, 3, 4$. Since $j = 0$ is the resonance, one of the variable in (10) appears to be a resonance parameter, say $l_0 = r_1'$, and we should replace it by $\bar{r_1}$ (where $\bar{r_1} = \sqrt{-4 - k_2^2}$, the arbitrary constant k_2 is the resonance parameter in the Painlevé test [2]), which satisfies the condition $\bar{r_1}^{-m_1} = r_1'$, that is, $\bar{r_1}^{-1} = r_1'$. Let us denote by $k_2 = r_1$ the resonance parameter, and hence we have $\bar{r_1} = \sqrt{-4 - r_1^2}$. Now, we refresh the leading order coefficients given by (10) as follows:

$$\begin{aligned} &l_0 = (\sqrt{-4 - r_1^2})^{-1}, \ w_{20} = \pm \frac{r_1}{\sqrt{-4 - r_1^2}}, \ b_{10} = \mp \frac{r_1}{(\sqrt{-4 - r_1^2})^2}, \\ &b_{20} = (\sqrt{-4 - r_1^2})^{-1}, \ b_{30} = \frac{2}{(\sqrt{-4 - r_1^2})^2}. \end{aligned} \tag{14}$$

2.2 Compatibility conditions

Further, we need to check the compatibility conditions for each resonance $j = 2, 3, 4$. We will do this for the first branch.

Case I: Consider the leading order coefficients

$$\begin{aligned} &l_0 = (\sqrt{-4 - r_1^2})^{-1}, \ w_{20} = \frac{r_1}{\sqrt{-4 - r_1^2}}, \ b_{10} = -\frac{r_1}{(\sqrt{-4 - r_1^2})^2}, \\ &b_{20} = (\sqrt{-4 - r_1^2})^{-1}, \ b_{30} = \frac{2}{(\sqrt{-4 - r_1^2})^2}. \end{aligned} \tag{15}$$

• **Compatibility condition at** $j = 1$.

As $j = 1$ is not resonance, we get the unique solution. Since the recursion relations (11, 12) remain valid when $j \geq 2$, we directly substitute the equations (15) into the equations (6) and then into (5). After that, equating the like powers of θ on both sides of the resulting expansion, we obtain the system of linear equations which determine the coefficients l_1, w_{21}, b_{11}, b_{21} and b_{31} uniquely as

$$\begin{aligned} l_1 &= \frac{(-f'+k_1)r_1}{\sqrt{-4-r_1^2}}, \quad w_{21} = \frac{2(f'-k_1)}{\sqrt{-4-r_1^2}}, \\ b_{11} &= \frac{4f'+k_1r_1^2}{r_1^2+4}, \quad b_{21} = \frac{k_1r_1}{\sqrt{-4-r_1^2}}, \quad b_{31} = \frac{2(f'-k_1)r_1}{4+r_1^2}. \end{aligned} \tag{16}$$

• **Compatibility condition at the resonance** $j = 2$.
Now $j = 2$ is a resonance so that one of the coefficients in the computation of the system (11) at this level is independent. Let b_{32} be independent and let $b_{32} = r_2$ (the arbitrary coefficient), where r_2 is the second resonance parameter so that the values of coefficients are given in terms of r_2, which are as follows:

$$\begin{aligned} l_2 &= \frac{(r_2 - f'k_1)}{2}\sqrt{-4-r_1^2}, \quad w_{22} = 0, \\ b_{12} &= \frac{r_1}{2}(f'k_1 - r_2), \quad b_{22} = \frac{1}{2}\left[r_2\sqrt{-4-r_1^2} + \frac{f'(4f'+k_1r_1^2)}{\sqrt{-4-r_1^2}}\right], \quad b_{32} = r_2. \end{aligned} \tag{17}$$

• **Compatibility condition at the resonance** $j = 3$.
To check the compatibility condition at $j = 3$, we substitute the equations (15, 16, 17) into the system of ODEs (5), then we obtain a system of linear equations. While solving that linear system, we found the variable b_{23} to be independent. Now assign the arbitrary value to b_{23}, say $b_{23} = r_3$, and solving the corresponding system we obtain the following solution. At this level of resonance, we have the third resonance parameter r_3:

$$\begin{aligned} & l_3 = r_3,\ w_{23} = \frac{-r_3}{r_1},\ b_{13} = -\frac{1}{\sqrt{-4-r_1^2}}\left(r_1r_3 + \frac{2r_3}{r_1}\right), \\ & b_{23} = r_3,\ b_{33} = \frac{r_3}{\sqrt{-4-r_1^2}}. \end{aligned} \tag{18}$$

• **Compatibility condition at the resonance** $j = 4$.
Now $j = 4$ is the fourth resonance and solving the system (11) for $j = 4$ involves the resonance parameter, say r_4. Solving the system (11) for this value of j, we obtain the following solution with b_{24} as an arbitrary constant with value r_4:

$$\begin{aligned} l_4 &= r_4 + \frac{f'r_3}{r_1}, \quad w_{24} = (k_1 - f')r_3, \\ b_{14} &= \frac{1}{\sqrt{-4-r_1^2}}\left[-r_1r_4 + (-2f'+k_1)r_3\right], \quad b_{24} = r_4, \\ b_{34} &= \frac{(f'-k_1)(2+r_1^2)r_3}{r_1\sqrt{-4-r_1^2}}. \end{aligned} \tag{19}$$

Substituting all the values of coefficients l_j, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 0, 1, 2, 3, 4 \ldots$ into the equations (6), we get

$$\begin{aligned}
\theta' &= \frac{1}{\sqrt{-4-r_1^2}} + \frac{(-f'+k_1)r_1}{\sqrt{-4-r_1^2}}\theta + \frac{1}{2}\sqrt{-4-r_1^2}(r_2 - f'k_1)\theta^2 + r_3\theta^3 \\
&+ (r_4 + \frac{f'r_3}{r_1})\theta^4 + \cdots, \\
w_2(t) &= \theta^{-1}\Big[\frac{r_1}{\sqrt{-4-r_1^2}} + \frac{2(f'-k_1)}{\sqrt{-4-r_1^2}}\theta - \frac{r_3}{r_1}\theta^3 + (k_1 - f')r_3\theta^4 + \cdots\Big], \\
b_1(t) &= \theta^{-2}\Big[-\frac{r_1}{(\sqrt{-4-r_1^2})^2} + (\frac{4f' + k_1 r_1^2}{r_1^2+4})\theta + \frac{r_1}{2}(f'k_1 - r_2)\theta^2 - \frac{1}{\sqrt{-4-r_1^2}} \\
&\left(r_1 r_3 + \frac{2r_3}{r_1}\right)\theta^3 + \frac{1}{\sqrt{-4-r_1^2}}\left(-r_1 r_4 + (-2f' + k_1)r_3\right)\theta^4 + \cdots\Big], \\
b_2(t) &= \theta^{-2}\Big[\frac{1}{\sqrt{-4-r_1^2}} + (\frac{k_1 r_1}{\sqrt{-4-r_1^2}})\theta + \frac{1}{2}\left(r_2\sqrt{-4-r_1^2} + \frac{f'(4f' + k_1 r_1^2)}{\sqrt{-4-r_1^2}}\right)\theta^2 \\
&+ r_3\theta^3 + r_4\theta^4 + \cdots\Big], \\
b_3(t) &= \theta^{-2}\Big[\frac{2}{(\sqrt{-4-r_1^2})^2} - (\frac{2(f'-k_1)r_1}{4+r_1^2})\theta + r_2\theta^2 + \frac{r_3}{\sqrt{-4-r_1^2}}\theta^3 \\
&+ \frac{(f'-k_1)(2+r_1^2)r_3}{r_1\sqrt{-4-r_1^2}}\theta^4 + \cdots\Big].
\end{aligned} \tag{20}$$

We have just finished the primary calculations of the system (11) and we have determined the resonance parameters, say r_1, r_2, r_3 and r_4. In the following subsection we obtain the mirror transformations and consequently, we determine the mirror system of (5).

2.3 Mirror system

In this subsection we will develop the mirror transformations by which we transform the system (5) to its mirror system. Thereby, we discuss the regularity of it.

Now the important step towards determining the mirror system is to introduce a new variable in which we develop the mirror system. Let us introduce the new variables ξ_1, ξ_2, ξ_3 and ξ_4 in the Laurent θ-series of w_2, b_1, b_2 and b_3 by successively truncating the expansion at the free parameters (resonance parameters) r_1, r_2, r_3 and r_4. Now we begin to truncate the θ-series of w_2 at the first resonance parameter r_1 by introducing the variable ξ_1 as

$$w_2(t) = \theta^{-1}\xi_1, \tag{21}$$

where

$$\xi_1 = \frac{r_1}{\sqrt{-4-r_1^2}} + \frac{2(f'-k_1)}{\sqrt{-4-r_1^2}}\theta - \frac{r_3}{r_1}\theta^3 + (k_1 - f')r_3\theta^4 + \cdots. \tag{22}$$

We convert this into

$$r_1 = \xi_1\bar{r_1} - 2(f'-k_1)\theta + \frac{r_3}{\xi_1}\theta^3 - r_3(f'-k_1)(\frac{2}{\xi_1^2\bar{r_1}} + \bar{r_1})\theta^4 + \cdots. \tag{23}$$

Upon substituting the value of r_1 in b_1, we get

$$\begin{aligned} b_1(t) &= -\frac{\xi_1}{\bar{r_1}}\theta^{-2} + \Big(\frac{-2f'-2k_1}{\bar{r_1}^2} - k_1{\xi_1}^2\Big)\theta^{-1} + \Big[\frac{1}{2}(f'k_1-r_2)\xi_1\bar{r_1} \\ &+ \frac{4k_1\xi_1(f'-k_1)}{\bar{r_1}}\Big] + \Big[\frac{-3r_3}{\xi_1\bar{r_1}^2} - \frac{4k_1(f'-k_1)^2}{\bar{r_1}^2} - (f'k_1-r_2)(f'-k_1) \\ &- \xi_1 r_3\Big]\theta + \Big[\frac{-2r_3(f'-k_1)}{{\xi_1}^2\bar{r_1}^3} - \xi_1 r_4 + \frac{(f'r_3-4k_1r_3)}{\bar{r_1}}\Big]\theta^2. \end{aligned} \tag{24}$$

Next we proceed to cut the θ-series of b_1 at r_2 by introducing the second variable, say ξ_2:

$$b_1(t) = -\frac{\xi_1}{\bar{r_1}}\theta^{-2} + \Big(\frac{-2f'-2k_1}{\bar{r_1}^2} - k_1{\xi_1}^2\Big)\theta^{-1} + \xi_2, \tag{25}$$

where

$$\begin{aligned} \xi_2 &= \Big[\frac{1}{2}(f'k_1-r_2)\xi_1\bar{r_1} + \frac{4k_1\xi_1(f'-k_1)}{\bar{r_1}}\Big] + \Big[\frac{-3r_3}{\xi_1\bar{r_1}^2} - \frac{4k_1(f'-k_1)^2}{\bar{r_1}^2} \\ &- (f'k_1-r_2)(f'-k_1) - \xi_1 r_3\Big]\theta + \Big[\frac{-2r_3(f'-k_1)}{{\xi_1}^2\bar{r_1}^3} - \xi_1 r_4 \\ &+ \frac{(f'r_3-4k_1r_3)}{\bar{r_1}}\Big]\theta^2 + \cdots. \end{aligned} \tag{26}$$

From the θ-series of ξ_2, we have

$$\begin{aligned} r_2 &= f'k_1 - \frac{2\xi_2}{\xi_1\bar{r_1}} + \frac{8k_1(f'-k_1)}{\bar{r_1}^2} + \frac{2}{\xi_1\bar{r_1}}\Big[\frac{-3r_3}{\xi_1\bar{r_1}^2} + \frac{4k_1(f'-k_1)^2}{\bar{r_1}^2} - \frac{2\xi_2(f'-k_1)}{\xi_1\bar{r_1}} \\ &- \xi_1 r_3\Big]\theta - \frac{2}{\xi_1\bar{r_1}}\Big[\frac{8r_3}{\xi_1^2\bar{r_1}^3}(f'-k_1) - \frac{8k_1(f'-k_1)^3}{\xi_1\bar{r_1}^3} + \frac{4\xi_2(f'-k_1)^2}{\xi_1^2\bar{r_1}^2} \\ &+ \frac{(f'+2k_1)r_3}{\bar{r_1}} + \xi_1 r_4\Big]\theta^2 + \cdots. \end{aligned} \tag{27}$$

Now, we substitute the value of r_2 into θ-series of b_2 and consequently, we update it. And then after cutting this series at the third resonance parameter r_3, we obtain the θ-series of b_2 as follows:

$$\begin{aligned} b_2(t) &= \frac{1}{\bar{r_1}}\theta^{-2} + k_1\xi_1\theta^{-1} + \Big[\frac{2k_1(f'-k_1)}{\bar{r_1}} + \frac{1}{2}f'k_1\bar{r_1} + \frac{1}{2}f'k_1\xi_1^2\bar{r_1} + \frac{2f'^2}{\bar{r_1}} \\ &- \frac{\xi_2}{\xi_1}\Big] + \xi_3\theta, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \xi_3 &= \Big[\frac{-3r_3}{\xi_1^2\bar{r_1}^2} + \frac{4k_1(f'-k_1)^2}{\xi_1\bar{r_1}^2} - \frac{2\xi_2(f'-k_1)}{\xi_1^2\bar{r_1}} - 2f'k_1\xi_1(f'-k_1)\Big] + \Big[\frac{-8r_3(f'-k_1)}{\xi_1^3\bar{r_1}^3} \\ &+ \frac{8k_1(f'-k_1)^3}{\xi_1^3\bar{r_1}^3} - \frac{4\xi_2(f'-k_1^2)}{\xi_1^3\bar{r_1}^2} - \frac{(f'+k_1)r_3}{\xi_1\bar{r_1}} + \frac{2f'k_1(f'-k_1^2)}{\bar{r_1}}\Big]\theta + \cdots. \end{aligned} \tag{29}$$

From the θ-series of ξ_3 we have

$$\begin{aligned} r_3 &= -\frac{\xi_1^2\bar{r_1}^2\xi_3}{3} + \frac{4}{3}k_1(f'-k_1)^2\xi_1 - \frac{2}{3}\xi_2\bar{r_1}(f'-k_1) - \frac{2}{3}f'k_1\xi_1^3\bar{r_1}^2(f'-k_1) \\ &- \Big[-\frac{8}{9}(f'-k_1)\xi_1\xi_3\bar{r_1} + \frac{8k_1(f'-k_1)^3}{9\bar{r_1}} - \frac{4\xi_2(f'-k_1)^2}{9\xi_1} - \frac{22}{9}(f'-k_1)^2f'k_1\xi_1^2\bar{r_1} \\ &- \frac{1}{9}(f'+k_1)\xi_1^3\xi_3\bar{r_1}^3 + \frac{4}{9}(f'+k_1)(f'-k_1)^2k_1\xi_1^2\bar{r_1} \\ &- \frac{2}{9}(f'+k_1)(f'-k_1)\xi_1\xi_2\bar{r_1}^2 - \frac{2}{9}f'k_1(f'+k_1)(f'-k_1)\xi_1^4\bar{r_1}^3\Big]\theta + \cdots. \end{aligned} \tag{30}$$

Similarly, we truncate the θ series of b_3 at the resonance parameter r_4 and we obtain the following θ-series:

$$\begin{aligned} b_3(t) &= \frac{2}{\bar{r_1}^2}\theta^{-2} - \frac{2(f'-k_1)\xi_1}{\bar{r_1}}\theta^{-1} + \Big[\frac{4(f'-k_1)(f'+k_1)}{\bar{r_1}^2} - \frac{2\xi_2}{\xi_1\bar{r_1}} + f'k_1\Big] \\ &+ \Big[\frac{2}{\bar{r_1}}\xi_3 + \frac{4f'k_1\xi_1(f'-k_1)}{\bar{r_1}} + \frac{\xi_1^2\xi_3\bar{r_1}}{3} - \frac{4k_1\xi_1(f'-k_1)^2}{3\bar{r_1}} + \frac{2}{3}\xi_2(f'-k_1) \\ &+ \frac{2}{3}f'k_1\xi_1^3\bar{r_1}(f'-k_1)\Big]\theta + \xi_4\theta^2. \end{aligned} \tag{31}$$

Hence, we have

$$\begin{aligned} \xi_4 &= \frac{2}{9}\xi_1\xi_3(7k_1-4f') - \frac{4k_1}{9\bar{r_1}^2}(f'-k_1)^2(7f'+8k_1) + \frac{4}{9\xi_1\bar{r_1}}(f'-k_1) \\ &\quad (4k_1-f')\xi_2 + \frac{4}{9}(f'-k_1)^2(-10f+k_1)k_1\xi_1^2 - \frac{2}{9}(2f'-k_1)\xi_1^3\xi_3\bar{r_1}^2 \\ &- \frac{2}{9}(f'-k_1)(2f'-k_1)\xi_1\xi_2\bar{r_1} - \frac{4}{9}f'k_1\xi_1^4\bar{r_1}^2(f'-k_1)(2f'-k_1) - \frac{2}{\bar{r_1}}r_4 \\ &+ \frac{4}{3}k_1\xi_1^2(f'-k_1)(f'^2-2f'k_1+k_1^2) + \cdots. \end{aligned} \tag{32}$$

Using (21), (25), (28) and (31) with $w_1 = \theta^{-1}$, we get the change of variables $(w_1, w_2, b_1, b_2, b_3) \longleftrightarrow (\theta, \xi_1, \xi_2, \xi_3, \xi_4)$. The following is the conversion of given system into the mirror system in terms of the new variables θ, ξ_1, ξ_2, ξ_3 and ξ_4:

$$\begin{aligned} \theta' &= \frac{1}{\bar{r_1}} + (k_1-f')\xi_1\theta + \Big[\frac{2(k_1f'-k_1^2+f'^2)}{\bar{r_1}} + \frac{1}{2}f'k_1\bar{r_1}(1+\xi_1^2) - \frac{\xi_2}{\xi_1}\Big]\theta^2 \\ &+ \xi_3\theta^3, \\ \xi_1' &= \Big[-(1+\xi_1^2)f' - \frac{2(f'+k_1)}{\bar{r_1}^2}\Big] + \Big[\frac{2(k_1f'-k_1^2+f'^2)}{\bar{r_1}} + \frac{1}{2}f'k_1\bar{r_1}\xi_1(1+\xi_1^2)\Big]\theta \\ &+ \xi_1\xi_3\theta^2, \\ \xi_2' &= \Big[\frac{(-1-\xi_1^2)(f'+k_1)}{\bar{r_1}} - \frac{4(f'+k_1)}{\bar{r_1}^3}\Big]\theta^{-2} + \Big[\frac{2\xi_1}{\bar{r_1}^2}(2f'^2-4k_1^2-3k_1f') \\ &- \frac{f'k_1\xi_1}{2}(3+5\xi_1^2) - k_1^2\xi_1 - (k_1-f')k_1\xi_1^3\Big]\theta^{-1} + \Big[-\frac{4k_1\xi_1^2(f'-k_1)^2}{3\bar{r_1}} - \frac{f'k_1^2\xi_1^4\bar{r_1}}{6} \\ &+ \frac{\xi_1\xi_3 + 5f'^2\xi_1^2k_1 - 3f'k_1^2\xi_1^2 - 3k_1^2f' - 3f'^2k_1 + 2k_1^3(1-\xi_1^2)}{\bar{r_1}} - \frac{f'k_1^2\bar{r_1}}{2} + \frac{k_1\xi_2}{\xi_1} \\ &+ \frac{1}{3}(\xi_1^3\xi_3\bar{r_1} + \xi_1\xi_2(2f'+k_1) + 2f'^2k_1\xi_1^4\bar{r_1}) - \frac{4(f'+k_1)(k_1f'-k_1^2+f'^2)}{\bar{r_1}^3} \\ &+ \frac{2\xi_2(f'+k_1)}{\xi_1\bar{r_1}^2}\Big] + \Big[\xi_1\xi_4 + k_1\xi_3(\xi_1^2-1) - \frac{2\xi_3(f'+k_1)}{\bar{r_1}^2}\Big]\theta, \end{aligned} \tag{33}$$

$$
\begin{aligned}
{\xi_3}' &= \Big[\frac{(-1-\xi_1^2)(f'+k_1)}{\xi_1\bar{r_1}} - \frac{4(f'+k_1)}{\xi_1\bar{r_1}^3}\Big]\theta^{-3} + \Big[\frac{-2k_1f'-8k_1^2+4f'^2}{\bar{r_1}^2} - k_1^2(\xi_1^2+1)\\
&- \frac{1}{2}f'k_1(\xi_1^2+1)\Big]\theta^{-2} + \Big[\frac{\xi_3+3f'^2k_1\xi_1+3f'k_1^2\xi_1}{\bar{r_1}} + k_1\xi_2 + \frac{1}{2}f'k_1^2\xi_1^3\bar{r_1} - \xi_3\\
&+ f'^2k_1\xi_1\bar{r_1}(\xi_1^2+1) - \frac{3k_1^2f'+3f'^2k_1-2k_1^3(1-\xi_1^2)}{\xi_1\bar{r_1}} - \frac{f'k_1^2\bar{r_1}}{2\xi_1} + \frac{k_1\xi_2+(1+\xi_1^2)f'\xi_2}{\xi_1^2}\\
&- \frac{4(f'+k_1)(k_1f'-k_1^2+f'^2)}{\xi_1\bar{r_1}^3} + \frac{4\xi_2(f'+k_1)}{\xi_1^2\bar{r_1}^2}\Big]\theta^{-1} + \Big[\frac{-2f'k_1\xi_1^2\bar{r_1}(k_1f'-k_1^2+f'^2)}{\bar{r_1}}\\
&- \frac{1}{2}f'^2k_1^2\bar{r_1}^2\xi_1^2(1+\xi_1^2) + \frac{k_1\xi_3(\xi_1^2-1)}{\xi_1} - \frac{2\xi_3(f'+k_1)}{\xi_1\bar{r_1}^2} - \frac{2\xi_2}{\xi_1\bar{r_1}}(k_1f'-k_1^2+f'^2)\\
&- \frac{f'k_1\bar{r_1}\xi_2(1+\xi_1^2)}{2\xi_1}\Big] + \Big[-f'k_1\xi_1^2\xi_3\bar{r_1} - \frac{\xi_2\xi_3}{\bar{r_1}}\Big]\theta,\\
{\xi_4}' &= \Big[\frac{\xi_1^2+1}{\bar{r_1}} + \frac{4}{\bar{r_1}^3}\Big]\theta^{-5} + \Big[k_1\xi_1(1+\xi_1^2) + \frac{4\xi_1}{\bar{r_1}^2}(-f'+2k_1) + \frac{4\xi_1}{3\bar{r_1}^2}(f'+k_1)\\
&+ \frac{\xi_1(\xi_1^2+1)(f'+k_1)}{3}\Big]\theta^{-4} + \Big[\frac{f'k_1\bar{r_1}}{2} + \frac{2f'k_1\xi_1^2\bar{r_1}}{3} - \frac{\xi_2}{\xi_1} + \frac{4}{\bar{r_1}^3}(f'^2-k_1^2)\\
&+ \frac{1}{\bar{r_1}}(4f'k_1-2k_1^2+2k_1^2\xi_1^2-2f'k_1\xi_1^2) - \frac{4\xi_2}{\xi_1\bar{r_1}^2} + \frac{1}{6}f'k_1\xi_1^4\bar{r_1} + \frac{8(f'^2-k_1^2)}{3\bar{r_1}^3}\\
&+ \frac{1}{3\bar{r_1}}(2k_1f'\xi_1^2+6k_1^2\xi_1^2-2f'^2\xi_1^2-2(k_1^2-f'^2)) + \frac{\xi_1^2\bar{r_1}k_1^2}{3}(\xi_1^2+1)\Big]\theta^{-3}\\
&+ \Big[\frac{2}{3}f'k_1^2\xi_1(1+\xi_1^2\bar{r_1}-4\xi_1^2) - \frac{1}{6}f'k_1^2\xi_1\bar{r_1}^2(\xi_1^4-1) - \frac{2}{3}f'^2k_1\xi_1^3(1+\bar{r_1})\\
&- \frac{1}{3}f'^2k_1\xi_1^3\bar{r_1}^2(1+\xi_1^2) - \frac{1}{3}f'\xi_2(2+\bar{r_1}) + \frac{1}{3}k_1\xi_2(2-\bar{r_1}) - \frac{1}{3}\xi_1^2\xi_2\bar{r_1}(f'+k_1)\\
&+ \frac{1}{\bar{r_1}^2}(4\xi_3+4f'^2k_1\xi_1-12f'k_1^2\xi_1) + \frac{1}{\bar{r_1}}(2\xi_2(f'-2k_1)-4f'k_1\xi_1(f'-k_1))\\
&+ \frac{1}{3\bar{r_1}^2}(20f'k_1\xi_1(f'+k_1)-28k_1^3\xi_1-4\xi_1f'^3) + \frac{1}{3}(\xi_1^2\xi_3-4k_1^3\xi_1)\\
&+ \frac{1}{3\bar{r_1}}(-2k_1\xi_2+4k_1\xi_1(f'-k_1)^2)\Big]\theta^{-2} + \Big[2f'^3\xi_1^2k_1\bar{r_1}(1+\xi_1^2)\\
&- f'^2k_1^2\xi_1^2\bar{r_1}(1+\frac{1}{9}\xi_1^2) + \frac{1}{\bar{r_1}}(-8f'k_1^3\xi_1^2+12f'^3k_1\xi_1^2+4f'^3k_1-4f'^2k_1^2)\\
&- \frac{4\xi_2(f'^2-k_1^2)}{3\xi_1\bar{r_1}^2} - \frac{2(f'-k_1)k_1\xi_2}{3\xi_1} + \frac{8f'k_1(f'^2-k_1^2)}{\bar{r_1}^3} + \frac{1}{9}f'k_1\xi_1^4\bar{r_1}(-7k_1^2+4f'^2)\\
&+ \frac{1}{6}f'^2k_1^2\xi_1^4\bar{r_1}^3(1+\xi_1^2) + \frac{1}{3}\xi_1^3\xi_3\bar{r_1}(f'-\frac{1}{3}k_1) + \xi_1\xi_3\bar{r_1}(2+\frac{1}{3}k_1)\\
&+ \frac{4}{3}k_1\xi_1\xi_2(2f'-k_1) + \frac{2}{3}f'^2\xi_1\xi_2 + \frac{1}{2}f'k_1\bar{r_1}^2\xi_1(\xi_2+\xi_1^2\xi_3) + 2\xi_4\xi_3\bar{r_1}\\
&+ f'k_1^2\bar{r_1}(f'-k_1) + \frac{1}{3\bar{r_1}}(4\xi_1\xi_3(f'+2k_1) + 2f'^3k_1(1-\frac{17}{3}\xi_1^2) + 2f'k_1^3(-7+\xi_1^2)\\
&+ 8f'^2k_1^2(1+2\xi_1^2) + 4k_1^4(1-\frac{5}{3}\xi_1^2) + \frac{8f'^2(f'^2-k_1^2)}{3\bar{r_1}^3})\Big]\theta^{-1} + \Big[\frac{1}{\bar{r_1}^2}(2\xi_2\xi_3\\
&- 8f'k_1\xi_1(f'-k_1)(k_1f'-k_1^2+f'^2)) + \frac{1}{3\bar{r_1}^2}(8k_1(k_1f'-k_1^2+f'^2)(f'-k_1)^2\xi_1\\
&+ 4\xi_3(f'^{-}k_1^2)) + \frac{1}{3}(-f'k_1\xi_1^2\xi_3\bar{r_1}^2 + 2k_1^2f'\xi_1(f'-k_1)^2(1+\xi_1^2) - \xi_3(\xi_1^2-1)\\
&- 2(f'-k_1)\xi_1\xi_4) - \frac{\xi_2\xi_3}{\xi_1} + 2\xi_1^2\xi_3(f'k_1+\xi_2) - f'^2k_1^2\xi_1(f'-k_1)(1+\xi_1^2)\\
&\quad (2+\xi_1^2\bar{r_1}^2) - 4\xi_1^2(k_1f'-k_1^2+f'^2)(\xi_3+f'k_1\xi_1(f'-k_1)) - 2\xi_1\xi_4(f'-k_1)\Big]\\
&+ \Big[-\frac{2\xi_1^2\xi_3\bar{r_1}}{3} + \frac{4k_1\xi_1\xi_3(f'-k_1)(-2f'-k_1)}{3\bar{r_1}} - \frac{4\xi_4(k_1f'-k_1^2+f'^2)}{\bar{r_1}}\\
&- 2f'k_1\xi_1^3\xi_3\bar{r_1} - f'k_1\xi_4\bar{r_1}(1+\xi_1^2) + \frac{2\xi_2\xi_4}{\xi_1}\Big]\theta - 2\xi_3\xi_4\theta^2.
\end{aligned}
\tag{34}
$$

By similar calculations, we can find the mirror system for the following branch of leading order coefficients:

$$\begin{aligned} &l_0 = (\sqrt{-4-r_1^2})^{-1}, \quad w_{20} = -\frac{r_1}{\sqrt{-4-r_1^2}}, \\ &b_{10} = \frac{r_1}{(\sqrt{-4-r_1^2})^2}, \quad b_{20} = (\sqrt{-4-r_1^2})^{-1}, \quad b_{30} = \frac{2}{(\sqrt{-4-r_1^2})^2}. \end{aligned} \tag{35}$$

The mirror system obtained so far for the present case of leading order coefficient is regular if and only if the following condition are satisfied:

$$\xi_1 = \frac{\sqrt{-4-\bar{r_1}^2}}{\bar{r_1}},\ \xi_2 = -\frac{26k_1^2\xi_1}{9\bar{r_1}},\ f' = k_1,\ \xi_3 = 0. \tag{36}$$

The most prominent thing for the singularity analysis is that the system is regular near $\theta = 0$, which corresponds to movable singularity of the system of six coupled ODEs (5).

3 Alternative Approach of the Convergence of Laurent Series in Painlevé Test

The convergence of Laurent series solution obtained by the Painlevé test is guaranteed by Kichenassamy and Littman [4]. But here we are going to present an alternative approach of the convergence of these series by making use of the mirror system and the Cauchy-Kowalevski theorem.

An ideal rotating, uniformly stratified system of six coupled ODEs (5) is completely integrable for the Rayleigh number $Ra = 0$. For $Ra = 0$, the Painlevé test produces the following formal solution of ODEs (5) for the first case of leading order coefficients:

$$\begin{aligned} w_1(t) &= \sqrt{-4-k_2^2}\tau^{-1} + \frac{(f'-k_1)k_2}{2} + \frac{\sqrt{-4-k_2^2}}{2}(-k_3+f'k_1)\tau \\ &+ \Big[-\frac{k_4}{2} + \frac{f'k_2}{4}(-k_3+f'k_1)\Big]\tau^2 \\ &+ \{-\frac{k_5}{3} + \frac{f'\sqrt{-4-k_2^2}}{12k_2}\Big[f'k_2(k_3-f'k_1)+2k_4\Big]\}\tau^3 \\ &+ \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\ w_2(t) &= k_2\tau^{-1} + \Big[\frac{\sqrt{-4-k_2^2}}{2}\Big(-f'+k_1\Big)\Big] + \frac{(-k_3k_2+f'k_2k_1)}{2}\tau \\ &+ \sqrt{-4-k_2^2}\Big[\frac{k_4}{2k_2} + \frac{f'}{4}\Big(k_3-f'k_1\Big)\Big]\tau^2 \\ &+ \Big[\frac{-k_5k_2}{3\sqrt{-4-k_2^2}} + \frac{f'}{12}\Big(f'k_2k_3+2k_4-f'^2k_2k_1\Big)\Big]\tau^3 + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\ b_1(t) &= -k_2\tau^{-2} + f'\sqrt{-4-k_2^2}\tau^{-1} + \frac{(-k_2k_3+f'^2k_2)}{2} + \frac{k_4\sqrt{-4-k_2^2}}{k_2}\tau \\ &- \frac{k_5k_2}{\sqrt{-4-k_2^2}}\tau^2 + \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \end{aligned} \tag{37}$$

$$\begin{aligned}
b_2(t) &= \sqrt{-4-k_2^2}\tau^{-2} + f'k_2\tau^{-1} + \Big[\frac{\sqrt{-4-k_2^2}}{2}(k_3 - f'^2)\Big] + k_4\tau + k_5\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 - \frac{4k_5}{3\sqrt{-4-k_2^2}}\tau^2 \\
&- \frac{1}{6k_2}\Big(f'^2k_2k_3 - 3k_2k_3^2 + 2f'k_4 - f'^3k_2k_1 + 3f'k_2k_3k_1 - 6k_4k_1\Big)\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned} \tag{38}$$

The above Laurent series contains the five arbitrary constant $w_{30} = k_1$, k_2, k_3, k_4 and k_5. Here $\tau = t - t_0$ and t_0 is an arbitrary position of singularity in complex domain. As we see, the above Laurent series has a movable pole type singularity, and using the Painlevé method we conclude that the above Laurent series (37) and (38) are convergent for small τ; and this convergence is guaranteed by Kichenassamy and Littman [4]. But for an alternative approach, we convert these series into an initial value problem for the mirror system (33) and (34). For this purpose we substitute the formal Laurent series (37) and (38) into the mirror transformation $w_1 = \theta^{-1}$, (21), (25), (28) and (31). After simplification, we obtain the following formal power series for θ, ξ_1, ξ_2, ξ_3 and ξ_4:

$$\begin{aligned}
\theta &= (\sqrt{-4-k_2^2})^{-1}\tau - \frac{(f'-k_1)k_2}{(\sqrt{-4-k_2^2})^2}\tau^2 + \frac{1}{4(\sqrt{-4-k_2^2})^3} \\
&\quad (-8k_3 - 2k_2^2k_3 + 8f'k_1 + f'^2k_2^2 + k_1^2k_2^2)\tau^3 \\
&+ \Big[\frac{1}{2(\sqrt{-4-k_2^2})^2}\Big(k_4 + (-k_3 + f'k_1)(\frac{f'k_2}{2} - k_1k_2)\Big) - \frac{(f'-k_1)^3k_2^3}{8(\sqrt{-4-k_2^2})^4}\Big]\tau^4 \\
&+ \Big[\frac{k_5}{3(\sqrt{-4-k_2^2})^2} + \frac{1}{\sqrt{-4-k_2^2}}(-\frac{f'^2}{12}(k_3 - f'k_1) - \frac{f'k_4}{6k_2} + \frac{(-k_3+f'k_1)^2}{4}) \\
&+ \frac{1}{2(\sqrt{-4-k_2^2})^3}(-k_4(f'-k_1)k_2 + \frac{(f'-k_1)k_2^2}{4}(-k_3 + f'k_1)(3k_1 - f')) \\
&+ \frac{(f'-k_1)^4k_2^4}{(16\sqrt{-4-k_2^2})^5}\Big]\tau^5 + \ldots, \\
\xi_1 &= \frac{k_2}{\sqrt{-4-k_2^2}} - \frac{2(f'-k_1)}{4+k_2^2}\tau - \frac{k_2(f'-k_1)^2}{(-4-k_2^2)^{\frac{3}{2}}}\tau^2 + \Big[\frac{1}{(-4-k_2^2)}\Big(\frac{k_2k_4}{2\sqrt{-4-k_2^2}} \\
&- \frac{1}{4}(f'k_1 - k_3)k_1k_2^2 + \frac{1}{8}(-f' + k_1)(8f'k_1 + f'^2k_2^2 + k_1^2k_2^2)\Big) \\
&+ \frac{k_4}{2k_2} - f'^2k_1 + \frac{k_1k_3}{4} - \frac{(f'-k_1)^3k_2^4}{8(-4-k_2^2)^2}\Big]\tau^3 + \ldots, \\
\xi_2 &= \frac{k_2}{2}(f'^2 - 3k_3 + 2f'k_1) + \frac{1}{(-4-k_2^2)}(f'-k_1)k_2(\frac{1}{4}k_2^2(f'+k_1) + 2(f'+2k_1)) \\
&+ \Big[\frac{1}{\sqrt{-4-k_2^2}}\Big(-\frac{k_2k_4}{2} + \frac{1}{4}(f'k_1 - k_3)(4f' - k_1)k_2^2 + 2f'^2k_1 - 3k_3(f'-k_1) \\
&- \frac{1}{4}f'k_1k_2^2(f'-k_1)\Big) + \sqrt{-4-k_2^2}(\frac{k_4}{2k_2} - f'^2k_1 + \frac{k_1k_3}{4}) + \frac{4k_1(f'-k_1)^2}{(-4-k_2^2)^{\frac{3}{2}}}\Big]\tau + \ldots,
\end{aligned} \tag{39}$$

$$
\begin{aligned}
\xi_3 &= \sqrt{-4-k_2^2}\Big(\frac{3}{2}k_4 - \frac{2k_4}{k_2} - \frac{k_4k_2}{2} + \frac{5f'^2k_1}{k_2} - \frac{k_1k_3}{2} + \frac{1}{2}f'^2k_1k_2 + \frac{1}{4}f'k_1^2k_2 - \frac{f'^3}{k_2} \\
&+ \frac{2f'k_1^2}{k_2}\Big) + \frac{(f'-k_1)}{\sqrt{-4-k_2^2}}\Big[-2f'k_1k_2 + \frac{1}{2}k_2(k_1^2 - f'^2) - \frac{3}{8}f'k_1k_2^3 + \frac{1}{4}k_1^2k_2^3 \\
&- \frac{4f'}{k_2}(f'-k_1)\Big] + \{-\frac{3}{8}k_1^2k_2^2k_3 + \frac{1}{2}f'k_2k_4 - \frac{3}{8}f'^2k_2^2k_3 + \frac{5}{8}f'^3k_1k_2^2 - \frac{15}{8}k_1k_2k_4 \\
&- \frac{1}{2}f'^2k_1^2 - 2f'k_1^3 + \frac{3}{2}f'^3k_1 + \frac{7}{2}f'k_1k_3 - \frac{3}{2}k_1^2k_3 - f'^2k_3 - \frac{5}{8}f'^2k_1^2k_2^2 \\
&+ f'k_1k_2^2k_3 - f'k_1(f'-k_1) + \frac{1}{4}k_1^3k_2^2(f' - \frac{1}{2}k_1^2) + \frac{1}{k_2}\Big[2f'^4 - 6f'k_3(f'-2k_1) \\
&- 6k_1^2(f'^2 + k_3 + 4f'k_1^3 + f'k_4 + \frac{2f'^3k_1}{k_2} + \frac{6k_3(f'-k_1)^2}{k_2} - k_1k_4 - \frac{2f'^2k_1^2}{k_2})\Big] \\
&+ (-4-k_2^2)\Big[-\frac{5}{12}f'^2k_3 + \frac{f'k_4}{6k_2} - \frac{13}{12}f'^3k_1 - \frac{1}{4}f'k_1k_3 + \frac{1}{2}(k_3^2 + f'^3k_1) - \frac{k_1k_4}{k_2} \\
&+ 2f'^2k_1^2 - \frac{k_1^2}{2}(4f'^2 - k_3) - \frac{1}{k_2^3}k_4(k_1 - 2f') + \frac{2f'^2k_1}{k_2^2}(f'-k_1) + \frac{k_1^2k_3}{2k_2^2}\Big] \\
&+ \sqrt{-4-k_2^2}\Big[\frac{5}{3}k_5 + \frac{1}{k_2}(6f'^2k_1 + 2k_1k_3 - 3f'k_3) - k_4 + \frac{7}{4}f'^2k_1k_2 - \frac{2k_4}{k_2^2} \\
&- f'k_2k_3\Big] + \frac{1}{(-4-k_2^2)}\Big[k_1(f'-k_1)^3(4 - \frac{8}{k_2^2} + \frac{k_2^4}{8} + \frac{1}{2}k_2^2) + \frac{4(f'-k_1)^3}{k_2} \\
&\quad (\frac{1}{4}k_2^2(f'+k_1) + 2(f'+2k_1))\Big]\}\tau + \ldots, \\
\xi_4 &= \frac{5}{18}k_5\sqrt{-4-k_2^2} + \frac{2}{3}(f'-k_1)^2(2f'+k_1)k_1 - 12f'k_1(f'-k_1)^2k_2^2 \\
&- \frac{1}{\sqrt{-4-k_2^2}}(f'-k_1)^3(f'+k_1)k_2^2 + \frac{1}{3}(f'-k_1)^2\sqrt{-4-k_2^2} + \frac{(f'+k_1)^2k_2^2}{3(4+k_2^2)} \\
&\quad [12f'k_1 + f'k_1k_2^2 - 3(f'-k_1)^2k_2^2] + \frac{1}{3}k_2(f'-k_1)^2(\frac{8}{k_2^3} - \frac{2}{k_2} + \frac{1}{2}k_2 \\
&- \frac{2(4+k_2^2)(f'k_1-k_3)}{k_2(f'-k_1)^2})\Big[\frac{f'^2+2f'k_1-(f'-k_1)(f'(8+k_2^2)+k_1(16+k_2^2))}{2(4+k_2^2)} - 3k_3\Big] \\
&+ 2(f'k_1-k_3)[3(f'-k_1)^2 - 2(f'^2-k_1^2)\sqrt{-4-k_2^2}] + \frac{7}{2}(4+k_2^2) \\
&\quad (f'k_1-k_3)^2 + \frac{1}{4}(f'-k_1)^2k_2^2k_3 - \frac{5}{2}(f'-k_1)k_2(f'^2k_1k_2 - f'k_2k_3 - 2k_4) \\
&+ \frac{72(f'-k_1)k_2(f'^2k_1k_2 - f'k_2k_3 - 2k_4)}{\sqrt{-4-k_2^2}} + 12(f'-k_1)(-4-k_2^2)\Big[-2f'^2k_1 \\
&+ \frac{1}{2}k_1k_3 + \frac{k_4}{k_2} - \frac{k_2^4(f'-k_1)^4}{(4+k_2^2)^2} + \frac{k_2^2(-f'^2 - f'k_1^2 + k_1^3 + k_1(f'^2+2k_3)) + 4k_2k_4}{\sqrt{-4-k_2^2}}\Big] \\
&+ \ldots,
\end{aligned}
\tag{40}
$$

Thus, we have the mirror system (33) and (34) with the following initial data

$$
\begin{aligned}
\theta(0) &= 0, \quad \xi_1(0) = \frac{k_2}{\sqrt{-4-k_2^2}}, \\
\xi_2(0) &= \frac{k_2}{2}(-3k_3 + f'^2 + 2f'k_1) + \frac{1}{(-4-k_2^2)}(f'-k_1)k_2\Big[\frac{1}{4}k_2^2(f'+k_1) \\
&+ 2(f'+2k_1)\Big], \\
\xi_3(0) &= \sqrt{-4-k_2^2}\Big(\frac{3}{2}k_4 - \frac{2k_4}{k_2} - \frac{k_4k_2}{2} + \frac{5f'^2k_1}{k_2} - \frac{k_1k_3}{2} + \frac{1}{2}f'^2k_1k_2 + \frac{1}{4}f'k_1^2k_2 \\
&- \frac{f'^3}{k_2} + \frac{2f'k_1^2}{k_2}\Big) + \frac{(f'-k_1)}{\sqrt{-4-k_2^2}}\Big[(-2f'k_1k_2 + \frac{1}{2}k_2(k_1^2 - f'^2) - \frac{3}{8}f'k_1k_2^3 \\
&+ \frac{1}{4}k_1^2k_2^3 - \frac{4f'}{k_2}(f'-k_1)\Big], \\
\xi_4(0) &= \frac{5}{18}k_5\sqrt{-4-k_2^2} + \frac{2}{3}(f'-k_1)^2(2f'+k_1)k_1 - 12f'k_1(f'-k_1)^2k_2^2 + \ldots.
\end{aligned}
\tag{41}
$$

Now we are ready to show the convergence of (37)and (38) by using the Cauchy theorem [10, p.150-151]. From the differential equations (33) and (34) and the initial conditions (41) we see that the coefficients of variable in (33), (34) and initial value conditions (41) are analytic functions provided that $r_1 = k_2 \neq \pm 2i$. Thus, the initial value problem (33) and (34) with initial conditions (41) has unique analytic solutions which are convergent in the neighbourhood of $\theta = 0$.

Substituting the series (39) and (40) back into $w_1 = \theta^{-1}$, (21), (25), (28) and (31), we obtain the convergent power series for w_1, w_2, b_1, b_2 and b_3 which was not just formal. Furthermore, with some computation we see that these series are exactly (37) and (38). Therefore, we come to the conclusion that the Laurent series (37) and (38) are convergent. Thus, we summarise these results in terms of the following theorem.

Theorem 3.1 *For the principal Laurent series solution of the ideal rotating, uniformly stratified system of six coupled ODEs (3), there is a change of variables of the form (6) such that the system of ODEs (3) is transformed into a regular system of ODEs (33) and (34) for the new variables* $(\theta, \xi_1, \xi_2, \xi_3, \xi_4)$. *Further, the Laurent series (37) and (38) in the principle dominant balance are converted into the power series (39) and (40) with initial data (41) which are the analytic functions in terms of new variables and thus, the series solutions (39)with (40) are convergent in the neighbourhood of* $\theta = 0$.

4 Conclusion

The reduced system of ODEs (3) which arose in the reduction of uniformly stratified fluid contained in the rotating box of dimension $L \times L \times H$ is completely integrable if the Rayleigh number $Ra = 0$. By taking $Ra = 0$, we have obtained the mirror system for both possible branches of leading ordered coefficients of system (3). The main feature in the singularity analysis is that the mirror system is regular near $\theta = 0$, which corresponds to the movable singularity of the system (3) provided (36) holds. Also, we have shown that the formal Laurent series solutions arising from successful application of the Painlevé test to the system of ODEs (3) are convergent.

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